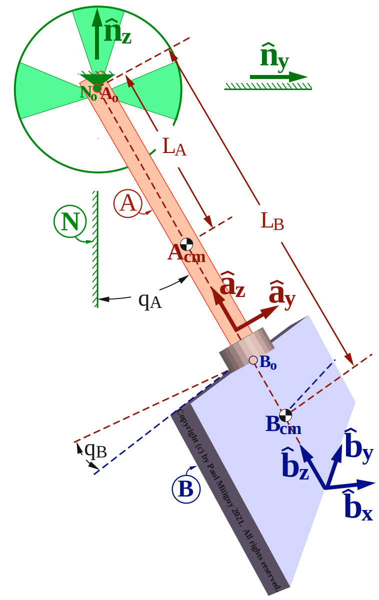
Motivating example (MIPSI): Babyboot



Model

Shown right is a mechanical model of a swinging babyboot (uniform plate B) attached by a shoelace (thin uniform rod A) to a fixed rigid support N. Rod A is attached to N by a revolute joint at point N_o of N. B is attached to A with a second revolute joint at point B_o so B can rotate freely about A's axis.

Note: The revolute joints' axes are ***perpendicular***, not parallel.

- **Bodies**: The rod and plate are **rigid** (inflexible/undeformable).
- **Connections**: The revolute joints are **ideal** (massless, frictionless, with no slop or flexibility).
- **Force**: Earth's gravity is uniform and constant. Other contact forces (e.g., air resistance, solar/light pressure) and distance forces (e.g., electromagnetic, other gravitational) are negligible.
- **Newtonian reference frame**: Earth

Identifiers

Right-handed sets of unit vectors $\hat{n}_x$, $\hat{n}_y$, $\hat{n}_z$; $\hat{a}_x$, $\hat{a}_y$, $\hat{a}_z$; $\hat{b}_x$, $\hat{b}_y$, $\hat{b}_z$ are fixed in N, A, B, respectively, with $\hat{n}_x = \hat{a}_x$ parallel to the revolute axis joining A to N, $\hat{n}_z$ vertically-upward, $\hat{a}_z = \hat{b}_z$ parallel to the rod's long axis (and the revolute axis joining B to A), and $\hat{b}_z$ perpendicular to plate B.

Quantity	Symbol	Type	Value
Earth's gravitational constant	g	Constant	9.81 m/s^2
Distance between N_o and A_{cm}	L_A	Constant	7.5 cm
Distance between N_o and B_{cm}	L_B	Constant	20 cm
Mass of A	m^A	Constant	0.01 kg
Mass of B	m^B	Constant	0.1 kg
A's moment of inertia about A_{cm} for $\hat{a}_x$	I^A	Constant	0.05 kg$*$cm^2
B's moment of inertia about B_{cm} for $\hat{b}_x$	I^B_x	Constant	2.5 kg$*$cm^2
B's moment of inertia about B_{cm} for $\hat{b}_y$	I^B_y	Constant	0.5 kg$*$cm^2
B's moment of inertia about B_{cm} for $\hat{b}_z$	I^B_z	Constant	2.0 kg$*$cm^2
Angle from $\hat{n}_z$ to $\hat{a}_z$ with $+\hat{n}_x$ sense	q_A	Dependent variable	Varies
Angle from $\hat{a}_y$ to $\hat{b}_y$ with $+\hat{a}_z$ sense	q_B	Dependent variable	Varies
Time	t	Independent variable	Varies

Physics

Physics from www.MotionGenesis.com ⇒ Get Started ⇒ Chaotic Pendulum (Babyboot).

