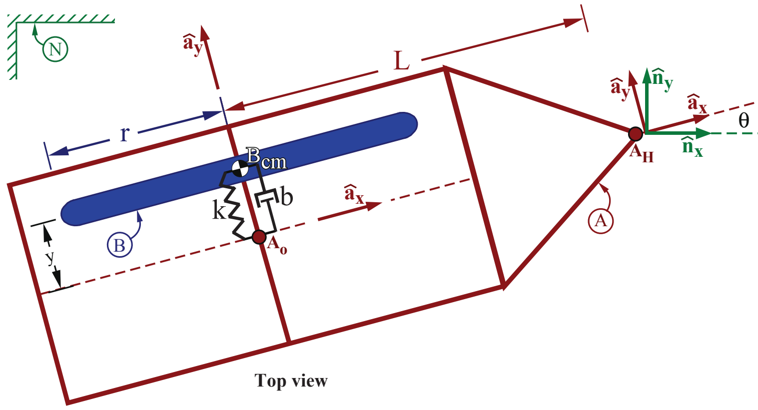
21.4 Motion of a single-wheeled trailer (bike-trailer, truck-trailer, ...).

Although single-wheel trailers sometimes behave poorly, it is not always clear why they do so. One possibility is tire flexibility and loose wheel mounting give rise to unstable behavior. To explore this, consider the system consisting of a rigid chassis A attached to a rigid wheel B in such a way that B rotates freely about A's axle and B_{cm} (B's center of mass) can translate along A's axle (B_{cm}'s translation along the axle is resisted by a translational spring/damper that models tire flexibility and loose wheel mounting).[5] The wheel is in single point contact with a flat horizontal road N (a Newtonian reference frame).

The trailer is attached to a vehicle (e.g., a bicycle or truck) at point A_H by an ideal pin joint whose revolute axis is vertical. The vehicle's motion ensures point A_H only moves in the $\hat{\mathbf{n}}_x$ direction.

Courtesy of Larry Liefer



Right-handed orthogonal unit vectors $\hat{\mathbf{n}}_x$, $\hat{\mathbf{n}}_y$, $\hat{\mathbf{n}}_z$ and $\hat{\mathbf{a}}_x$, $\hat{\mathbf{a}}_y$, $\hat{\mathbf{a}}_z$ are fixed in N and A, respectively, with $\hat{\mathbf{n}}_x$ horizontally-right, $\hat{\mathbf{n}}_z = \hat{\mathbf{a}}_z$ vertically-upward (also a **principal** inertia axes for A) $\hat{\mathbf{a}}_x$ directed forward along A's centerline, and $\hat{\mathbf{a}}_y$ parallel to the wheel's axle.

Quantity	Symbol	Type	(Initial) value
Distance from A_H (hitch point) to the wheel's axle	L	Constant	1.5 m
Wheel radius	r	Constant	0.25 m
Linear spring constant modeling tire flexibility	k	Constant	9000 N/m
Linear damping constant modeling tire flexibility	b	Constant	900 N s/m
B's mass	m	Constant	20 kg
A's moment of inertia about A_H for $\hat{\mathbf{a}}_z$	I^A	Constant	400 kg m^2
B's moment of inertia about B_{cm} for $\hat{\mathbf{a}}_y$	I^B	Constant	$\frac{1}{2}\, m\, r^2$
B's moment of inertia about B_{cm} for **any** line perpendicular to $\hat{\mathbf{a}}_y$	J^B	Constant	$\frac{1}{4}\, m\, r^2$
$\hat{\mathbf{n}}_x$ measure of A_H's velocity in N	v	Constant	10 m/s
Angle from $\hat{\mathbf{n}}_x$ to $\hat{\mathbf{a}}_x$ with $+\hat{\mathbf{n}}_z$ sense	θ	Variable	1°
$\hat{\mathbf{a}}_y$ measure of B_{cm}'s position vector from A_o	y	Variable	1 cm
$\hat{\mathbf{a}}_y$ measure of B's angular velocity in A	ω	Variable	Not Applicable

(a) Form the angular velocities/accelerations of A in N and B in N in terms of $\hat{\mathbf{a}}_x$, $\hat{\mathbf{a}}_y$, $\hat{\mathbf{a}}_z$.
Form the ${}^{a}R^{n}$ rotation table that relates $\hat{\mathbf{a}}_x$, $\hat{\mathbf{a}}_y$, $\hat{\mathbf{a}}_z$ to $\hat{\mathbf{n}}_x$, $\hat{\mathbf{n}}_y$, $\hat{\mathbf{n}}_z$.
Form B_{cm}'s velocity and acceleration in N.

