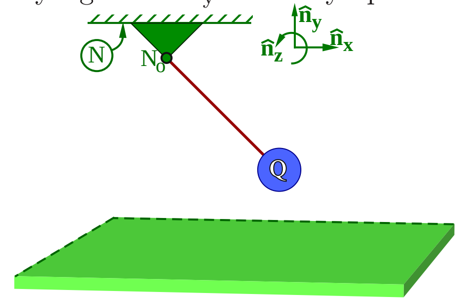
10.23 Optional**: Spherical (3D) pendulum simulation with slack/taut string

The simulation of cable-loads in a ***wrecking-ball*** or the motion of a ***tether-ball*** can be modeled with a rope that connects a particle Q to a point N_o (fixed in a Newtonian reference frame N). Right-handed orthogonal unit vectors $\widehat{\mathbf{n}}_x$, $\widehat{\mathbf{n}}_y$, $\widehat{\mathbf{n}}_z$ are fixed in N with $\widehat{\mathbf{n}}_x$ horizontally-right and $\widehat{\mathbf{n}}_y$ vertically-upward.

Quantity	Symbol	Type	Value
Mass of Q	m	Constant	1 kg
Earth's gravitational constant	g	Constant	$9.8\ \frac{\text{m}}{\text{s}^2}$
Natural spring length	L_n	Constant	0.3 m
Linear spring constant	k	Constant	$100\ \frac{\text{N}}{\text{m}}$
Damping constant	ζ	Constant	0 NoUnits
Aerodynamic damping constant	b_{air}	Constant	$0\ \frac{\text{N s}}{\text{m}}$



*Tabulated numerical values represent one scenario. Simulations should provide the ability to vary these values.

Extensional stiffness in the rope is modeled with a linear spring that is active when the rope is **taut** - but not **slack**. Damping in the rope is modeled with a linear damper $b = 2\,\zeta\,\sqrt{m\,k}$ that is active when the rope is both **taut** and **elongating** - but not **shortening**. Aerodynamic damping is modeled with a force of magnitude $b_{air}\,|{}^N\vec{\mathbf{v}}^Q|$ that opposes Q's motion in N.

Produce **one** computer code (**M**otion**G**enesis and/or MATLAB® file) that continuously simulates Q's motion in N, whether the rope is taut or slack, for any realizable initial conditions or parameter values. Use a numerical integrator step tStep $\le$ 0.01 sec and integrator error tolerances $\le 1 \times 10^{-7}$.

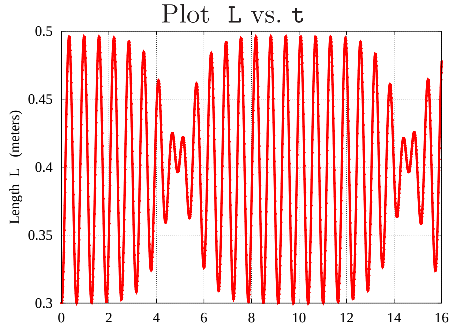
The simulation code should be able to test free-fall and projectile motion (with/without air-resistance), spring-damper motions, impact (transitions from free-flight to a taut rope), and should track whether mechanical energy is conserved, lost, or gained. Output a file that reports the following in SI units:

`t`	Time in seconds
`q`	Angle in degrees between $-\widehat{\mathbf{n}}_y$ and the line connecting N_o and Q (or 0° if undefined)
`L`	Distance between Q and N_o
`Ldot`	Time-derivative of L (i.e., $\dot{L}$)
`x`	$\widehat{\mathbf{n}}_x$ measure of Q's position from N_o
`y`	$\widehat{\mathbf{n}}_y$ measure of Q's position from N_o
`z`	$\widehat{\mathbf{n}}_z$ measure of Q's position from N_o
`Tension`	Tension in the rope (due to spring and damper)
`KE`	Q's kinetic energy in N
`PEGravity`	Q's gravitational potential energy in N
`PESpring`	Potential energy due to stiffness in the spring
`WorkDamper`	Work done on Q from $t=0$ by damping in the rope
`WorkAir`	Work done on Q from $t=0$ by air-resistance
`EnergySum` = `KE` + `PEGravity` + `PESpring` − `WorkDamper` − `WorkAir`	

