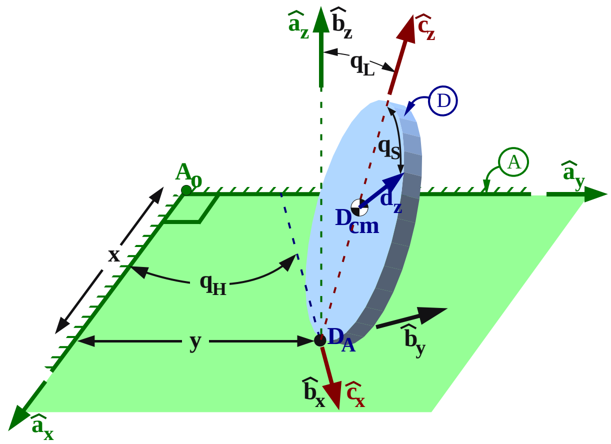
22.1 Dynamics of a rolling disk (refers to Homework 6.17 and Homework 10.10).

The figure to the right shows a thin uniform circular disk D rotating on a horizontal plane A (Newtonian reference frame). Right-handed orthogonal unit vectors $\hat{\mathbf{a}}_x$, $\hat{\mathbf{a}}_y$, $\hat{\mathbf{a}}_z$ are fixed in A with $\hat{\mathbf{a}}_y$ horizontally right and $\hat{\mathbf{a}}_z$ vertically upward.

Right-handed orthogonal unit vectors $\hat{\mathbf{d}}_x$, $\hat{\mathbf{d}}_y$, $\hat{\mathbf{d}}_z$ are fixed in D with $\hat{\mathbf{d}}_y$ parallel to the disk's axis. D's orientation in A is determined by initially setting $\hat{\mathbf{d}}_i = \hat{\mathbf{a}}_i$ $(i = x, y, z)$ and then subjecting D to a sequence of right-handed rotations in A characterized by $q_H\,\hat{\mathbf{d}}_z$, $-q_L\,\hat{\mathbf{d}}_x$, $q_S\,\hat{\mathbf{d}}_y$.



Description	Symbol	Type	(Initial) value
Earth's gravity	g	Constant	$9.8\ \frac{m}{s^2}$
Mass of D	m	Constant	2 kg
Radius of D	r	Constant	34.29 cm
D's axial moment of inertia	I	Constant	$0.5\, m\, r^2$
D's transverse moment of inertia	J	Constant	$0.25\, m\, r^2$
Heading angle (angle from $\hat{\mathbf{a}}_x$ to $\hat{\mathbf{b}}_x$ with $+\hat{\mathbf{a}}_z$ sense)	q_H	Variable	$0°$
Lean angle (angle from $\hat{\mathbf{b}}_z$ to $\hat{\mathbf{c}}_z$ with $-\hat{\mathbf{b}}_x$ sense)	q_L	Variable	$0°$
Spin angle (angle from $\hat{\mathbf{c}}_z$ to $\hat{\mathbf{d}}_z$ with $+\hat{\mathbf{c}}_y$ sense)	q_S	Variable	$10°$
$\hat{\mathbf{a}}_x$ measure of P's position vector from A_o	x	Variable	0 m
$\hat{\mathbf{a}}_y$ measure of P's position vector from A_o	y	Variable	0 m
$\hat{\mathbf{c}}_x$ measure of D's angular velocity in A	ω_x	Variable	$0\ \frac{rad}{sec}$
$\hat{\mathbf{c}}_y$ measure of D's angular velocity in A	ω_y	Variable	$0\ \frac{rad}{sec}$
$\hat{\mathbf{c}}_z$ measure of D's angular velocity in A	ω_z	Variable	$5\ \frac{rad}{sec}$

Note: Point P is the path point described in Section 12.11 (the point in space that tracks contact between A and D).