The ODEs (ordinary differential equations) governing the motion of this mechanical system are[4]

$$\ddot{q}_A = \frac{2\,\dot{q}_A\,\dot{q}_B\,\sin(q_B)\,\cos(q_B)\,\left(\mathrm{I}^B_x - \mathrm{I}^B_y\right) \;-\; \left(m^A L_A + m^B L_B\right) g \sin(q_A)}{\mathrm{I}^A + m^A L_A^2 + m^B L_B^2 + \mathrm{I}^B_x \cos^2(q_B) + \mathrm{I}^B_y \sin^2(q_B)}$$

$$\ddot{q}_B = \frac{-\dot{q}_A^2\,\sin(q_B)\,\cos(q_B)\,(\mathrm{I}^B_x - \mathrm{I}^B_y)}{\mathrm{I}^B_z}$$

[4]Four methods to form equations of motion are: ***Free-body diagrams*** of A and B (which is inefficient as it introduces up to 10 unknown force/torque measures); D'Alembert's method (***MG road-maps*** of Section 23.1) which efficiently forms the two equations shown for $\ddot{q}_A$ and $\ddot{q}_B$ (but requires a clever selection of systems, points, and unit vectors); ***Lagrange's equations*** (an energy-based method that automates D'Alembert's cleverness); ***Kane's equations*** (a modern efficient blend of D'Alembert and Lagrange).

Simplify and solve

Computers has revolutionized the solution of differential equations. There are a many numerical algorithms for solving nonlinear, coupled, variable coefficient, ODEs (ordinary differential equations) including Euler's method, predictor-corrector, Runga-Kutta, etc. In addition, there are many programs (MATLAB®, **M**otion**G**enesis, WolframAlpha, etc.) that make it easy to solve ODEs.

Computer (numerical) solution of ODEs with MotionGenesis (with plotting)

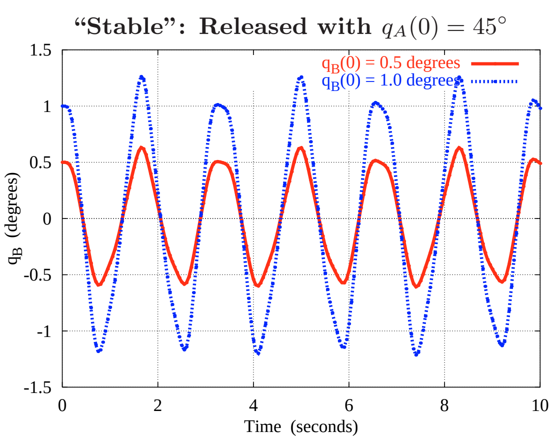
```
Variable  qA'', qB''      % Angles and first/second time-derivatives.
%--------------------------------------------------------------------
qA'' = 2*( 508.89*sin(qA) - sin(qB)*cos(qB)*qA'*qB' ) / (-21.556 + sin(qB)^2)
qB'' = -sin(qB)*cos(qB)*qA'^2
%--------------------------------------------------------------------
Input  tFinal = 10 sec, tStep = 0.02 sec, absError = 1.0E-07
Input  qA = 90 deg,  qB = 1.0 deg,  qA' = 0.0 rad/sec,  qB' = 0.0 rad/sec
OutputPlot  t sec,  qA degrees,  qB degrees
%--------------------------------------------------------------------
ODE() solveBabybootODE
Quit
```

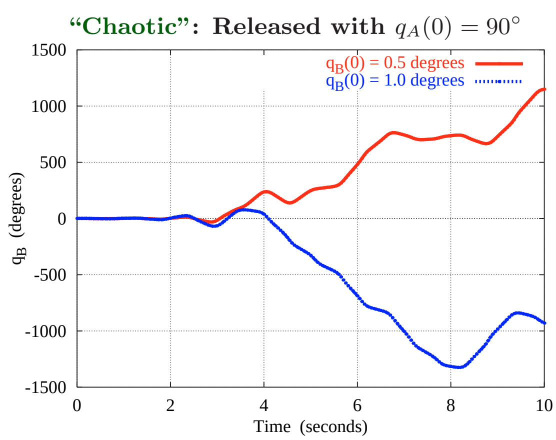
Interpret

This simple system has strange, non-intuitive motion. For certain initial values of q_A, plate B's motion is well-behaved and "stable" whereas for other initial values of q_A, B's motion is **"chaotic"** – meaning that a small variation in the initial value of q_B or numerical integration inaccuracies lead to dramatically different results (these ODEs are used to test the accuracy of numerical integrators – the plots below required a numerical integrator error of absError $= 1 \times 10^{-7}$).

