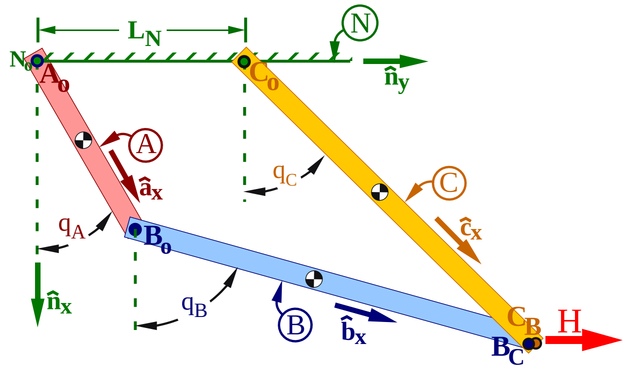
22.5 Four-bar linkage statics (refers to Hw 10.7).



The figure to the right shows a planar four-bar linkage consisting of frictionless-pin-connected uniform rigid links A, B, C and ground N.

- Link A connects to N and B at points A_o and A_B
- Link B connects to A and C at points B_o and B_C
- Link C connects to N and B at points C_o and C_B
- Point N_o of N is coincident with A_o
- Point N_C of N is coincident with C_o

Right-handed orthogonal unit vectors $\hat{\mathbf{a}}_i$, $\hat{\mathbf{b}}_i$, $\hat{\mathbf{c}}_i$, $\hat{\mathbf{n}}_i$ $(i = x, y, z)$ are fixed in A, B, C, N, with:

- $\hat{\mathbf{a}}_x$ directed from A_o to A_B
- $\hat{\mathbf{b}}_x$ directed from B_o to B_C
- $\hat{\mathbf{c}}_x$ directed from C_o to C_B
- $\hat{\mathbf{n}}_x$ vertically-downward
- $\hat{\mathbf{n}}_y$ directed from N_o to N_C
- $\hat{\mathbf{a}}_z = \hat{\mathbf{b}}_z = \hat{\mathbf{c}}_z = \hat{\mathbf{n}}_z$ parallel to pin axes

Quantity	Symbol	Value
Length of link A	L_A	1 m
Length of link B	L_B	2 m
Length of link C	L_C	2 m
Distance between N_o and N_C	L_N	1 m
Mass of A	m^A	10 kg
Mass of B	m^B	20 kg
Mass of C	m^C	20 kg
Earth's gravitational acceleration	g	9.81 $\frac{m}{s^2}$
$\hat{\mathbf{n}}_y$ measure of force applied to C_B	H	200 N
Angle from $\hat{\mathbf{n}}_x$ to $\hat{\mathbf{a}}_x$ with $^+\hat{\mathbf{n}}_z$ sense	q_A	Variable
Angle from $\hat{\mathbf{n}}_x$ to $\hat{\mathbf{b}}_x$ with $^+\hat{\mathbf{n}}_z$ sense	q_B	Variable
Angle from $\hat{\mathbf{n}}_x$ to $\hat{\mathbf{c}}_x$ with $^+\hat{\mathbf{n}}_z$ sense	q_C	Variable

As in Hw 10.7, create the following "***loop equation***" and dot-product with $\hat{\mathbf{n}}_x$ and $\hat{\mathbf{n}}_y$.

$$L_A\,\hat{\mathbf{a}}_x + L_B\,\hat{\mathbf{b}}_x - L_C\,\hat{\mathbf{c}}_x - L_N\,\hat{\mathbf{n}}_y = \vec{\mathbf{0}}$$

Complete the following ***MG road-map*** to determine this systems's ***static configuration***.
Make a "cut" between points B_C and C_B and introduce a constraint force $\vec{\mathbf{F}}^{C_B}$ on C_B from B_C.

