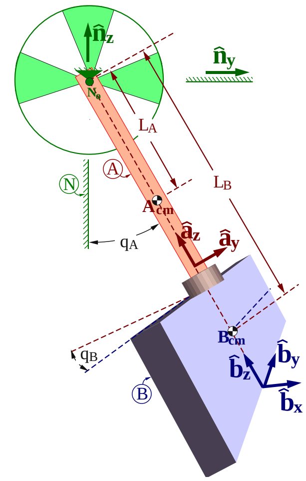
20.5 Kane/Lagrange methods for chaotic motion of a double pendulum.



The figure to the right is a schematic representation of a swinging babyboot attached by a shoelace to a rigid support. The mechanical model of the babyboot consists of a thin uniform rod A attached to a fixed support N by a revolute joint, and a uniform plate B connected to A with a second revolute joint so that B can rotate freely about A's axis.
Note: The revolute joints' axes are ***perpendicular***, not parallel.

Modeling considerations

- The plate, rod, and support are rigid.
- The revolute joints are ideal (massless, frictionless, no slop/flexibility).
- Earth is a Newtonian reference frame.
- Air resistance is negligible.
- Forces due to Earth's gravitation are uniform and constant.
- Other distance forces (electromagnetic and gravitational) are negligible.

Right-handed sets of unit vectors $\hat{\mathbf{n}}_x, \hat{\mathbf{n}}_y, \hat{\mathbf{n}}_z$; $\hat{\mathbf{a}}_x, \hat{\mathbf{a}}_y, \hat{\mathbf{a}}_z$; $\hat{\mathbf{b}}_x, \hat{\mathbf{b}}_y, \hat{\mathbf{b}}_z$ are fixed in N, A, B, respectively, with $\hat{\mathbf{n}}_x = \hat{\mathbf{a}}_x$ parallel to the revolute axis joining A to N, $\hat{\mathbf{n}}_z$ vertically upward, $\hat{\mathbf{a}}_z = \hat{\mathbf{b}}_z$ parallel to the rod's long axis (and the revolute axis joining B to A), and $\hat{\mathbf{b}}_z$ perpendicular to plate B.

Identifiers – for describing this system's motion

Description	Symbol	Type	Value
Earth's gravitational constant	g	Constant	9.81 m/s^2
Distance between N_o and A_{cm}	L_A	Constant	7.5 cm
Distance between N_o and B_{cm}	L_B	Constant	20 cm
Mass of A	m^A	Constant	0.01 kg
Mass of B	m^B	Constant	0.1 kg
A's moment of inertia about A_{cm} for $\hat{\mathbf{a}}_x$	I^A	Constant	0.05 kg∗cm^2
B's moment of inertia about B_{cm} for $\hat{\mathbf{b}}_x$	I^B_x	Constant	2.5 kg∗cm^2
B's moment of inertia about B_{cm} for $\hat{\mathbf{b}}_y$	I^B_y	Constant	0.5 kg∗cm^2
B's moment of inertia about B_{cm} for $\hat{\mathbf{b}}_z$	I^B_z	Constant	2.0 kg∗cm^2
Angle from $\hat{\mathbf{n}}_z$ to $\hat{\mathbf{a}}_z$ with $+\hat{\mathbf{n}}_x$ sense	q_A	Variable	Varies
Angle from $\hat{\mathbf{a}}_y$ to $\hat{\mathbf{b}}_y$ with $+\hat{\mathbf{a}}_z$ sense	q_B	Variable	Varies

(a) Use the ***road maps*** of Section 22.6 to form equations for $\ddot{q}_A$ and $\ddot{q}_B$.

(b) Form ***Kane's equations*** (see Chapter 25) for the ***generalized speeds*** $\dot{q}_A$ and $\dot{q}_B$ or ***Lagrange's equations*** (see Chapter 26) for the ***generalized coordinates*** q_A and q_B.
Result: (Note: Problem solution at www.MotionGenesis.com ⇒ Get Started ⇒ Babyboot).

$$\ddot{q}_A = \frac{2\,\dot{q}_A\,\dot{q}_B\,\sin(q_B)\,\cos(q_B)\,\left(\mathrm{I}^B_x - \mathrm{I}^B_y\right) \;-\; \left(m^A\,L_A + m^B\,L_B\right) g\sin(q_A)}{\mathrm{I}^A + m^A\,L_A^2 + m^B\,L_B^2 + \mathrm{I}^B_x\cos^2(q_B) + \mathrm{I}^B_y\sin^2(q_B)}$$

$$\ddot{q}_B = \frac{-\dot{q}_A{}^2\,\sin(q_B)\,\cos(q_B)\,(\mathrm{I}^B_x - \mathrm{I}^B_y)}{\mathrm{I}^B_z}$$

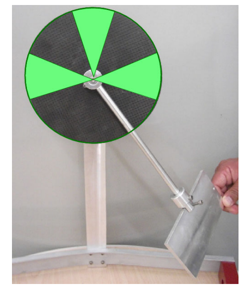
(c) **Optional:** Use separate ***free-body diagrams*** of A and B and Newton/Euler methods to form equations for $\ddot{q}_A$ and $\ddot{q}_B$.