Result:

$${}^N\vec{\boldsymbol{\omega}}^A = \dot{\theta}\,\hat{\mathbf{a}}_z \qquad {}^N\vec{\boldsymbol{\alpha}}^A = \ddot{\theta}\,\hat{\mathbf{a}}_z$$

$${}^N\vec{\boldsymbol{\omega}}^B = \omega\,\hat{\mathbf{a}}_y + \dot{\theta}\,\hat{\mathbf{a}}_z \qquad {}^N\vec{\boldsymbol{\alpha}}^B = -\omega\,\dot{\theta}\,\hat{\mathbf{a}}_x + \dot{\omega}\,\hat{\mathbf{a}}_y + \ddot{\theta}\,\hat{\mathbf{a}}_z$$

${}^{a}R^{n}$	$\hat{\mathbf{n}}_x$	$\hat{\mathbf{n}}_y$	$\hat{\mathbf{n}}_z$
$\hat{\mathbf{a}}_x$	$\cos(\theta)$	$\sin(\theta)$	0
$\hat{\mathbf{a}}_y$	$-\sin(\theta)$	$\cos(\theta)$	0
$\hat{\mathbf{a}}_z$	0	0	1

[5] For this analysis, the spring/damper force on B_{cm} can be modeled as having an equal and opposite force on point A_o (the intersection of the axle and the line parallel to $\hat{\mathbf{a}}_x$ that passes through A_H) **or** $A_{B_{cm}}$ (point of A coincident with B_{cm}).

$$^N\vec{\mathbf{v}}^{B_{cm}} = v\,\widehat{\mathbf{n}}_x \;-\; y\,\dot\theta\,\widehat{\mathbf{a}}_x \;+\; (\dot y - L\,\dot\theta)\,\widehat{\mathbf{a}}_y$$

$$^N\vec{\mathbf{a}}^{B_{cm}} = (L\,\dot\theta^2 - y\,\ddot\theta - 2\,\dot\theta\,\dot y)\,\widehat{\mathbf{a}}_x \;+\; (\ddot y \;-\; y\,\dot\theta^2 - L\,\ddot\theta)\widehat{\mathbf{a}}_y$$

(b) Use the rolling constraint to express $\omega,\ y,\ \dot\omega,\ \dot y,$ in terms of $r,\ \theta,\ \dot\theta,\ \ddot\theta,\ \dot y,\ v$.

Result:

$$\omega = [v\,\cos(\theta) - y\,\dot\theta]\;/\;r \qquad\qquad \dot y = v\,\sin(\theta) + L\,\dot\theta$$

$$\dot\omega = [\text{-}v\,\sin(\theta)\,\dot\theta \;-\; \dot y\,\dot\theta \;-\; y\,\ddot\theta]\;/\;r \qquad\qquad \ddot y = v\,\cos(\theta)\,\dot\theta + L\,\ddot\theta$$

(c) Determine $^N\vec{\mathbf{F}}^B$ (B's effective force in N), $^N\vec{\mathbf{M}}^{A/A_H}$ (A's moment of effective forces about A_H in N), and $^N\vec{\mathbf{M}}^{B/B_{cm}}$ (B's moment of effective forces about B_{cm} in N).

Result:

$$^N\vec{\mathbf{F}}^B = m\,^N\vec{\mathbf{a}}^{B_{cm}} = m\,(L\,\dot\theta^2 - y\,\ddot\theta - 2\,\dot\theta\,\dot y)\,\widehat{\mathbf{a}}_x \;+\; m*(\ddot y \;-\; y\,\dot\theta^2 - L\,\ddot\theta)\widehat{\mathbf{a}}_y$$

$$^N\vec{\mathbf{M}}^{A/A_H} \underset{(18.7)}{=} I^A\,\ddot\theta\,\widehat{\mathbf{a}}_z \qquad\qquad ^N\vec{\mathbf{M}}^{B/B_{cm}} \underset{(18.7)}{=} \text{-}I^B\,\dot\theta\,\omega\,\widehat{\mathbf{a}}_x + I^B\,\dot\omega\,\widehat{\mathbf{a}}_y + J^B\ddot\theta\,\widehat{\mathbf{a}}_z$$

(d) Accounting for rolling and how ω and $\dot y$ relate to $\dot\theta$, find the following ***partial angular velocities*** and ***partial velocities*** for the ***generalized speed*** $\dot\theta$.