Kinematical differential equations
Results from Hw 6.17 and 10.10.

$$\dot{q}_L = -\omega_x$$
$$\dot{q}_S = \omega_y + \tan(q_L)\,\omega_z$$
$$\dot{q}_H = \frac{\omega_z}{\cos(q_L)}$$

$$^{A}\vec{\boldsymbol{\omega}}^{D} = \omega_x\,\hat{\mathbf{c}}_x + \omega_y\,\hat{\mathbf{c}}_y + \omega_z\,\hat{\mathbf{c}}_z$$
$$^{A}\vec{\boldsymbol{\omega}}^{C} = \omega_x\,\hat{\mathbf{c}}_x - \tan(q_L)\,\omega_z\,\hat{\mathbf{c}}_y + \omega_z\,\hat{\mathbf{c}}_z$$
$$^{A}\vec{\boldsymbol{\alpha}}^{D} = (\dot{\omega}_x - \omega_z\,\dot{q}_S)\,\hat{\mathbf{c}}_x + \dot{\omega}_y\,\hat{\mathbf{c}}_y + (\omega_x\,\dot{q}_S + \dot{\omega}_z)\,\hat{\mathbf{c}}_z$$

Rolling: $\dot{x} = r\cos(q_H)\,\dot{q}_S = r\cos(q_H)\,[\omega_y + \tan(q_L)\,\omega_z]$

Rolling: $\dot{y} = r\sin(q_H)\,\dot{q}_S = r\sin(q_H)\,[\omega_y + \tan(q_L)\,\omega_z]$

(a) Determine the $\hat{\mathbf{c}}_x$, $\hat{\mathbf{c}}_y$, $\hat{\mathbf{c}}_z$ measures of D_{cm}'s velocity and acceleration in A.
Result: (Use the fact that D <u>**rolls**</u> on A and D_{cm} is D's center of mass).

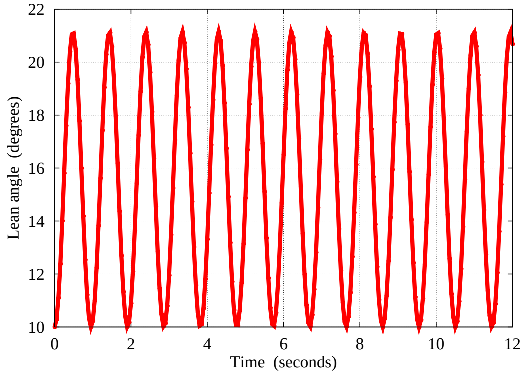
	$\hat{\mathbf{c}}_x$	$\hat{\mathbf{c}}_y$	$\hat{\mathbf{c}}_z$
$^{A}\vec{\mathbf{v}}^{D_{cm}}$	$r\,\omega_y$	$-r\,\omega_x$	0
$^{A}\vec{\mathbf{a}}^{D_{cm}}$	$r\,(\omega_x\,\omega_z + \dot{\omega}_y)$	$r\,(\omega_y\,\omega_z - \dot{\omega}_x)$	$-r\,[\omega_x^2 - \tan(q_L)\,\omega_y\,\omega_z]$

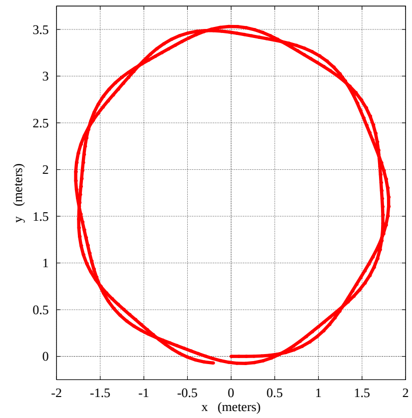
(b) Knowing D <u>**rolls**</u> on A, find ***Kane's equations*** for the ***generalized speeds*** ω_x, ω_y, ω_z and/or use a free-body diagram (calculate moment about the point of D in contact with A).
Result:

$$\dot{\omega}_x = \frac{(I + m\,r^2)\,\omega_y\,\omega_z + J\,\omega_z\,(\dot{q}_S - \omega_y) - m\,g\,r\,\sin(q_L)}{J + m\,r^2}$$

$$\dot{\omega}_y = \frac{-m\,r^2\,\omega_x\,\omega_z}{I + m\,r^2} \qquad \dot{\omega}_z = -\omega_x\,[\tan(q_L)\,\omega_z + \frac{I}{J}\,\omega_y] = \omega_x\,(\omega_y - \frac{I}{J}\,\omega_y - \dot{q}_S)$$

(c) **Optional****: Run a simulation of a thin uniform disk D with the constants and initial values provided in the previous table. Use a variable-step 4^{th}-order Runga-Kutta-Merson numerical integrator with an integration step of 0.05 sec and absolute and relative error tolerances of 1.0×10^{-8}. Plot the first 12 seconds of the lean angle versus t and y versus x.[1]





Euler disk: Courtesy/purchase from Arbor Scientific www.arborsci.com

[1]**Note: Problem solution at www.MotionGenesis.com ⇒ Get Started ⇒ Rolling disk.**