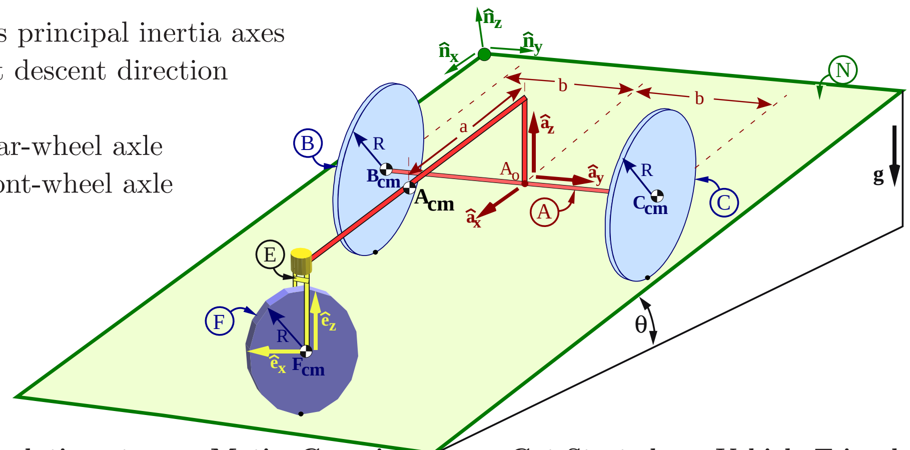
22.7 Braking torque and motion of a three-wheeled vehicle (statics and dynamics)

The figure below shows a ***tricycle***[7] S consisting of a rigid chassis A, a light (massless) rigid steering assembly E, and three **identical** uniform rigid wheels B, C, and F. All three wheels are in rolling contact with a planar road N (a Newtonian reference frame) that is inclined with angle θ to the local horizontal. The left-rear wheel C is free to rotate relative to A about the rear axle whose midpoint is A_o. The rotation of the right-rear wheel B relative to A about the rear axle is resisted by a ***braking torque*** characterized by T_B.[8] The front wheel F is free to rotate about the front axle. which itself is fixed in steering assembly E. The rotation of steering assembly E relative to the chassis A about an axis normal to N can be actuated by a ***steering torque*** characterized by $\mathrm{T}_{\mathrm{Steer}}$.

Right-handed, orthogonal, unit vectors $\hat{\mathbf{n}}_x, \hat{\mathbf{n}}_y, \hat{\mathbf{n}}_z$, $\hat{\mathbf{a}}_x, \hat{\mathbf{a}}_y, \hat{\mathbf{a}}_z$, and $\hat{\mathbf{e}}_x, \hat{\mathbf{e}}_y, \hat{\mathbf{e}}_z$ are fixed in N, A, and E, respectively, and directed with $\hat{\mathbf{n}}_z = \hat{\mathbf{a}}_z = \hat{\mathbf{e}}_z$ perpendicular to the road and

- $\hat{\mathbf{a}}_x, \hat{\mathbf{a}}_y, \hat{\mathbf{a}}_z$ parallel to A's principal inertia axes
- $\hat{\mathbf{n}}_x$ in the road's steepest descent direction
- $\hat{\mathbf{a}}_x$ vehicle-forward
- $\hat{\mathbf{a}}_y$ leftward along the rear-wheel axle
- $\hat{\mathbf{e}}_y$ leftward along the front-wheel axle



Problem solution at www.MotionGenesis.com ⇒ Get Started ⇒ Vehicle Tricycle.