Variable	Translate/ Rotate	Direction (unit vector)	System S	FBD of S	About point	***MG road-map equation***	Additional Unknowns
q_A	Rotate	$\hat{\mathbf{n}}_z$	A, B	**Draw**	A_o	$\hat{\mathbf{a}}_x \cdot \vec{\mathbf{M}}^{S/A_o} = 0$	$F_x^{C_B}$, $F_y^{C_B}$
q_B	Rotate	$\hat{\mathbf{n}}_z$	B	**Draw**	B_o	$\hat{\mathbf{a}}_y \cdot \vec{\mathbf{M}}^{B/B_o} = 0$	$F_x^{C_B}$, $F_y^{C_B}$
q_C	Rotate	$\hat{\mathbf{n}}_z$	C	**Draw**	C_o	$\hat{\mathbf{a}}_y \cdot \vec{\mathbf{M}}^{C/C_o} = 0$	$F_x^{C_B}$, $F_y^{C_B}$
* Additional scalar constraint equation:						$-L_A\,\sin(q_A)\,\dot{q}_A - L_B\,\sin(q_B)\,\dot{q}_B + L_C\,\sin(q_C)\,\dot{q}_C = 0$	
* Additional scalar constraint equation:						$L_A\,\cos(q_A)\,\dot{q}_A + L_B\,\cos(q_B)\,\dot{q}_B - L_C\,\cos(q_C)\,\dot{q}_C = 0$	

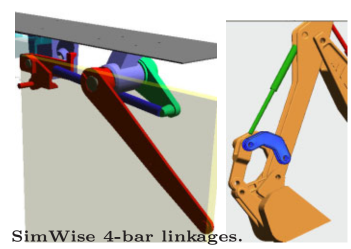
Hint: If you use efficient replacement of gravity forces, MG road-maps results can be identical to those in Hw 22.6.

Using the ***MG road-map***, determine the ***static equilibrium*** values of q_A, q_B, q_C. Using your intuition (**guess**), circle the **stable** solution.

Solution 1	$q_A \approx 20.0°$	$q_B \approx 71.7°$	$q_C \approx 38.3°$
Solution 2	$q_A \approx 249.3°$	$q_B \approx 140.2°$	$q_C \approx 199.1°$
Solution 3	$q_A \approx 30.7°$	$q_B \approx 226.1°$	$q_C \approx 254.7°$

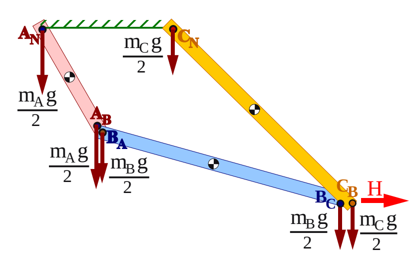
Determine at least one solution. Solutions are for $H = 200$ N.

Solution at **www.MotionGenesis.com** ⇒ **Get Started** ⇒ **Four-bar linkage**



SimWise 4-bar linkages.
Courtesy Design Simulation Technology

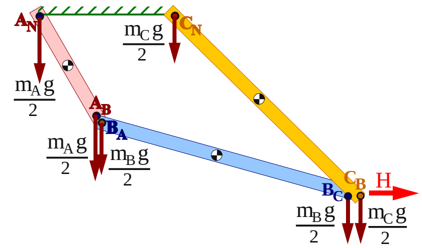
Efficient replacement of gravity forces: **Real** gravity forces are complicated as there is a gravity force on each of the $\approx 6.02 \times 10^{23}$ particles of the rod. Although it is common to replace the **real** gravitational forces on each rod with "$m\,g$" at each link's center of mass, an efficient alternative (similar to Hw 21.7) replaces gravity forces on each rod with half the gravity force at each end as shown right.



22.6 Augmented Kane/Lagrange statics for a four-bar linkage (refers to Hw 22.5).

Form the 3 generalized forces $\mathcal{F}_{\dot{q}_A}$, $\mathcal{F}_{\dot{q}_B}$, $\mathcal{F}_{\dot{q}_A}$ for Hw 22.5 when the linkage is "cut" between points B_C and C_B so there is a constraint force $\vec{\mathbf{F}}^{C_B} = F_x\,\widehat{\mathbf{n}}_x + F_y\,\widehat{\mathbf{n}}_y$ on C_B from B_C.

Form ***Kane/Lagrange*** statics equations with these 3 generalized forces, augmenting by the 2 scalar constraint (loop) equations.



Lagrange multipliers are related to the constraint force $\vec{\mathbf{F}}^{C_B}$.

