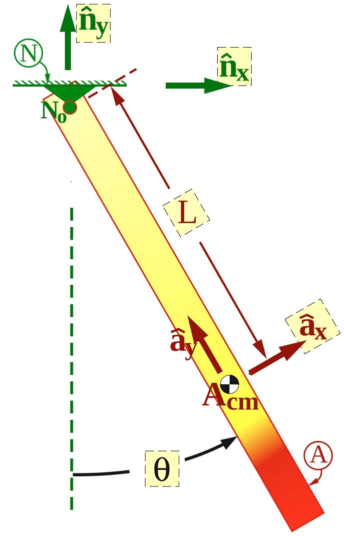
MIPSI: Rigid-body pendulum



Modeling (assumptions)

The figure to the right shows a non-uniform density rigid rod A attached by a frictionless revolute/pin joint to Earth N (Newtonian reference frame). The only external forces on rod A are contact forces at N_o (the point of N in contact with A) and Earth's gravitational forces (modeled as local and uniform).

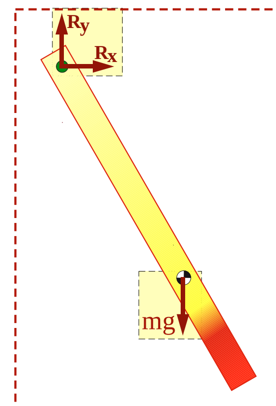
Identifiers (label L, θ, and unit vectors $\hat{\mathbf{n}}_x$, $\hat{\mathbf{n}}_y$, $\hat{\mathbf{n}}_z$, $\hat{\mathbf{a}}_x$ on the figure)

Right-handed orthogonal unit vectors $\hat{\mathbf{n}}_x$, $\hat{\mathbf{n}}_y$, $\hat{\mathbf{n}}_z$ and $\hat{\mathbf{a}}_x$, $\hat{\mathbf{a}}_y$, $\hat{\mathbf{a}}_z$ are fixed in N and A, respectively, with $\hat{\mathbf{n}}_x$ horizontally-right, $\hat{\mathbf{n}}_y$ vertically-upward, $\hat{\mathbf{a}}_y$ directed from A_{cm} (A's center of mass) to N_o, and $\hat{\mathbf{a}}_z = \hat{\mathbf{n}}_z$ parallel to the pin.

Description	Symbol	Type	Value
Earth's gravitational constant	g	Constant	9.8 m/s^2
Distance between N_o and A_{cm}	L	Constant	7.5 cm
Mass of rod A	m	Constant	10 grams
A's moment of inertia about A_{cm} for $\hat{\mathbf{a}}_z$	I	Constant	50 g $*$ cm^2
Angle from $\hat{\mathbf{n}}_y$ to $\hat{\mathbf{a}}_y$ with $^+\hat{\mathbf{n}}_z$ sense	θ	Variable	

Physics

Answers at www.MotionGenesis.com $\Rightarrow$ Get Started $\Rightarrow$ Pendulum.



Complete rod A's ***free-body diagram (FBD)*** (to the right).
Form an **ODE** (ordinary differential equation) governing this pendulum's motion.

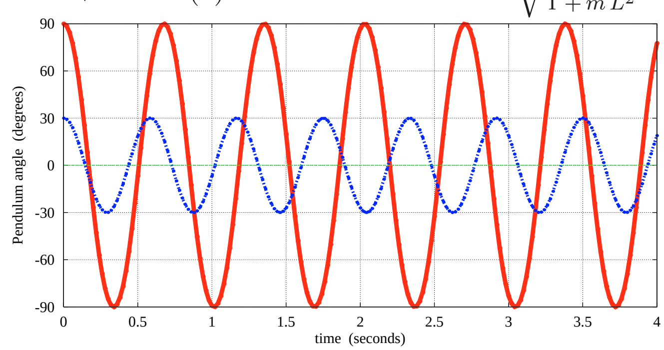
Euler's 2D rigid body equation together with the moment of inertia parallel axis theorem and subsequent dot-multiplication with $\hat{\mathbf{n}}_z$ and rearrangement gives:

$$\vec{\mathbf{M}}_z \underset{\mathbf{2D}}{=} \mathrm{I}_{zz}\,\vec{\boldsymbol{\alpha}} \;\Rightarrow\; \vec{\mathbf{M}}_z^{A/N_o} \underset{\mathbf{(FBD)}}{=} -m\,g\,L\,\sin(\theta)\,\hat{\mathbf{n}}_z \;=\; \left(\mathrm{I} + m\,L^2\right) * \ddot{\theta}\,\hat{\mathbf{n}}_z$$

Result: $\left(\mathrm{I} + m\,L^2\right)\ddot{\theta} \;+\; m\,g\,L\,\sin(\theta) \;=\; 0 \;\;\Rightarrow\;\; \ddot{\theta} \;+\; \frac{m\,g\,L}{\mathrm{I}+m\,L^2}\,\sin(\theta) \;=\; 0$

Simplify and solve

$$\ddot{\theta} \;+\; \omega^2 \sin(\theta) \;=\; 0 \qquad \text{where } \omega \;\triangleq\; \sqrt{\frac{m\,g\,L}{\mathrm{I} + m\,L^2}}$$



By defining the constant ω (shown right), the previous ODE is written in a form that clearly shows it is **nonlinear** due to $\sin(\theta)$.