Description	Symbol	Type	(Initial) value
Radius of wheels B, C, and F	R	Constant	0.35 m
Distance from A_o to B_{cm} (B's center of mass)	b	Constant	0.75 m
Distance from A_o to F_{cm} (F's center of mass)	e	Constant	2.0 m
$\hat{\mathbf{a}}_x$ measure of position vector from A_o to A_{cm} (A's center of mass)	a	Constant	1.64 m
$\hat{\mathbf{a}}_z$ measure of position vector from A_o to A_{cm}	h	Constant	0.35 m
Inclination angle of road from local horizontal	θ	Constant	15°
Earth's gravitational constant	g	Constant	9.8 m/s^2
Mass of A	m^A	Constant	640 kg
Mass of B or C or F	m	Constant	30 kg
A's moment of inertia about A_{cm} for $\hat{\mathbf{a}}_z$	I^A_{zz}	Constant	166.6 kg m^2
Axial moment of inertia of wheel (B or C or F)	J	Constant	2 kg m^2
Transverse moment of inertia of wheel (B or C or F)	K	Constant	1 kg m^2
Angle from $\hat{\mathbf{n}}_x$ to $\hat{\mathbf{a}}_x$ with $+\hat{\mathbf{n}}_z$ sense (heading angle)	q_A	Variable	0° or 180°
$\hat{\mathbf{a}}_y$ measure of B's angular velocity in A	ω_B	Variable	Constrained
$\hat{\mathbf{a}}_y$ measure of C's angular velocity in A	ω_C	Variable	Constrained
$\hat{\mathbf{e}}_y$ measure of F's angular velocity in E	ω_F	Variable	Constrained
$\hat{\mathbf{a}}_x$ measure of A_o's velocity in N	v	Variable	0 m/s
Angle from $\hat{\mathbf{a}}_x$ to $\hat{\mathbf{e}}_x$ with $-\hat{\mathbf{a}}_z$ sense (steering angle)	q_E	Variable	0° or 0.1°
$\hat{\mathbf{a}}_y$ measure of breaking torque on B from A	T^B	**Variable**	**Unknown**
$-\hat{\mathbf{a}}_z$ measure of steering torque on E from A	$\mathrm{T}_{\mathrm{Steer}}$	**Variable**	**Unknown**

(a) Find all the **independent** scalar constraints when wheels B, C, and F ***roll*** on N. Classify each constraint as a ***configuration constraint*** or ***motion constraint***. Write each constraint as $\boldsymbol{a} \cdot \boldsymbol{b} = s$, where s is a scalar and $\boldsymbol{a}$ and $\boldsymbol{b}$ are either a unit vector, position vector, velocity vector, or angular velocity vector. Subsequently, express your configuration constraint(s) in the form $f(q_A, q_E) = s$ and your motion constraint(s) as $f(\dot{q}_A, \omega_B, \omega_C, \dot{q}_E, \omega_F, v) = s$.

[7] ***Tricycle*** dynamics are important for designing trailers, three-wheeled motorized vehicles, and landing gear on aircraft.

[8] The set of forces on B from A across the brakes can be replaced by an equivalent set consisting of a force $\vec{\mathbf{F}}^{B_A}$ applied to the point B_A **of** B contacting the wheel's axle and a couple of torque $\vec{\mathbf{T}}^{B/A}$. ***Braking torque*** is characterized by $T_B \triangleq \vec{\mathbf{T}}^{B/A} \cdot \hat{\mathbf{a}}_y$.

Result:[9]

Constraint type	$\boldsymbol{a} \cdot \boldsymbol{b} = s$	Scalar constraint equation
	$\;\cdot\; = 0$	$= 0$
	$\;\cdot\; = 0$	$= 0$
	$\;\cdot\; = 0$	$= 0$
	$\;\cdot\; = 0$	$= 0$

(b) Solve the motion constraints for $\dot{q}_A,\ \omega_B,\ \omega_C,\ \omega_F$ in terms of $v,\ \dot{q}_E$.
The solution below is singular (divide by zero error) when $\cos(q_E) = $ ____.

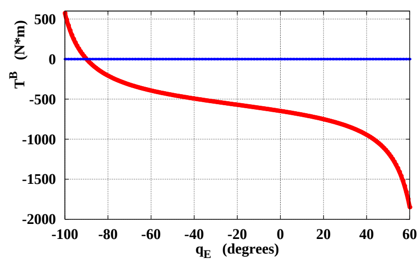
Result:

$$\dot{q}_A = \frac{-1}{e}\frac{\sin(q_E)}{\cos(q_E)}\, v \qquad \omega_B = (1 + b\,\dot{q}_A)\,\frac{v}{R} = \left[1 - \frac{b}{e}\frac{\sin(q_E)}{\cos(q_E)}\right]\frac{v}{R}$$

$$\omega_F = \frac{1}{R\,\cos(q_E)}\, v \qquad \omega_C = (1 - b\,\dot{q}_A)\,\frac{v}{R} = \left[1 + \frac{b}{e}\frac{\sin(q_E)}{\cos(q_E)}\right]\frac{v}{R}$$

(c) Accounting for the constraints, calculate the braking torque and steering torque required to keep this system at rest in N (**static equilibrium**). Express results in terms of the constants m^S (the system's mass) and s_x (the $\hat{\mathbf{a}}_x$ measure of the position vector from A_o to S's center of mass).

