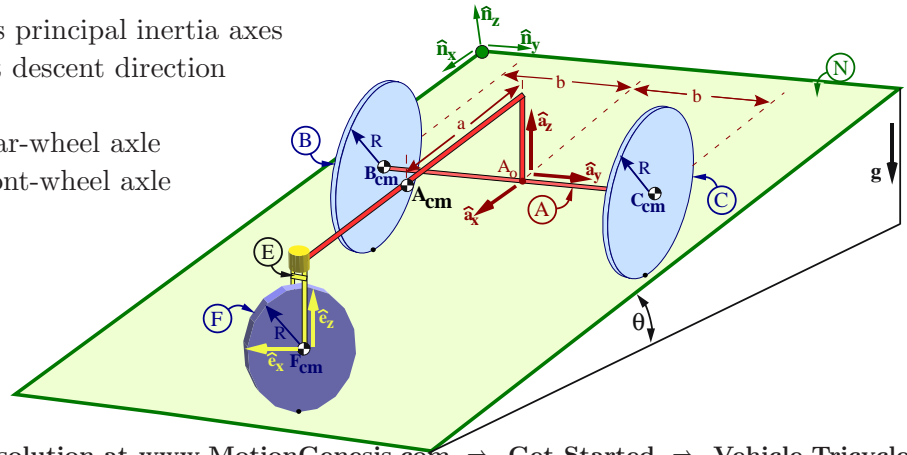


22.7 Braking torque and motion of a three-wheeled vehicle (statics and dynamics)

The figure below shows a *tricycle*⁷ S consisting of a rigid chassis A , a light (massless) rigid steering assembly E , and three **identical** uniform rigid wheels B , C , and F . All three wheels are in rolling contact with a planar road N (a Newtonian reference frame) that is inclined with angle θ to the local horizontal. The left-rear wheel C is free to rotate relative to A about the rear axle whose midpoint is A_o . The rotation of the right-rear wheel B relative to A about the rear axle is resisted by a **braking torque** characterized by T_B .⁸ The front wheel F is free to rotate about the front axle, which itself is fixed in steering assembly E . The rotation of steering assembly E relative to the chassis A about an axis normal to N can be actuated by a **steering torque** characterized by T_{Steer} .

Right-handed, orthogonal, unit vectors $\hat{n}_x, \hat{n}_y, \hat{n}_z, \hat{a}_x, \hat{a}_y, \hat{a}_z,$ and $\hat{e}_x, \hat{e}_y, \hat{e}_z$ are fixed in $N, A,$ and $E,$ respectively, and directed with $\hat{n}_z = \hat{a}_z = \hat{e}_z$ perpendicular to the road and

- $\hat{a}_x, \hat{a}_y, \hat{a}_z$ parallel to A 's principal inertia axes
- \hat{n}_x in the road's steepest descent direction
- \hat{a}_x vehicle-forward
- \hat{a}_y leftward along the rear-wheel axle
- \hat{e}_y leftward along the front-wheel axle



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Description	Symbol	Type	(Initial) value
Radius of wheels $B, C,$ and F	R	Constant	0.35 m
Distance from A_o to B_{cm} (B 's center of mass)	b	Constant	0.75 m
Distance from A_o to F_{cm} (F 's center of mass)	e	Constant	2.0 m
\hat{a}_x measure of position vector from A_o to A_{cm} (A 's center of mass)	a	Constant	1.64 m
\hat{a}_z measure of position vector from A_o to A_{cm}	h	Constant	0.35 m
Inclination angle of road from local horizontal	θ	Constant	15°
Earth's gravitational constant	g	Constant	9.8 m/s ²
Mass of A	m^A	Constant	640 kg
Mass of B or C or F	m	Constant	30 kg
A 's moment of inertia about A_{cm} for \hat{a}_z	I_{zz}^A	Constant	166.6 kg m ²
Axial moment of inertia of wheel (B or C or F)	J	Constant	2 kg m ²
Transverse moment of inertia of wheel (B or C or F)	K	Constant	1 kg m ²
Angle from \hat{n}_x to \hat{a}_x with $+\hat{n}_z$ sense (heading angle)	q_A	Variable	0° or 180°
\hat{a}_y measure of B 's angular velocity in A	ω_B	Variable	Constrained
\hat{a}_y measure of C 's angular velocity in A	ω_C	Variable	Constrained
\hat{e}_y measure of F 's angular velocity in E	ω_F	Variable	Constrained
\hat{a}_x measure of A_o 's velocity in N	v	Variable	0 m/s
Angle from \hat{a}_x to \hat{e}_x with $-\hat{a}_z$ sense (steering angle)	q_E	Variable	0° or 0.1°
\hat{a}_y measure of braking torque on B from A	T^B	Variable	Unknown
$-\hat{a}_z$ measure of steering torque on E from A	T_{Steer}	Variable	Unknown