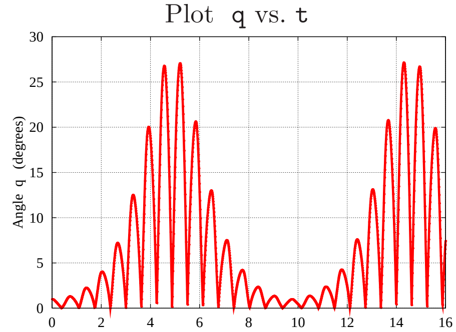
(a) Simulate 16 seconds of motion using the values above and the initial values below.[7]

$\vec{\mathbf{r}}^{\,Q/N_o}{}_{(t=0)} = L_n \sin(1°)\,\widehat{\mathbf{n}}_x - L_n \cos(1°)\,\widehat{\mathbf{n}}_y$ ${}^N\vec{\mathbf{v}}^Q{}_{(t=0)} = \vec{\mathbf{0}}$ (Q starts from **rest** in N)

Plot `L` vs. `t`



Plot `q` vs. `t`



(b) Simulate 2 seconds of **free-fall** with $b_{air} = 24.5\ \frac{\text{N s}}{\text{m}}$, $k = \zeta = 0$, $\vec{\mathbf{r}}^{\,Q/N_o}{}_{(t=0)} = {}^N\vec{\mathbf{v}}^Q{}_{(t=0)} = \vec{\mathbf{0}}$.

[7]More information on this non-intuitive "***swinging-spring***" can be found in [18, pgs. 62, 129-137] and [44].

To **7+** significant digits, ***terminal velocity*** is $\dot{L}$ = 0.4 $\frac{m}{s}$.
To **1+** significant digits, the first occurrence of $\left|{}^{N}\vec{\mathbf{v}}^{Q}\right| \geq 0.39\ \frac{m}{s}$ is t = 0.151 sec.

(c) Simulate 2 seconds of **free-fall and bounce** motion with tStep $\approx$ 0.0002 sec and[8]

$k = 10000\ \frac{N}{m}$ $\quad \zeta = 1 \quad b_{air} = 0 \quad \vec{\mathbf{r}}^{Q/N_o}{}_{(t=0)} = \vec{\mathbf{0}} \quad {}^{N}\vec{\mathbf{v}}^{Q}{}_{(t=0)} = \vec{\mathbf{0}}$

Guess and check: Use your engineering intuition, then check with simulation:
The ratio of maximum tension to tension at static equilibrium is approximately 49.
Increasing k by a factor of 9 causes maximum tension to **decrease/increase** by a factor of 3.
Increasing ζ to 2 causes maximum tension to **decrease/increase** by a factor of 2.

(d) Simulate 2 seconds of **simple pendulum motion** with tStep $\leq$ 0.01 sec and

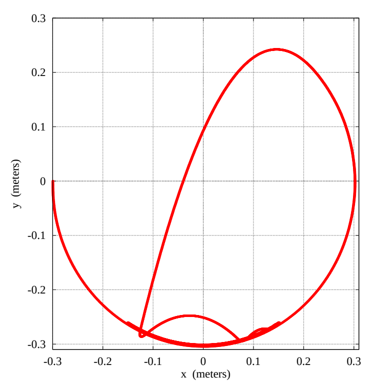
$k = 10000\ \frac{N}{m} \quad \zeta = 1 \quad b_{air} = 0 \quad \vec{\mathbf{r}}^{Q/N_o}{}_{(t=0)} = -0.3\,\widehat{\mathbf{n}}_x \quad {}^{N}\vec{\mathbf{v}}^{Q}{}_{(t=0)} = \vec{\mathbf{0}}$