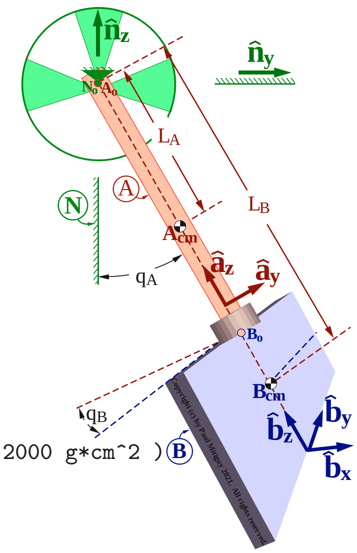
The following chart and figure to the right shows this system's regions of stability (**black**) and instability (**green**). Notice the **"chaotic"** plot below shows q_B is ***very*** sensitive to initial values. A 0.5° change in the initial value of $q_{B}(0)$ results in more than a 2000° difference in the value of $q_{B}(t{=}10)$!

Initial value of q_A			Stability	
0°	$\le q_A(0) \le$	71.3°	Stable	black
71.4°	$\le \mathbf{q_A(0)} \le$	**111.77°**	**Unstable**	**green**
111.78°	$\le q_A(0) \le$	159.9°	Stable	black
160.0°	$\le \mathbf{q_A(0)} \le$	**180.0°**	**Unstable**	**green**

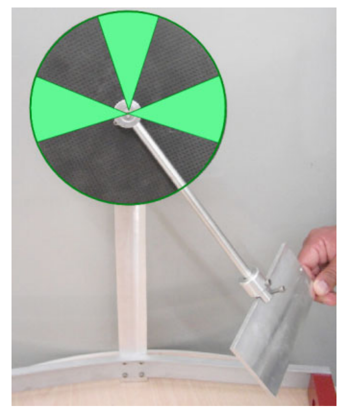




More information about this problem is in "Mechanical Demonstration of Mathematical Stability and Instability", *International Journal of Engineering Education (Journal of Mechanical Engineering Education)*, Vol. 2, No. 4, 1974, pp. 45-47, by Thomas Kane. Or visit **www.MotionGenesis.com ⇒ Get Started ⇒ Chaotic Pendulum (Babyboot)**.

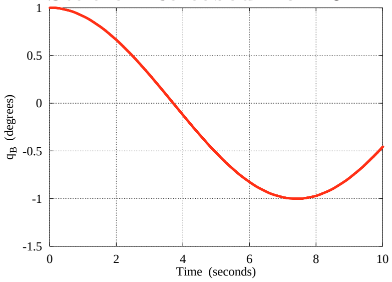



```
% MotionGenesis file:  MGBabybootDynamics.txt
% Problem: Analysis of 3D chaotic double pendulum.
% Copyright (c) 2009 Motion Genesis LLC.  All rights reserved.
%--------------------------------------------------------------------
SetDigits( 5 )              % Number of digits displayed for numbers.
%--------------------------------------------------------------------
NewtonianFrame  N           % Earth.
RigidBody       A           % Upper rod.
RigidBody       B           % Lower plate.
%--------------------------------------------------------------------
Variable   qA''             % Pendulum angle and its time-derivatives.
Variable   qB''             % Plate angle and its time-derivative.
Constant   LA = 7.5 cm      % Distance from pivot to A's mass center.
Constant   LB = 20 cm       % Distance from pivot to B's mass center.
Constant   g = 9.81 m/s^2   % Earth's gravitational acceleration.
A.SetMassInertia( mA =  10 grams,  IAx = 50 g*cm^2,  IAy,  IAz )
B.SetMassInertia( mB = 100 grams,  IBx = 2500 g*cm^2, IBy = 500 g*cm^2, IBz = 2000 g*cm^2 )
%--------------------------------------------------------------------
%   Rotational kinematics.
A.RotateX( N, qA )
B.RotateZ( A, qB )
%--------------------------------------------------------------------
%   Translational kinematics.
Acm.Translate( No, -LA*Az> )
Bcm.Translate( No, -LB*Az> )
%--------------------------------------------------------------------
%   Add relevant contact/distance forces.
System.AddForceGravity( -g*Nz> )
%--------------------------------------------------------------------
%   Equations of motion via free-body-diagrams (MG road-maps).
Dynamics[1] = Dot( Ax>,   System(A,B).GetDynamics(No)   )
Dynamics[2] = Dot( Bz>,             B.GetDynamics(Bcm) )
%--------------------------------------------------------------------
%   Kinetic and potential energy.
KE = System.GetKineticEnergy()
PE = System.GetForceGravityPotentialEnergy( -g*Nz>, No )
MechanicalEnergy = KE + PE
%--------------------------------------------------------------------
%   Optional: Equations of motion with Kane's method.
SetGeneralizedSpeed( qA', qB' )
KaneDynamics = System.GetDynamicsKane()
%--------------------------------------------------------------------
%   Optional: Equations of motion with Lagranges's method.
SetGeneralizedCoordinates( qA, qB )
LagrangeDynamics = System.GetDynamicsLagrange( SystemPotential = PE )
%--------------------------------------------------------------------
%   Solve dynamics equations for qA'', qB''.
Solve( Dynamics = 0,   qA'',  qB'')
%--------------------------------------------------------------------
%   Integration parameters and initial values.
Input  tFinal = 10 sec,  tStep = 0.02 sec,  absError = 1.0E-07,  relError = 1.0E-07
Input  qA = 90 deg,  qA' = 0.0 rad/sec,  qB = 1.0 deg,  qB' = 0.0 rad/sec
%--------------------------------------------------------------------
%   List output quantities and solve ODEs.
Output  t sec,  qA deg,  qB deg,  MechanicalEnergy Joules
ODE()  MGBabybootDynamics
%--------------------------------------------------------------------
%   Record input together with responses
Save MGBabybootDynamics.html
Quit
```