- (a) Find all the **independent** scalar constraints when wheels $B, C,$ and F **roll** on N . Classify each constraint as a **configuration constraint** or **motion constraint**. Write each constraint as $\mathbf{a} \cdot \mathbf{b} = s$, where s is a scalar and \mathbf{a} and \mathbf{b} are either a unit vector, position vector, velocity vector, or angular velocity vector. Subsequently, express your configuration constraint(s) in the form $f(q_A, q_E) = s$ and your motion constraint(s) as $f(\dot{q}_A, \omega_B, \omega_C, \dot{q}_E, \omega_F, v) = s$.

⁷ *Tricycle* dynamics are important for designing trailers, three-wheeled motorized vehicles, and landing gear on aircraft.

⁸ The set of forces on B from A across the brakes can be replaced by an equivalent set consisting of a force $\vec{F}^{B/A}$ applied to the point B_A of B contacting the wheel's axle and a couple of torque $\vec{T}^{B/A}$. **Braking torque** is characterized by $T_B \triangleq \vec{T}^{B/A} \cdot \hat{a}_y$.

Result: ⁹	Constraint type	$\mathbf{a} \cdot \mathbf{b} = s$	Scalar constraint equation
	Velocity	$N_{\vec{v}}^{B_N} \cdot \hat{\mathbf{a}}_x = 0$	$v + b\dot{q}_A - R\omega_B = 0$
	Velocity	$N_{\vec{v}}^{C_N} \cdot \hat{\mathbf{a}}_x = 0$	$v - b\dot{q}_A - R\omega_C = 0$
	Velocity	$N_{\vec{v}}^{F_N} \cdot \hat{\mathbf{e}}_x = 0$	$\cos(q_E)v - R\omega_F - e \sin(q_E)\dot{q}_A = 0$
	Velocity	$N_{\vec{v}}^{F_N} \cdot \hat{\mathbf{e}}_y = 0$	$\sin(q_E)v + e \cos(q_E)\dot{q}_A = 0$

- (b) Solve the motion constraints for $\dot{q}_A, \omega_B, \omega_C, \omega_F$ in terms of v, \dot{q}_E .
 The solution below is singular (divide by zero error) when $\cos(q_E) = 0$.

Result:

$$\dot{q}_A = \frac{-1 \sin(q_E)}{e \cos(q_E)} v \quad \omega_B = (1 + b\dot{q}_A) \frac{v}{R} = \left[1 - \frac{b \sin(q_E)}{e \cos(q_E)} \right] \frac{v}{R}$$

$$\omega_F = \frac{1}{R \cos(q_E)} v \quad \omega_C = (1 - b\dot{q}_A) \frac{v}{R} = \left[1 + \frac{b \sin(q_E)}{e \cos(q_E)} \right] \frac{v}{R}$$

Accounting for the constraints, calculate the braking torque and steering torque required to keep this system at rest in N (**static equilibrium**). Express results in terms

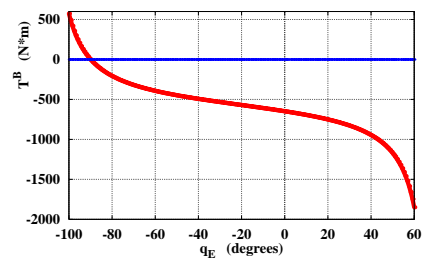
- (c) of the constants m^S (the system's mass) and s_x (the $\hat{\mathbf{a}}_x$ measure of the position vector from A_o to S 's center of mass).

