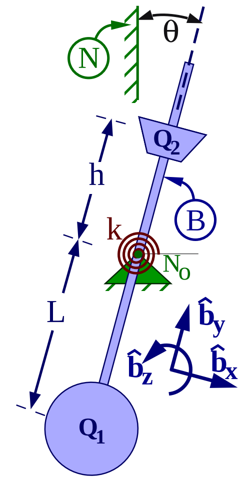
14.2 FE/EIT Review – Frequency analysis of a metronome $\boxed{\vec{\mathrm{M}}_z \underset{\mathbf{2D}}{=} \mathrm{I}_{zz}\, \vec{\boldsymbol{\alpha}}}$



The figure to the right shows the main mechanism of a metronome.
Part of the metronome is modeled as a rigid body B that consists of a light (massless) rod and two particles, namely Q_1 and Q_2.

The rod is attached to an immovable base N at point N_o by a linear torsional spring whose stiffness is k and whose natural (undeformed) angle is $\theta = 0$.

Particle Q_1 has mass m_1 and is located a distance L from N_o.
Particle Q_2 has mass m_2 and is located a distance h from N_o.

Right-handed orthogonal unit vectors $\widehat{\mathbf{b}}_x$, $\widehat{\mathbf{b}}_y$, $\widehat{\mathbf{b}}_z$ are fixed in B with $\widehat{\mathbf{b}}_z$ parallel to B's axis of rotation in N and $\widehat{\mathbf{b}}_y$ pointing from Q_1 to N_o to Q_2.

(a) Form B's angular velocity in N in terms of the time-derivative of θ.
Write the definition of B's angular acceleration in N and express it in terms of $\ddot{\theta}$ and $\widehat{\mathbf{b}}_z$.

Result:

$$ {}^N\vec{\boldsymbol{\omega}}^B = \boxed{-\dot{\theta}}\ \widehat{\mathbf{b}}_z \qquad {}^N\vec{\boldsymbol{\alpha}}^B \triangleq \boxed{\frac{{}^N d\ {}^N\vec{\boldsymbol{\omega}}^B}{dt}} = \boxed{-\ddot{\theta}}\ \widehat{\mathbf{b}}_z $$

(b) Express I_{zz} (B's moment of inertia about a line L passing through N_o and parallel to $\widehat{\mathbf{b}}_z$) in terms of m_1, m_2, L, and h. Note: The moment of inertia of a particle about line L is $\mathrm{m} * \mathrm{distance}^2$, where m is the mass of the particle and distance is the distance between the particle and line L.

Result:

$$ \mathrm{I}_{zz} = \boxed{m_1\, L^2} + \boxed{m_2\, h^2} $$

(c) Neglecting damping, determine the $\widehat{\mathbf{b}}_z$ component of the moment of all forces on B about N_o (due to gravity and the $k\,\theta\,\widehat{\mathbf{b}}_z$ torque associated with the torsional spring).
Note: The moment of a force on an arbitrary point P about a point N_o is defined $\vec{\mathbf{r}}^{\,P/N_o} \times \vec{\mathbf{F}}^P$ where $\vec{\mathbf{r}}^{\,P/N_o}$ is P's position vector from N_o and $\vec{\mathbf{F}}^P$ is the force on P.

Result:

$$ \vec{\mathrm{M}}_z^{B/N_o} \underset{\mathbf{FBD}}{=} \vec{\mathrm{M}}_{gravity} + k\,\theta\,\widehat{\mathbf{b}}_z = \left[(\boxed{m_1\, L - m_2\, h})\, g\, \sin(\theta) + k\,\theta \right] \widehat{\mathbf{b}}_z $$

(d) Form an equation for the metronome's motion in terms of θ, $\dot{\theta}$, $\ddot{\theta}$, m_1, m_2, L, h, k, g.

Result:

$$ \boxed{\vec{\mathrm{M}}_z^{B/N_o} \underset{(21.6)}{=} \mathrm{I}_{zz}\, \vec{\boldsymbol{\alpha}}} \Rightarrow \left(\boxed{m_1\, L^2} + m_2\, h^2\right) \ddot{\theta} + \boxed{k}\,\theta + \left(m_1\, L - \boxed{m_2\, h}\right) g\, \sin(\theta) = 0 $$

(e) This system ***conserves mechanical energy***, i.e., the sum of its kinetic energy K and potential energy U is constant (Section 22.2, example in Section 23.3.6). Form the metronome's mechanical energy. Verify the previous metronome equation of motion with Lagrange's method (given below).