(d) Comment on the relative difficulty of forming this system's equations of motion with each of the four methods. Also comment on the relative efficiency of the resulting ODEs.
Result: Forming dynamics equations with ***road maps*** or ***Kane's or Lagrange's equations*** produces a minimal number (2) of efficient ODEs and is significantly easier than employing Newton/Euler methods with separate ***free-body diagrams*** of both A and B (which introduces many unknown reaction force/torque measures - and generates a larger number of coupled algebraic-differential equations).

Interpret

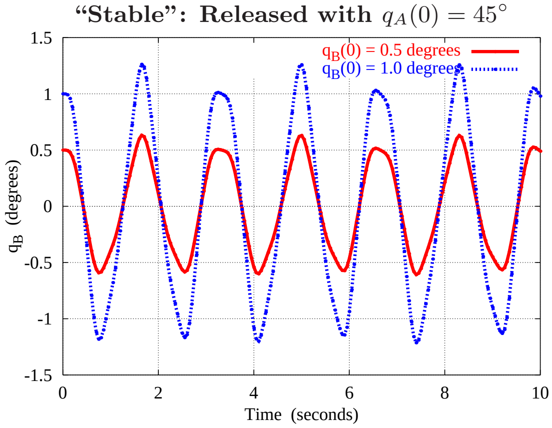
The solution to these differential equations reveals this simple system has strange, non-intuitive motion.[1] For certain initial values of q_A, the motion of plate B is well-behaved and "stable". Alternately, for other initial values of q_A, B's motion is **"chaotic"** – meaning that a small variation in the initial value of q_B or numerical integration inaccuracies lead to dramatically different results (these ODEs are used to test the accuracy of numerical integrators – the plots below required a numerical integrator error of absError $= 1 \times 10^{-7}$).

The following chart and figure to the right shows this system's regions of stability (**black**) and instability (**green**). Notice the **"chaotic"** plot below shows q_B is ***very*** sensitive to initial values. A 0.5° change in the initial value of $q_B(0)$ results in more than a 2000° difference in the value of $q_B(t = 10)$!

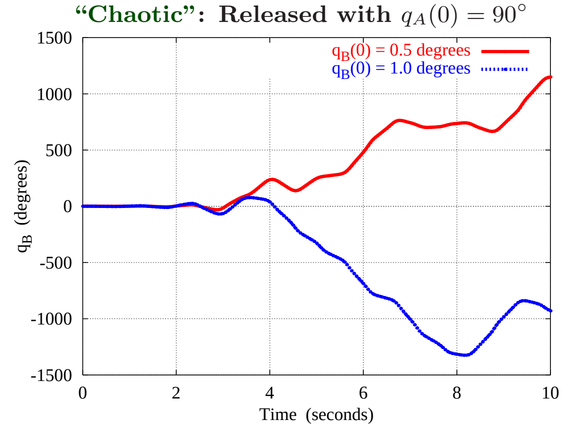


Initial value of q_A			Stability	
0°	$\leq q_A(0) \leq$	71.3°	**Stable**	**black**
71.4°	$\leq \mathbf{q_A(0)} \leq$	**111.77°**	**Unstable**	**green**
111.78°	$\leq q_A(0) \leq$	159.9°	**Stable**	**black**
160.0°	$\leq \mathbf{q_A(0)} \leq$	**180.0°**	**Unstable**	**green**

"Stable": Released with $q_A(0) = 45°$



"Chaotic": Released with $q_A(0) = 90°$



[1] More information about this problem is in "Mechanical Demonstration of Mathematical Stability and Instability", *International Journal of Engineering Education (Journal of Mechanical Engineering Education)*, Vol. 2, No. 4, 1974, pp. 45-47, by Thomas R. Kane. Or visit www.MotionGenesis.com ⇒ Get Started ⇒ `Chaotic Pendulum (Babyboot)`.