Optional:** Plot T^B vs. q_E for $-100^\circ \leq q_E \leq 60^\circ$.

Result: (in terms of m^S and s_x - but **not** m^A and m)

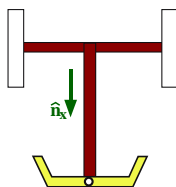
Note: The solution for T^B is singular (divide by zero error) when $q_E = 69.4^\circ$ or $q_E = -110.6^\circ$.¹⁰

$$T^B = \frac{m^S g R \sin(\theta) \left[\cos(q_A) \cos(q_E) + \frac{s_x}{e} \sin(q_A) \sin(q_E) \right]}{\frac{b}{e} \sin(q_E) - \cos(q_E)} \quad T_{\text{Steer}} = 0$$

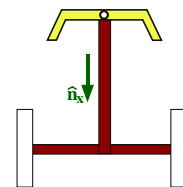
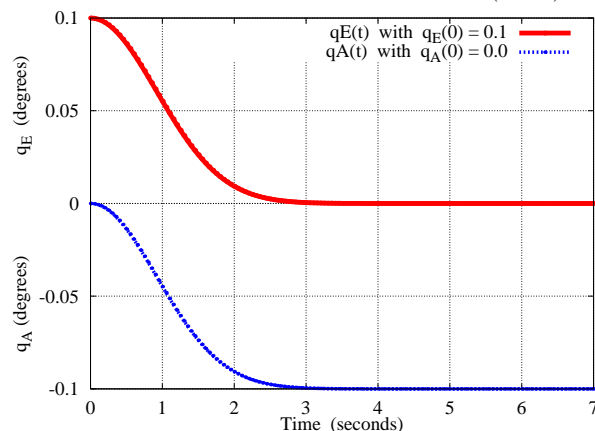


- (d) **Optional**:** Plot $q_A(t)$ and $q_E(t)$ when the brake is released ($T^B = 0$ and $T_{\text{Steer}} = 0$).

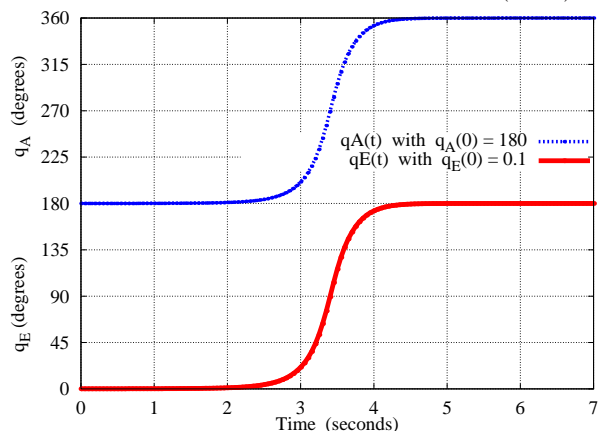
Simulate with a numerical integration step size 0.002 sec, absolute error tolerance 1×10^{-7} , and $q_E(0) = 0.1^\circ$.



Initially rolling forward downhill ($v \geq 0$)



Initially rolling backward downhill ($v \leq 0$)



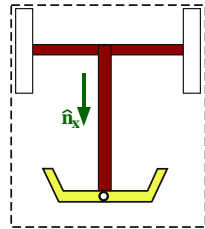
⁹Alternate expressions for rolling at F : $N_{\vec{v}}^{F_N} \cdot \hat{\mathbf{a}}_x = v - R\omega_F \cos(q_E) = 0$ and $N_{\vec{v}}^{F_N} \cdot \hat{\mathbf{a}}_y = e\dot{q}_A + R\omega_F \sin(q_E) = 0$.

¹⁰Hint: Use "clever" **road-maps** to pick systems and do an analysis that eliminates all but four constraint force measures (e.g., $F_x^B, F_x^C, F_x^F, F_y^F$). Alternately, use **Kane's method** to calculate **generalized forces** $\mathcal{F}_{u_r}^S$ for $u_r = v$ and \dot{q}_E .