Optional:** Plot T^B vs. q_E for $-100° \le q_E \le 60°$.

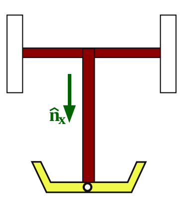


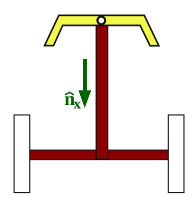
Result: (in terms of m^S and s_x - but **not** m^A and m)

Note: The solution for T^B is singular (divide by zero error) when $q_E = 69.4°$ or $q_E = -110.6°$. [10]

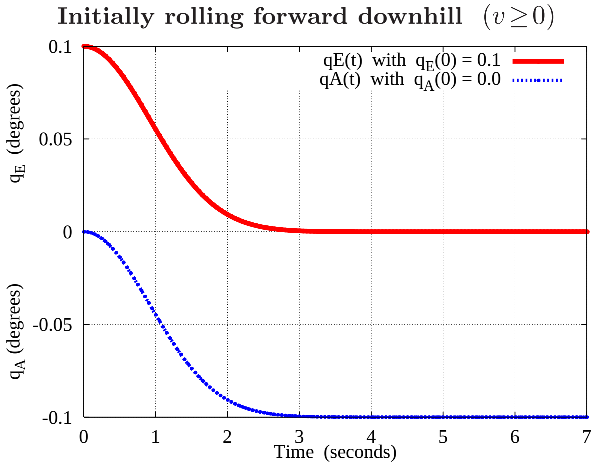
$$\mathrm{T}^B = \frac{m^S\, g\, R\, \sin(\theta)\left[\cos(q_A)\,\cos(q_E) + \frac{s_x}{e}\,\sin(q_A)\,\sin(q_E)\right]}{\frac{b}{e}\,\sin(q_E) - \cos(q_E)} \qquad \mathrm{T}_{\text{Steer}} = 0$$

(d) **Optional**:** Plot $q_A(t)$ and $q_E(t)$ when the brake is released ($\mathrm{T}^B = 0$ and $\mathrm{T}_{\text{Steer}} = 0$).
Simulate with a numerical integration step size 0.002 sec, absolute error tolerance 1×10^{-7}, and $q_E(0) = 0.1°$.

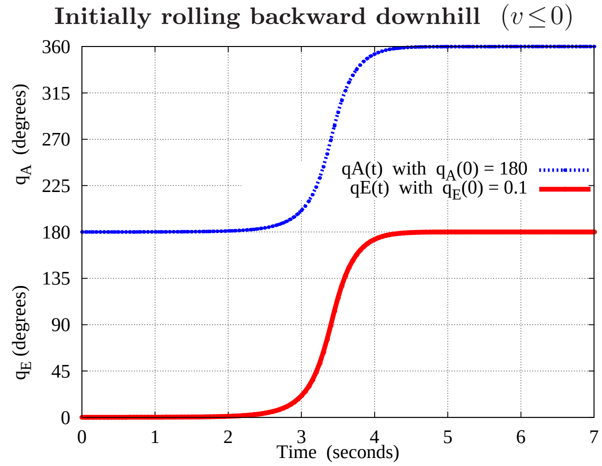




Initially rolling forward downhill $(v \ge 0)$



Initially rolling backward downhill $(v \le 0)$



[9] Alternate expressions for rolling at F: ${}^{N}\vec{\mathbf{v}}^{F_N} \cdot \hat{\mathbf{a}}_x = v - R\,\omega_F \cos(q_E) = 0$ and ${}^{N}\vec{\mathbf{v}}^{F_N} \cdot \hat{\mathbf{a}}_y = e\,\dot{q}_A + R\,\omega_F \sin(q_E) = 0$.