Since the ODE is **nonlinear**, a closed-form (analytical/symbolic) solution is difficult. The plot shows a numerical (computer) solution to this nonlinear ODE when the pendulum is released from rest with $\theta(t\!=\!0) = 90°$ or $\theta(t\!=\!0) = 30°$.

Alternatively, making the small-angle approximation $\sin(\theta) \approx \theta$ produces the **linear** ODE:
$\ddot{\theta} \;+\; \omega^2\,\theta \;\approx\; 0$ whose closed-form (analytical/symbolic) solution is $\boxed{\theta(t) \;\approx\; \theta(0)\,\cos(\omega\,t)}$.

Interpret (communicate)

The pendulum swings back and forth with an amplitude that depends on starting angle (when released from 90°, it swings higher than when released from 30°). The solution to the **linear** ODE says the pendulum's oscillation frequency ω does not depend on release angle. However, the plot of the **nonlinear** ODE shows the pendulum has less-frequent oscillations when released from 90° than when released from 30°. This is due to the linear ODE only approximating the nonlinear ODE accurately at small angles ($\theta \ll 1$ radians).

```
% MotionGenesis file: MGRigidBodyPendulumDynamics.txt
% Copyright (c) 2009 Motion Genesis LLC.  All rights reserved.
%--------------------------------------------------------------------
NewtonianFrame  N            % Earth.
RigidBody       A            % Rod.
%--------------------------------------------------------------------
Variable   theta’’           % Angle and 1st/2nd time-derivatives.
Constant   g = 9.8 m/s^2     % Earth’s gravitational acceleration.
Constant   L = 7.5 cm        % Distance between No and Acm.
A.SetMassInertia( m = 10 grams,  IAx,  IAy,  I = 50 g*cm^2 )
%--------------------------------------------------------------------
%   Rotational kinematics.
A.RotateZ( N, theta )
%--------------------------------------------------------------------
%   Translational kinematics.
Acm.Translate( No, -L*Ay> )
%--------------------------------------------------------------------
%   Add relevant contact/distance forces.
Acm.AddForce( -m*g*Ny> )
%--------------------------------------------------------------------
%   Equation of motion with Euler.
Dynamics = Dot( Nz>,   A.GetDynamics(No) )
%-------------------------------------------------------------------
%   Check conservation of kinetic and potential energy.
KE = System.GetKineticEnergy()
PE = System.GetForceGravityPotentialEnergy( -g*Ny>, No )
MechanicalEnergy = KE + PE
%------------------------------------------------------------------------
%   Optional: Equation of motion with Kane’s method.
SetGeneralizedSpeed( theta’ )
KaneDynamics = System.GetDynamicsKane()
%------------------------------------------------------------------------
%   Optional: Equation of motion with Lagranges’s method.
SetGeneralizedCoordinates( theta )
LagrangeDynamics = System.GetDynamicsLagrange( SystemPotential = PE )
%------------------------------------------------------------------------
%   Solve dynamics equation for theta’’.
Solve( Dynamics = 0,   theta’’)
%--------------------------------------------------------------------
%   Integration parameters and initial values.
Input  tFinal = 4 sec,  tStep  = 0.02 sec,  absError = 1.0E-07
Input  theta = 90 deg,  theta’ = 0.0 rad/sec
%--------------------------------------------------------------------
%   List output quantities and solve ODEs.
Output  t seconds,  theta degrees,  MechanicalEnergy Joules
ODE()  MGRigidBodyPendulumDynamics
%--------------------------------------------------------------------
Save MGRigidBodyPendulumDynamics.html
Quit
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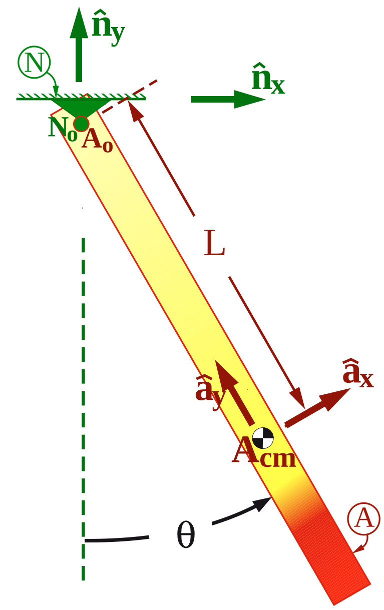

[3]The Shortt clock pictured here is the most accurate mechanical clock ever made, with an error rate of 1 sec in 12 years. The clock is so sensitive it detects tiny changes in gravity due to tidal distortions on Earth caused by the Sun and Moon’s gravity. Paul Heyl from NIST used it for a new determination of the gravitational constant.