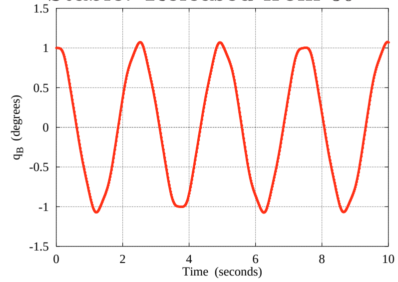


Investigation of stability: More simulation results

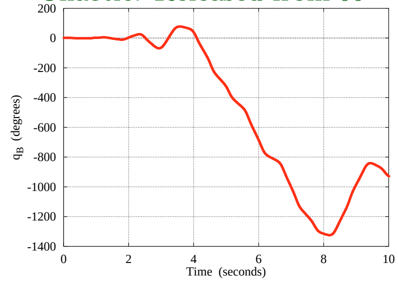
Stable: Released from 5°

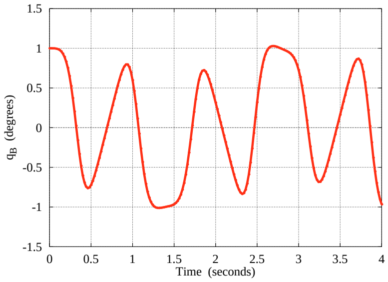


Stable: Released from 30°

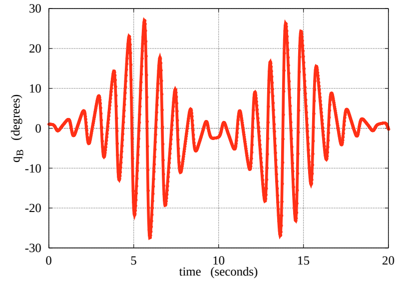


Chaotic: Released from 90°

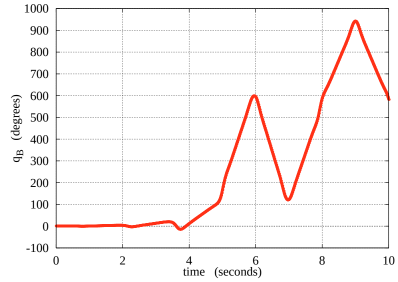




Stable: Released from 135°



Beat: Released from 158°



Chaotic: Released from 177°