Result: (replace gravity forces as in Hw 22.5 and use ${}^N\vec{\mathbf{v}}^{B_C} = L_C\,\dot{q}_C\,\widehat{\mathbf{c}}_y$)

$$\mathcal{F}_{\dot{q}_A} = 0 \;=\; L_A\,[F_x\,\sin(q_A) \;-\; F_y\,\cos(q_A) \;-\; 0.5\,(m^A + m^B)\,g\,\sin(q_A)]$$
$$\mathcal{F}_{\dot{q}_B} = 0 \;=\; L_B\,[Fx\,\sin(q_B) \;-\; F_y\,\cos(q_B)]$$
$$\mathcal{F}_{\dot{q}_B} = 0 \;=\; L_C\,[H\,\cos(q_C) \;+\; F_y\,\cos(q_C) \;-\; F_x\,\sin(q_C) \;-\; 0.5\,(m^B + m^C)\,g\,\sin(q_C)]$$

Scalar constraint (loop) equation: $-L_A\,\sin(q_A)\,\dot{q}_A \;-\; L_B\,\sin(q_B)\,\dot{q}_B \;+\; L_C\,\sin(q_C)\,\dot{q}_C \;=\; 0$

Scalar constraint (loop) equation: $L_A\,\cos(q_A)\,\dot{q}_A \;+\; L_B\,\cos(q_B)\,\dot{q}_B \;-\; L_C\,\cos(q_C)\,\dot{q}_C \;=\; 0$

Solution at **www.MotionGenesis.com** ⇒ **Get Started** ⇒ **Four-bar linkage**

22.7 Embedded Kane statics for a four-bar linkage (refers to Hw 22.5, 22.6).

Form this system's generalized force $\mathcal{F}_{\dot{q}_A}$ via ***Kane/Lagrange***. To use the "embedded method" which accounts for constraints and eliminates all constraint/reaction forces, differentiate the loop equation and solve for $\dot{q}_B$ and $\dot{q}_C$ in terms of $\dot{q}_A$ (shown right).

$$\dot{q}_B \;=\; \frac{-L_A\,\sin(q_A - q_C)}{L_B\,\sin(q_B - q_C)}\,\dot{q}_A$$
$$\dot{q}_C \;=\; \frac{-L_A\,\sin(q_A - q_B)}{L_C\,\sin(q_B - q_C)}\,\dot{q}_A$$

Result: (replace gravity forces as in Hw 22.5 and use ${}^N\vec{\mathbf{v}}^{B_C} = L_C\,\dot{q}_C\,\widehat{\mathbf{c}}_y$)

$$\mathcal{F}_{\dot{q}_A} \;=\; \frac{-L_A}{2}\,\{\,(m^A + m^B)\,g\,\sin(q_A) \;+\; \frac{\sin(q_A - q_B)}{\sin(q_B - q_C)}\,[2\,H\,\cos(q_C) \;-\; (m^B + m^C)\,g\,\sin(q_C)\,]\,\} \;=\; 0$$

Solution at **www.MotionGenesis.com** ⇒ **Get Started** ⇒ **Four-bar linkage**

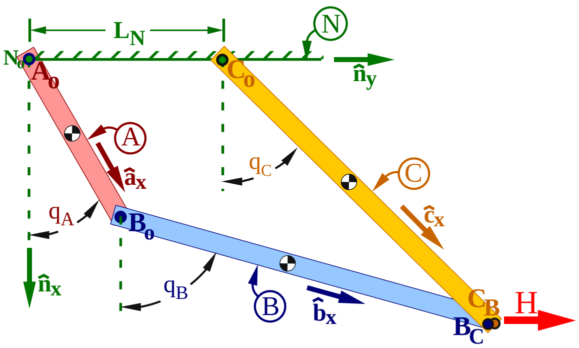
22.8 Four-bar linkage dynamics (modify Hw 22.5 - 22.7 for dynamics).

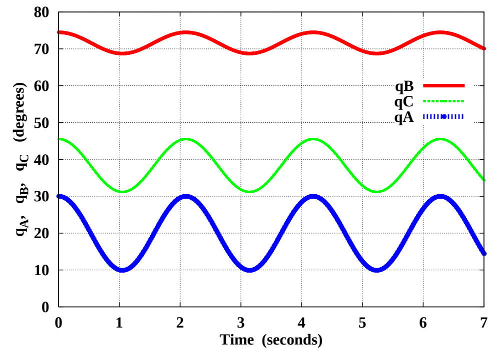
Form this system's dynamics equations with two or more of the following.