Result:

$$\frac{\partial\,^N\vec{\boldsymbol{\omega}}^A}{\partial\dot\theta} = \widehat{\mathbf{a}}_z \qquad\qquad \frac{\partial\,^N\vec{\mathbf{v}}^{A_H}}{\partial\dot\theta} = \vec{\mathbf{0}}$$

$$\frac{\partial\,^N\vec{\boldsymbol{\omega}}^B}{\partial\dot\theta} = \widehat{\mathbf{a}}_z - \frac{y}{r}\,\widehat{\mathbf{a}}_y \qquad\qquad \frac{\partial\,^N\vec{\mathbf{v}}^{B_{cm}}}{\partial\dot\theta} = \text{-}y\,\widehat{\mathbf{a}}_x$$

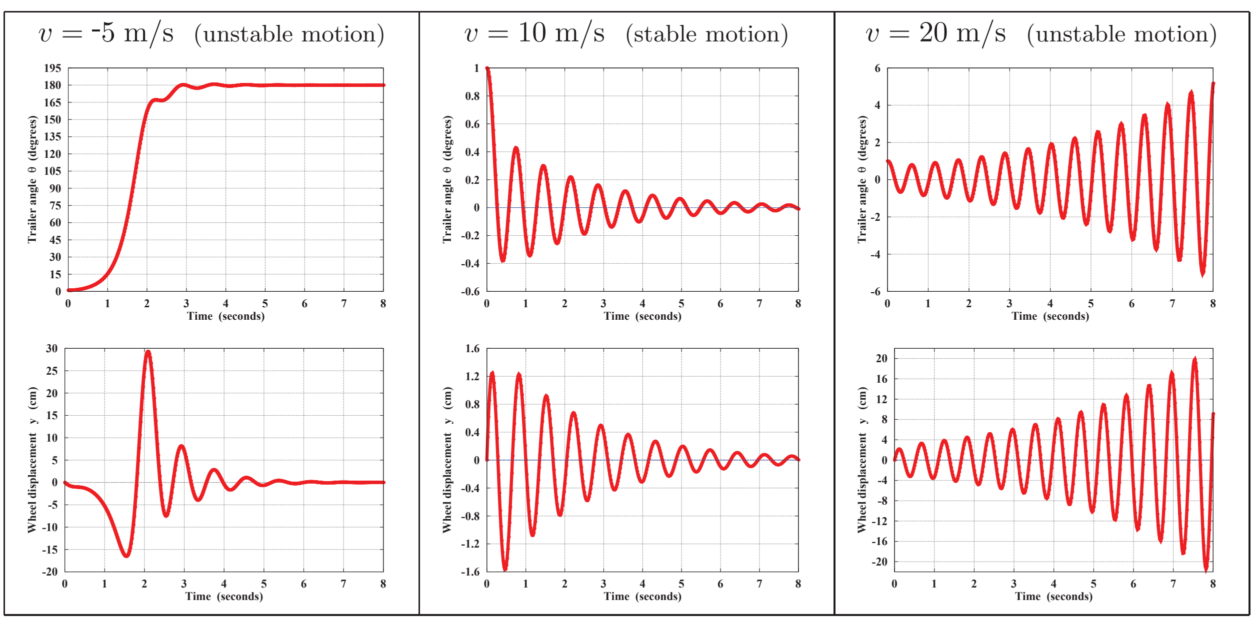
(e) Form ***Kane's equations of motion*** for the ***generalized speed*** $\dot\theta$.

Optional**: Investigate the inefficiencies of doing this problem with free-body diagrams for each rigid body.

Result: (Note: The spring is undeformed when $y = 0$).

$$L\,\big(k\,y + b\,\dot y\big) - L\,m\,y\,\dot\theta^2 + \frac{I^B}{r^2}\,v\,y\,\sin(\theta)\,\dot\theta + \Big(2\,m + \frac{I^B}{r^2}\Big)\,y\,\dot\theta\,\dot y + \Big[I^A + J^B + \Big(m + \frac{I^B}{r^2}\Big)\,y^2\Big]\,\ddot\theta = 0$$

(f) To simulate the trailer's motion, initial values should be assigned to the variables: $\theta,\ \dot\theta,\ y$.
Simulate the trailer's motion for the following values of v.[6]



[6]Note: Problem solution at www.MotionGenesis.com ⇒ Get Started ⇒ Single wheel trailer.