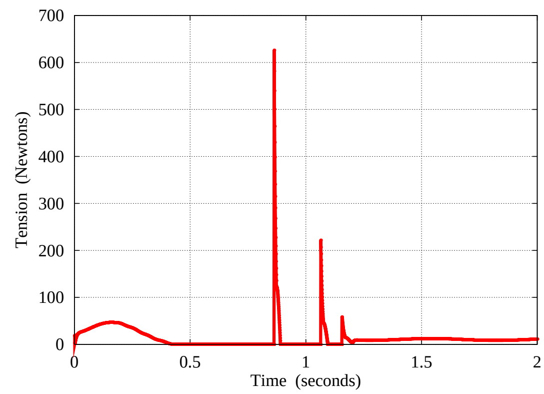
The ratio of maximum tension to tension at static equilibrium is approximately 3.
These simulations suggest stress for a wrecking ball in free-fall are more than 15x higher than those for swinging motions and nearly $50x$ higher than static equilibrium. **True/False**.

(e) Plot y vs. x and tension vs. t $0 \leq t \leq 2$ sec with tStep $\approx$ 0.0002 sec and

$k = 10000\ \frac{N}{m} \quad \zeta = 1 \quad b_{air} = 0 \quad \vec{\mathbf{r}}^{Q/N_o}{}_{(t=0)} = -0.3\,\widehat{\mathbf{n}}_x \quad {}^{N}\vec{\mathbf{v}}^{Q}{}_{(t=0)} = -2.35\,\widehat{\mathbf{n}}_y$

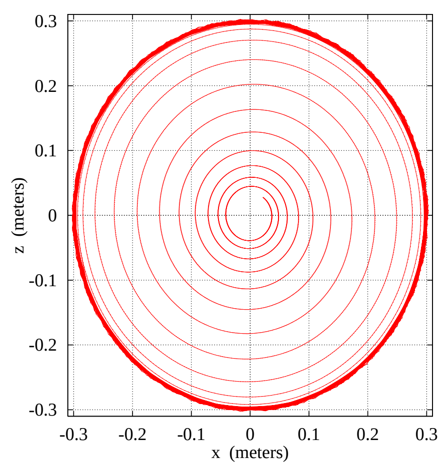
To **1+** significant digits, the maximum value of tension is 647 N, at time t = 0.863 s.

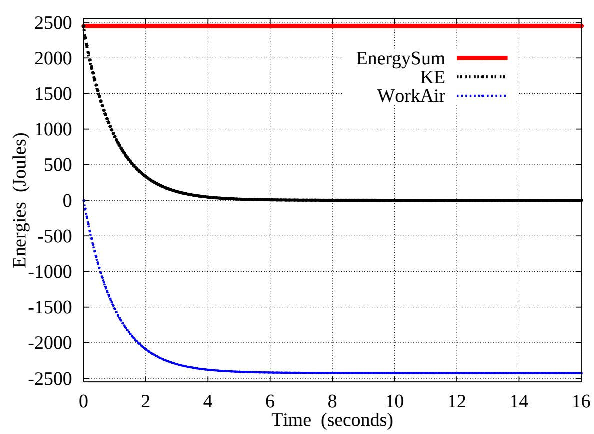




(f) Simulate 16 seconds of motion using with ith tStep $\leq$ 0.01 sec and the values below.
Plot z vs. x and `EnergySum`, `KE`, and `WorkAir` vs. t. `EnergySum` is conserved. **True/False**.

$k = 5\,\text{x}\,10^6\ \frac{N}{m} \quad \zeta = 1 \quad b_{air} = 0.5\ \frac{N\,s}{m} \quad \vec{\mathbf{r}}^{Q/N_o}{}_{(t=0)} = -0.3\,\widehat{\mathbf{n}}_x \quad {}^{N}\vec{\mathbf{v}}^{Q}{}_{(t=0)} = -70\,\widehat{\mathbf{n}}_z$





[8] $\zeta = 1$ loosely corresponds to critical damping in the rope, which may be more reasonable for nylon than steel.
[8] Consider using $\epsilon \approx 10^{-8}$ with $L = \left|\vec{\mathbf{r}}^{Q/N_o}\right| + \epsilon$ and implicitly-differentiate $L^2 = \vec{\mathbf{r}}^{Q/N_o} \cdot \vec{\mathbf{r}}^{Q/N_o}$ to solve for $\dot{L}$.
[8] **Note: Problem solution at www.MotionGenesis.com ⇒ Get Started ⇒ Pendulum.**