18.12 Road-map, FBDs, Lagrange, and simulation: Dynamicist on a turntable

A dynamics instructor stands on a spinning turntable and swings a heavy dumbbell inward and outward to change his spin-rate (similar to the ice-skater).

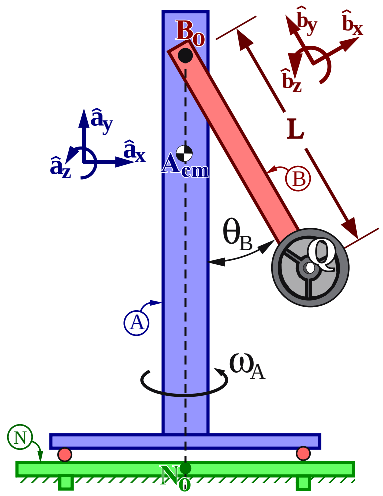
The schematic (below-right) shows a rigid body A (modeling the instructor's legs, torso, and head) that rotates (without friction) relative to Earth (a Newtonian reference frame N) about a vertical axis that is fixed in both A and N and which passes through both point N_o of N and point A_{cm} (A's center of mass).

A massless rigid arm B (modeling the instructor's arms and hands) attaches to A by a revolute motor (shoulder/muscles) whose revolute axis is horizontal and located at point B_o of B (B_o lies on the vertical axis connecting N_o and A_{cm}).

The motor (muscles) **specifies** B's angle θ_B relative to A to change in a known (prescribed) manner from 0 to π rad in 4 seconds $(\theta_B = \pi \frac{t}{4})$.

A heavy dumbbell Q (modeled as a particle) is rigidly attached (welded) to the end of B (the instructor's hands).

Right-handed orthogonal unit vectors $\hat{\mathbf{a}}_x$, $\hat{\mathbf{a}}_y$, $\hat{\mathbf{a}}_z$ and $\hat{\mathbf{b}}_x$, $\hat{\mathbf{b}}_y$, $\hat{\mathbf{b}}_z$ are fixed in A and B, respectively, with $\hat{\mathbf{a}}_y$ vertically-upward, $\hat{\mathbf{b}}_z = \hat{\mathbf{a}}_z$ parallel to the revolute motor's axis, and $\hat{\mathbf{b}}_y$ directed from Q to B_o.



Quantity	Symbol	Type	Value
Earth's gravitational constant	g	Constant	$9.8 \frac{\text{m}}{\text{s}^2}$
Distance between Q and B_o	L	Constant	0.7 m
Mass of Q	m	Constant	12 kg
A's moment of inertia about line $\overline{A_{cm} B_o}$	I_{yy}	Constant	0.6 kg m^2
Angle from $\hat{\mathbf{a}}_y$ to $\hat{\mathbf{b}}_y$ with $+\hat{\mathbf{a}}_z$ sense	θ_B	**Specified**	$\mathbf{0.25}\,\pi\,\mathbf{t}$ rad
$\hat{\mathbf{a}}_y$ measure of A's angular velocity in N	ω_A	Variable	

Complete the ***road map*** for the turntable's "spin-rate" ω_A (Note: The "about point" is not unique)

Variable	Translate/ Rotate	Direction (unit vector)	System S	FBD of S	About point*	***Road-map equation***
ω_A	Rotate	$\hat{\mathbf{a}}_y$	A, B, Q	**Draw**	B_o	$\hat{\mathbf{a}}_y \cdot (\vec{\mathbf{M}}^{S/B_o} = \frac{{}^N d\, {}^N\vec{\mathbf{H}}^{S/B_o}}{dt})$

Student/Instructor version at **www.MotionGenesis.com** ⇒ **Textbooks** ⇒ **Resources**

(a) Use the road map to form $\dot{\omega}_A = \dfrac{-2\, m\, L^2 \sin(\theta_B) \cos(\theta_B)\, \dot{\theta}_B\, \omega_A}{I_{yy} + m\, L^2 \sin^2(\theta_B)}$.

(b) Form the kinetic energy K of the system S formed by A, B, Q and the partial derivative of K with respect to ω_A. Complete the following modified ***Lagrange's equation*** associated with ω_A.
Is $\dot{\omega}_A$ produced by the road-map equal to $\dot{\omega}_A$ produced by Lagrange's equation? **Yes/No**
Result:

$$K = \frac{1}{2} I_{yy}\, \omega_A^2 + \frac{1}{2} m L^2\, [\dot{\theta}_B^2 + \sin^2(\theta_B)\, \omega_A^2] \qquad \frac{\partial K}{\partial \omega_A} = [I_{yy} + m L^2 \sin^2(\theta_B)]\, \omega_A$$

$$\frac{d}{dt} \frac{\partial K}{\partial \omega_A} = 0 \quad \Rightarrow \quad [I_{yy} + m\, L^2 \sin^2(\theta_B)]\, \dot{\omega}_A + 2\, m\, L^2 \sin(\theta_B) \cos(\theta_B)\, \dot{\theta}_B\, \omega_A = 0$$

(c) Determine the gravitational potential energy of the system S formed by A, B, Q. [See equation (24.5)].
Result:

$$U_{\text{gravity}} \underset{(24.5)}{=} -m\, g\, L \cos(\theta_B) + \text{Constant}$$

Is $K + U_{\text{gravity}}$ constant (mechanical energy is conserved)? **Yes/No**. (See Section 23.4).
If not conserved, where is work done? *Work is done by the torque on B from A.*

(d) Consider $\vec{\mathbf{M}}^{\bar{S}/P}$, (moment of all forces on a generic system $\bar{S}$ about a point P fixed in a Newtonian reference frame N) and a unit vector $\hat{\mathbf{u}}$ fixed in N. The ***conservation of angular momentum checklist*** (Section 22.8) states: If $\vec{\mathbf{M}}^{\bar{S}/P} \cdot \hat{\mathbf{u}} = 0$, then ${}^N\vec{\mathbf{H}}^{\bar{S}/P} \cdot \hat{\mathbf{u}} = \text{Constant}$. Use this to complete the following blanks.

System $\bar{S}$	(not unique) Point P	$\widehat{\mathbf{u}}$	Symbolic expression for ${}^{N}\vec{\mathrm{H}}^{\bar{S}/P} \cdot \widehat{\mathbf{u}}$	Numerical value of Constant
A, B, Q	B_o or N_o or ...	$\widehat{\mathbf{a}}_y$	$\left[\mathrm{I_{yy}} + m\,L^2 \sin^2(\theta_B)\right] \omega_A$	$3.6 \text{ kg} \frac{\text{m}^2}{\text{s}}$

(e) $\left|{}^{N}\vec{\mathrm{H}}^{\bar{S}/P}\right|$, the magnitude of $\bar{S}$'s angular momentum about P in N is constant. **True/False**

(f) Use a computer to numerically integrate $\dot{\omega}_A$ [solve the ODE for ω_A with initial value $\omega_A(t{=}0) = 6 \frac{\text{rad}}{\text{sec}}$]. Graph ω_A for $0 \le t \le 4$ sec. **Solution at www.MotionGenesis.com ⇒ Get Started ⇒ Momentum.**