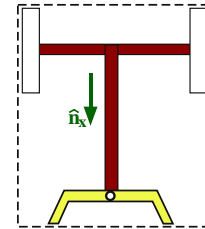
- (e) **Interpret:** In view of $q_A(t)$ and $q_E(t)$, complete each statement below and draw a picture that clearly shows the orientation of the tricycle's chassis (A) and handlebars (E) at $t = 7$ sec.

Initially rolling forward downhill $v(0) \geq 0$		Initially rolling backward downhill $v(0) \leq 0$	
Handlebars turn completely around	True/False	Handlebars turn completely around	True/False
Ends up rolling forward downhill ($v \geq 0$)	True/False	Ends up rolling forward downhill ($v \geq 0$)	True/False
Dangerous, particularly for a child	True/False	Dangerous, particularly for a child	True/False
"Unstable"	True/False	"Unstable"	True/False

Orientation at $t = 7$ sec



Orientation at $t = 7$ sec



Note: This three-wheel vehicle is also unstable in a braking turn.

In April 1988, the United States placed a 10-year ban on new sales of three-wheeled ATVs.

- (f) The number of degrees of freedom this system has when B , C , and F roll on N is: **2**.
- (g) One way to analyze this system is with a *free-body diagram* of each rigid object (A , B , C , E , F). The total number of scalar equations produced by these 5 FBDs (one for each rigid object) is: **30**. The total number of unknown constraint force/torque measures in these 5 FBDs is: **29**. It is possible to uniquely solve for each unknown constraint force/torque measure: **True/False**. Doing a *free-body diagram* of each rigid object is an efficient and effective way to analyze this system. **True/False**.

- (h) **Optional**:** † Determine the first time t_c (if any) any of the wheels lose contact with the ground and the minimum coefficient of static friction μ_s to ensure the front-wheel rolls for $0 \leq t \leq t_c$.

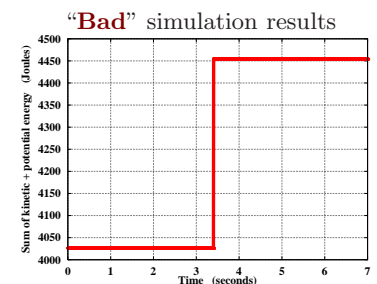
Result: (Simulate initially rolling backward downhill $v(0) \leq 0$ with $0 \leq t \leq 7$)

Optional:** Simulate again with a higher chassis center of mass with $h = 0.45$ m.

$h = 0.35$ m	$t_c \approx$ 3.66 sec	Wheel losing contact: C	$(\mu_s)_{\text{minimum}} \approx$ 3.84
$h = 0.45$ m	$t_c \approx$ 3.054 sec	Wheel losing contact: B	$(\mu_s)_{\text{minimum}} \approx$ 0.0426

- (i) † **Optional**:** Plot mechanical energy (kinetic + potential energy) vs. time (which should be \approx **constant** to numerical integrator error.) The plot to the right shows a **non-constant** energy, with a discontinuity at $t \approx 3.4$ sec. This signals **problems** with the equations of motion and/or how they are solved (e.g., singularities associated with $q_E = 90^\circ$ at $t \approx 3.4$ sec.)

Note: This singularity was fixed by augmenting the equations of motion with the time-derivatives of the constraint equations.



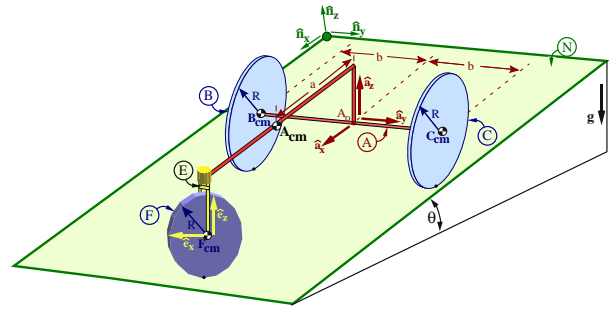
23.3 Dynamics, control, and simulation of three-wheeled vehicle (also see Homework 22.9)