Result: [**Optional:** Verify equation of motion via Kane's method in Chapter 25 with generalized speed $\dot{\theta}$.]

$$ \boxed{K + U = \text{Constant}} \Rightarrow \tfrac{1}{2}(m_1\, L^2 + m_2\, h^2)\,\dot{\theta}^2 + \tfrac{1}{2}\, k\,\theta^2 - (m_1\, L - m_2\, h)\, g\, \cos(\theta) $$

$$ \boxed{-\frac{\partial U}{\partial \theta} \underset{(26.1)}{=} \frac{d}{dt}\frac{\partial K}{\partial \dot{\theta}} - \frac{\partial K}{\partial \theta}} \Rightarrow \left(\boxed{m_1\, L^2} + m_2\, h^2\right) \ddot{\theta} + \boxed{k}\,\theta + \left(m_1\, L - \boxed{m_2\, h}\right) g\, \sin(\theta) = 0 $$

(f) Use the ***small angle approximation*** $\sin(\theta) \approx \theta$ to form an equation governing small values of θ. Write the ODE using an "***equivalent mass***" m_{eq} and "***equivalent stiffness***" k_{eq}.

Result:

$$ \boxed{m_{\mathrm{eq}}\, \ddot{\theta} + k_{\mathrm{eq}}\, \theta = 0} \qquad m_{\mathrm{eq}} = m_1\, L^2 + \boxed{m_2\, h^2} \qquad k_{\mathrm{eq}} = k + m_1\, L g - \boxed{m_2\, h\, g} $$

(g) **Optional:** Rewrite the ODE in part (2f) in a more **mathematically useful** form by dividing the ODE by m_{eq} and defining ***natural frequency*** $\omega_n \triangleq \sqrt{\frac{k_{\mathrm{eq}}}{m_{\mathrm{eq}}}}$. Next, knowing metronomes are built with $k > 0$ and $m_1\, L > m_2\, h$, express $\theta(t)$ in terms of m_{eq} and k_{eq} when the metronome is ***released from rest*** at $t = 0$ from a small angle θ_0.

Result:

$$ \omega_n \triangleq \sqrt{\frac{k_{\mathrm{eq}}}{m_{\mathrm{eq}}}} \qquad \ddot{\theta} + \omega_n^2\, \theta = 0 \qquad \theta(t) = \theta_0\, \cos(\omega_n\, t) $$

(h) **Optional:** Determine the relationship between the metronome frequency f_n (in cycles per second) and the adjustment height h of the upper particle.
Result:

$$f_n \triangleq \frac{\omega_n}{2\,\pi} = \frac{1}{\tau_{\text{period}}} = \frac{1}{2\,\pi}\sqrt{\frac{k + m_1\,L\,g - \boxed{m_2\,h\,g}}{m_1\,L^2 + \boxed{m_2\,h^2}}}$$

Explain the effect of increasing h on the metronome's frequency f_n.

- Increasing h **increases**/**decreases** the numerator of f_n.
- Increasing h **increases**/**decreases** the denominator of f_n.
- Increasing h **increases**/**decreases** the frequency f_n.

(i) **Optional:** Given $m_1 = 0.1$ kg, $m_2 = 0.02$ kg, $g = 9.8\,\frac{\text{m}}{\text{s}^2}$, $L = 0.2$ m, $k = 2\,\frac{\text{N}*\text{m}}{\text{rad}}$, the expression for f_n simplifies as shown below (you do not need to show this).
Find two values of h that correspond to $f_n = 3.5\,\frac{\text{cycles}}{\text{sec}}$.
Result: (circle the more physically meaningful value of h).

$$0.78957\,f_n^2\,h^2 + 0.196\,h + \left(0.158\,f_n^2 - 2.196\right) = 0 \quad \Rightarrow \quad \boxed{\text{h} = 0.1546\text{ m}}, \quad h = \boxed{-0.175}\text{ m}$$

Solution and video link at **www.MotionGenesis.com** ⇒ **Get Started** ⇒ **Pendulum** ⇒ **Metronome**.