Type	*Kane's method*	*Lagrange's method*	*MG road-map*
Augmented	*generalized speeds* $\dot{q}_A, \dot{q}_B, \dot{q}_C$	*generalized coordinates* $\dot{q}_A, \dot{q}_B, \dot{q}_C$	$\dot{q}_A, \dot{q}_B, \dot{q}_C$
Embedded	*generalized speed* $\dot{q}_A$	No	No

Simulate the motion of the four-bar linkage of Hw 22.5. Use initial values $q_A = 30°$, $\dot{q}_A = 0$. Plot q_A, q_B, q_C for 7 seconds. Use the plot to estimate the four-bar linkage's oscillation period as $\tau_{\text{period}} \approx 2$ sec. Note: $q_B = 74.47751219°$, $q_C = 45.52248781°$ satisfy the "*loop equation*" when $q_A = 30°$.

†Optional: Determine τ_{period} by linearizing the dynamics about the static equilibrium solution.





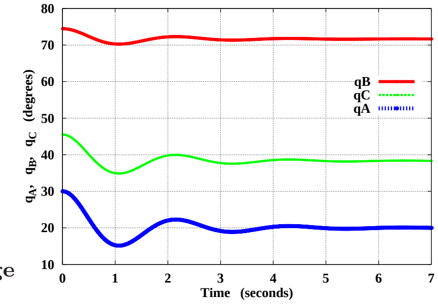
Stable statics via dynamics with damping.

One way to find a **stable static** solution is to simulate the dynamic system with damping (e.g., $H = 200 - 80\,\dot{q}_C$) until the system settles (stops moving).

Determine q_A, q_B, q_C when the system stops moving.

Result: $q_A \approx 20.0°$ $q_B \approx 71.7°$ $q_C \approx 38.3°$

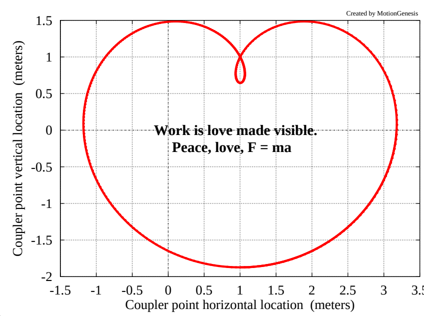
Solution at **www.MotionGenesis.com** ⇒ **Get Started** ⇒ **Four-bar linkage**

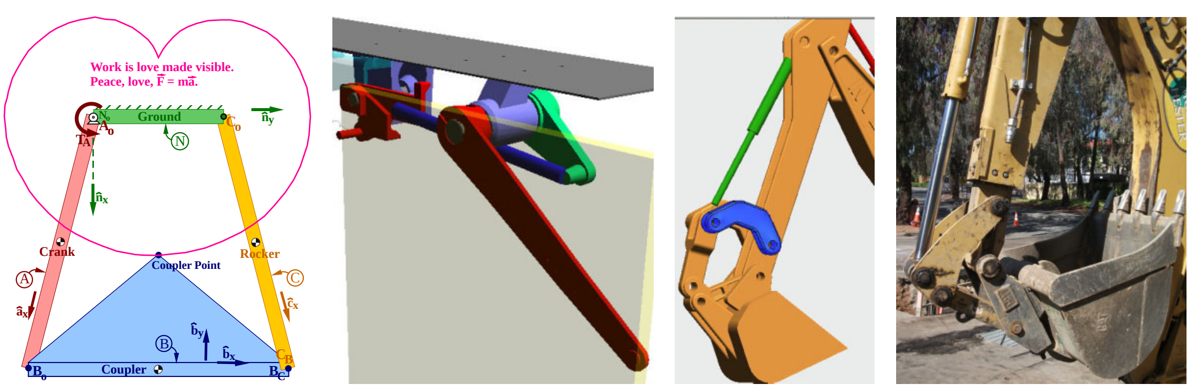


Work is love made visible.

Modify the four-bar so $L_N = 2$ m and $L_A = L_B = L_C = 4$ m. Use $m^A = m^B = m^C = 20$ kg. Remove the horizontal force H and add a motor of torque $T_A = 9600\,(\omega_{\text{des}} - \dot{q}_A)$ on crank-link A, where $\omega_{\text{des}} = 60\,\frac{\text{deg}}{\text{sec}}$. Start from rest with coupler-link B horizontal ($q_B = 90°$) and simulate for 7 seconds. Knowing the "coupler-point" position from B_o is $2\,\widehat{\mathbf{b}}_x + 2\,\widehat{\mathbf{b}}_y$, plot the coupler point's vertical vs. horizontal location.

Solution at **www.MotionGenesis.com** ⇒ **Get Started** ⇒ **Four-bar linkage**





Examples of 4-bar linkages: Courtesy Design Simulation Technology (SimWise)