Referring to Homework 22.9, design a control system for a rear-right-wheel **driving torque** T_B and steering torque T_{Steer} for the desired motion

$$v_{\text{Desired}} = \left(15 \frac{\text{km}}{\text{hour sec}}\right) * t \quad q_{E\text{Desired}} = 5^\circ$$

The control system should regulate error in v and q_E , defined as $\tilde{v} \triangleq v - v_{\text{Desired}}$, and $\tilde{q}_E \triangleq q_E - q_{E\text{Desired}}$, with

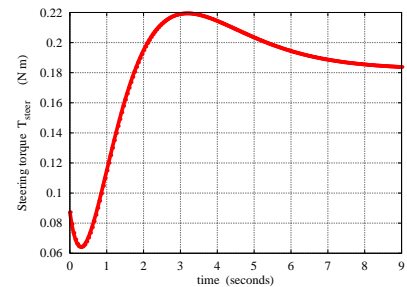
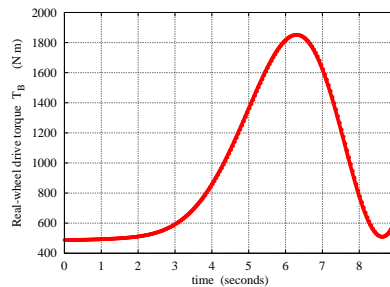
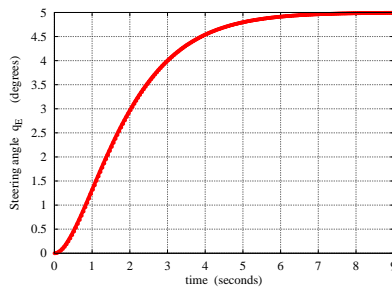
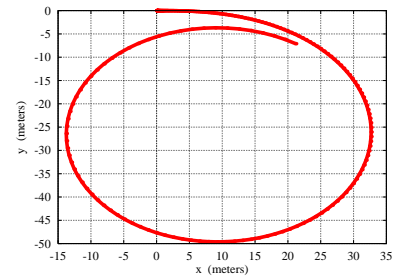
$$\frac{d\tilde{v}}{dt} + k_p \tilde{v} = 0 \quad (k_p = 1) \quad \frac{d^2 \tilde{q}_E}{dt^2} + 2\zeta \omega_n \frac{d\tilde{q}_E}{dt} + \omega_n^2 \tilde{q}_E = 0 \quad (\zeta = 1 \quad \omega_n = 1 \frac{\text{rad}}{\text{sec}})$$



Description (also see Homework 22.9)	Symbol	Type	(Initial) value
$\hat{\mathbf{a}}_x$ measure of A_o 's velocity in N	v	Variable	0 m/s
Angle from $\hat{\mathbf{n}}_x$ to $\hat{\mathbf{a}}_x$ with $+\hat{\mathbf{n}}_z$ sense (heading angle)	q_A	Variable	0°
Angle from $\hat{\mathbf{a}}_x$ to $\hat{\mathbf{e}}_x$ with $-\hat{\mathbf{a}}_z$ sense (steering angle)	q_E	Variable	0°
$\hat{\mathbf{a}}_y$ measure of driving torque on B from A	T^B	Variable	Unknown
$-\hat{\mathbf{a}}_z$ measure of steering torque on E from A	T_{Steer}	Variable	Unknown

Description	Symbol	Type	Initial
$\hat{\mathbf{n}}_x$ measure of A_o 's position vector from N_o	x	Variable	0 m
$\hat{\mathbf{n}}_y$ measure of A_o 's position vector from N_o	y	Variable	0 m

Knowing the system starts at **rest**, plot 9 seconds of y vs. x (in meters). Also plot the time-histories of steering angle q_E (in degrees), drive torque T_B (Nm), and steering torque T_{Steer} (Nm).



Problem solution at www.MotionGenesis.com ⇒ [Get Started](#) ⇒ [Vehicle tricycle](#).