[10] Hint: Use "clever" ***road-maps*** to pick systems and do an analyses that eliminates all but four constraint force measures (e.g., $F^B_x,\ F^C_x,\ F^F_x,\ F^F_y$). Alternately, use ***Kane's method*** to calculate ***generalized forces*** $\mathcal{F}^S_{u_r}$ for $u_r = v$ and $\dot{q}_E$.

(e) **Interpret:** In view of $q_A(t)$ and $q_E(t)$, complete each statement below and draw a picture that clearly shows the orientation of the tricycle's chassis (A) and handlebars (E) at $t = 7$ sec.

Initially rolling forward downhill $v(0) \geq 0$

Handlebars turn completely around	**True/False**
Ends up rolling forward downhill $(v \geq 0)$	**True/False**
Dangerous, particularly for a child	**True/False**
"Unstable"	**True/False**

Initially rolling backward downhill $v(0) \leq 0$

Handlebars turn completely around	**True/False**
Ends up rolling forward downhill $(v \geq 0)$	**True/False**
Dangerous, particularly for a child	**True/False**
"Unstable"	**True/False**

Orientation at $t = 7$ sec

Orientation at $t = 7$ sec

Note: This three-wheel vehicle is also unstable in a braking turn.
In April 1988, the United States placed a 10-year ban on new sales of three-wheeled ATVs.

(f) The number of degrees of freedom this system has when B, C, and F roll on N is: ___.

(g) One way to analyze this system is with a ***free-body diagram*** of each rigid object (A, B, C, E, F).
The total number of scalar equations produced by these 5 FBDs (one for each rigid object) is: ___
The total number of unknown constraint force/torque measures in these 5 FBDs is: ___
It is possible to uniquely solve for each unknown constraint force/torque measure: **True/False**.
Doing a ***free-body diagram*** of each rigid object is an efficient and effective way to analyze this system. **True/False**.

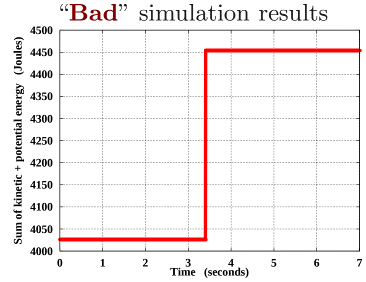
(h) **Optional**:** **†**Determine the first time t_c (if any) any of the wheels lose contact with the ground and the minimum coefficient of static friction μ_s to ensure the front-wheel rolls for $0 \leq t \leq t_c$.
Result: (Simulate initially rolling backward downhill $v(0) \leq 0$ with $0 \leq t \leq 7$)
Optional:** Simulate again with a higher chassis center of mass with $h = 0.45$ m.

$h = 0.35$ m	$t_c \approx$ 3.66 sec	Wheel losing contact: ___	$(\mu_s)_{\text{minimum}} \approx$ 3.84
$h = 0.45$ m	$t_c \approx$ 3.054 sec	Wheel losing contact: ___	$(\mu_s)_{\text{minimum}} \approx$ 0.0426

(i) **† Optional**:** Plot mechanical energy (kinetic + potential energy) vs. time (which should be $\approx$ <u>**constant**</u> to numerical integrator error.) The plot to the right shows a **non-constant** energy, with a discontinuity at $t \approx 3.4$ sec. This signals ***problems*** with the equations of motion and/or how they are solved (e.g., singularities associated with $q_E = 90°$ at $t \approx 3.4$ sec.)

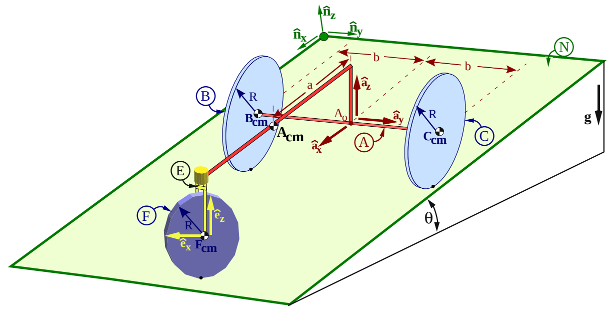
Note: This singularity was fixed by augmenting the equations of motion with the time-derivatives of the constraint equations.

