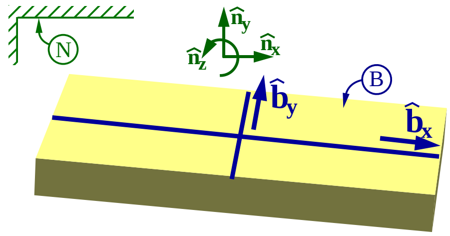
7.8 Euler parameters (quaternions) for 3D orientation

As shown in Homework 6.26, the $\widehat{\mathbf{b}}_x$, $\widehat{\mathbf{b}}_y$, $\widehat{\mathbf{b}}_z$ angular velocity measures for the "torque-free" motion of a rigid body B (e.g., a rotating spacecraft/aircraft or spinning book/tennis racquet) in a Newtonian reference frame N are governed by the following ordinary differential equations:

$$\dot{\omega}_x = \frac{(I_{yy} - I_{zz})\,\omega_z\,\omega_y}{I_{xx}}$$

$$\dot{\omega}_y = \frac{(I_{zz} - I_{xx})\,\omega_x\,\omega_z}{I_{yy}}$$

$$\dot{\omega}_z = \frac{(I_{xx} - I_{yy})\,\omega_y\,\omega_x}{I_{zz}}$$



Quantity	Symbol	Type	(Initial) value
B's central principal moment of inertia for $\widehat{\mathbf{b}}_x$	I_{xx}	Constant	$1\ \text{kg m}^2$
B's central principal moment of inertia for $\widehat{\mathbf{b}}_y$	I_{yy}	Constant	$2\ \text{kg m}^2$
B's central principal moment of inertia for $\widehat{\mathbf{b}}_z$	I_{zz}	Constant	$3\ \text{kg m}^2$
$\widehat{\mathbf{b}}_x$ measure of B's angular velocity in N	ω_x	Variable	0.2 rad/sec
$\widehat{\mathbf{b}}_y$ measure of B's angular velocity in N	ω_y	Variable	7.0 rad/sec
$\widehat{\mathbf{b}}_z$ measure of B's angular velocity in N	ω_z	Variable	0.2 rad/sec

Referring to Section 8.6.2, use ***Euler parameters*** (***quaternions***) ϵ_0, ϵ_1, ϵ_2, ϵ_3 to characterize the ${}^{n}R^{b}$ relating right-handed orthogonal unit vectors fixed in B to right-handed orthogonal unit vectors $\widehat{\mathbf{n}}_x$, $\widehat{\mathbf{n}}_y$, $\widehat{\mathbf{n}}_z$ fixed in N. Using Section 9.2, relate the time-derivative of the Euler parameters to the angular velocity measures ω_x, ω_y, ω_z.[4]

Result:

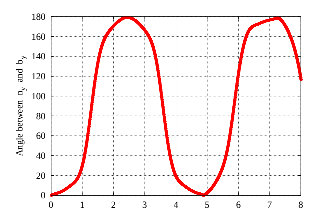
${}^{n}R^{b}$	$\widehat{\mathbf{b}}_x$	$\widehat{\mathbf{b}}_2$	$\widehat{\mathbf{b}}_z$
$\widehat{\mathbf{n}}_x$	$2\,\epsilon_1^2 + 2\,\epsilon_0^2 - 1$	$2(\epsilon_1\,\epsilon_2 - \epsilon_3\,\epsilon_0)$	$2(\epsilon_1\,\epsilon_3 + \epsilon_2\,\epsilon_0)$
$\widehat{\mathbf{n}}_y$	$2(\epsilon_1\,\epsilon_2 + \epsilon_3\,\epsilon_0)$	$2\,\epsilon_2^2 + 2\,\epsilon_0^2 - 1$	$2(\epsilon_2\,\epsilon_3 - \epsilon_1\,\epsilon_0)$
$\widehat{\mathbf{n}}_z$	$2(\epsilon_1\,\epsilon_3 - \epsilon_2\,\epsilon_0)$	$2(\epsilon_2\,\epsilon_3 + \epsilon_1\,\epsilon_0)$	$2\,\epsilon_3^2 + 2\,\epsilon_0^2 - 1$

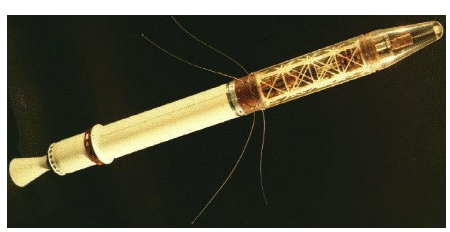
$$\begin{bmatrix} \dot{\epsilon}_0 \\ \dot{\epsilon}_1 \\ \dot{\epsilon}_2 \\ \dot{\epsilon}_3 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -\epsilon_1 & -\epsilon_2 & -\epsilon_3 \\ \epsilon_0 & -\epsilon_3 & \epsilon_2 \\ \epsilon_3 & \epsilon_0 & -\epsilon_1 \\ -\epsilon_2 & \epsilon_1 & \epsilon_0 \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

Knowing $\widehat{\mathbf{b}}_i = \widehat{\mathbf{n}}_i$ $(i = x, y, z)$ at $t = 0$, determine the angle θ between $\widehat{\mathbf{n}}_y$ and $\widehat{\mathbf{b}}_y$ at $t = 8$ sec. **Optional**:** Graph the time-history of θ.

Result:

$$\theta(t = 8) = 116.6°$$





[4]Note: **M**otion**G**enesis solves this problem at www.MotionGenesis.com ⇒ Get Started ⇒ 3D Spin stability of rigid body.