"**Bad**" simulation results



23.3 Dynamics, control, and simulation of three-wheeled vehicle (also see Homework 22.9)

Referring to Homework 22.9, design a control system for a rear-right-wheel ***driving torque*** T_B and steering torque $\mathrm{T_{Steer}}$ for the desired motion



$$v_{\text{Desired}} = \left(15 \,\frac{\text{km}}{\text{hour sec}}\right) * t \qquad\qquad q_{E\text{Desired}} = 5^\circ$$

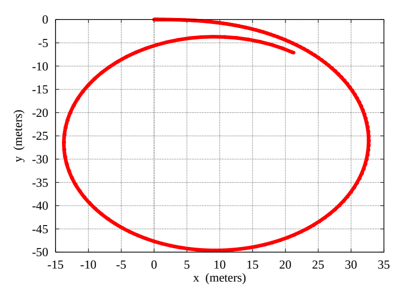
The control system should regulate error in v and q_E, defined as $\widetilde{v} \triangleq v - v_{\text{Desired}}$, and $\widetilde{q_E} \triangleq q_E - q_{E\text{Desired}}$, with

$$\frac{d\,\widetilde{v}}{dt} + k_p\,\widetilde{v} = 0 \quad (k_p = 1) \qquad\qquad \frac{d^2\,\widetilde{q}_E}{dt^2} + 2\,\zeta\,\omega_n\,\frac{d\,\widetilde{q}_E}{dt} + \omega_n^2\,\widetilde{q}_E = 0 \quad (\zeta = 1 \quad \omega_n = 1\,\frac{\text{rad}}{\text{sec}})$$

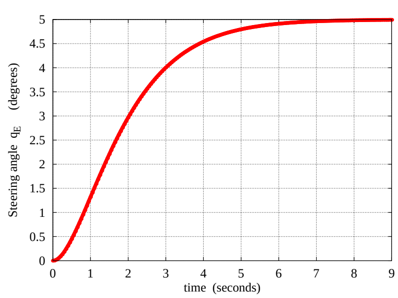
Description (also see Homework 22.9)	Symbol	Type	(Initial) value
$\widehat{\mathbf{a}}_x$ measure of A_o's velocity in N	v	Variable	0 m/s
Angle from $\widehat{\mathbf{n}}_x$ to $\widehat{\mathbf{a}}_x$ with $+\widehat{\mathbf{n}}_z$ sense (heading angle)	q_A	Variable	0°
Angle from $\widehat{\mathbf{a}}_x$ to $\widehat{\mathbf{e}}_x$ with $-\widehat{\mathbf{a}}_z$ sense (steering angle)	q_E	Variable	0°
$\widehat{\mathbf{a}}_y$ measure of driving torque on B from A	T^B	**Variable**	**Unknown**
$-\widehat{\mathbf{a}}_z$ measure of steering torque on E from A	$\mathrm{T_{Steer}}$	**Variable**	**Unknown**

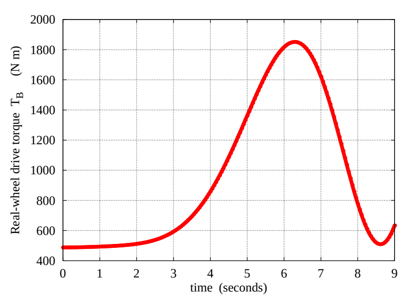
Description	Symbol	Type	Initial
$\widehat{\mathbf{n}}_x$ measure of A_o's position vector from N_o	x	Variable	0 m
$\widehat{\mathbf{n}}_y$ measure of A_o's position vector from N_o	y	Variable	0 m

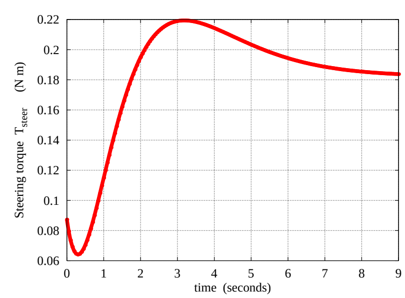
Knowing the system starts at **rest**, plot 9 seconds of y vs. x (in meters). Also plot the time-histories of steering angle q_E (in degrees), drive torque T_B ($N\,m$), and steering torque $\mathrm{T_{Steer}}$ ($N\,m$).



Result:







Problem solution at **www.MotionGenesis.com** ⇒ **Get Started** ⇒ **Vehicle tricycle**.