

4.1 ♣ **Soh Cah Toa: Sine, cosine, tangent as ratios of sides of a right triangle.** (Section 1.4)

The following shows a *right triangle* with one of its angles labeled as  $\theta$ .

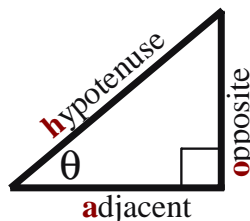
Note: A *right triangle* is a triangle with a  $90^\circ$  angle.

Write definitions for sine, cosine, and tangent in terms of:

- **h**ypotenuse – the triangle’s longest side (opposite the  $90^\circ$  angle)
- **o**pposite – the side opposite to  $\theta$
- **a**djacent – the side adjacent to  $\theta$

Note: A mnemonic for these definitions is “Soh Cah Toa”.

I can draw a triangle with a negative-length side **True/False**  
 Using the limited definition shown right, **True/False**  
 the sine of an angle can be negative.



$\sin(\theta) \triangleq$

$\cos(\theta) \triangleq$

$\tan(\theta) \triangleq$

4.2 ♣ **Pythagorean theorem and law of cosines - memorize.** (Section 1.4.1).

Draw a right-triangle with a hypotenuse of length  $c$  and other sides of length  $a$  and  $b$ . Relate  $c$  to  $a$  and  $b$  with the *Pythagorean theorem*.

Result:

$$c^2 = \text{ } + \text{ }$$

A non-right-triangle has angles  $\alpha, \beta, \phi$  opposite sides  $a, b, c$ , respectively.

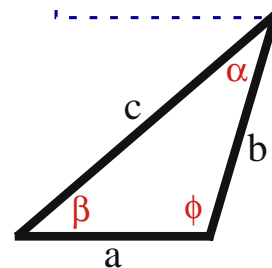
Use the *law of cosines* to complete each formula below.

Result:

$$c^2 = a^2 + b^2 - 2ab \cos(\phi)$$

$$a^2 = \text{ } + \text{ } - \text{ }$$

$$b^2 = \text{ } + \text{ } - \text{ }$$



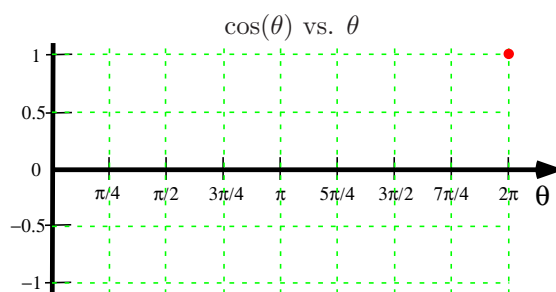
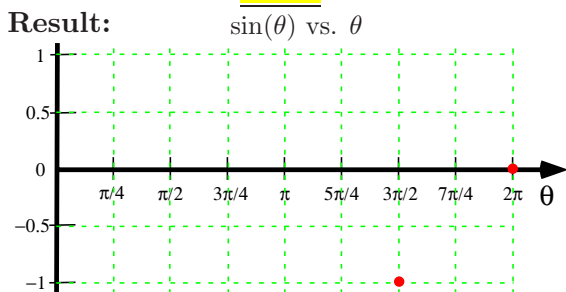
The *Pythagorean theorem* is a special case of the *law of cosines*. **True/False**. (circle one).

4.3 ♣ **Graphing sine and cosine - (a now-obvious invention from 1730 A.D.)** (Section 1.4.2)

Graph sine and cosine as functions of the angle  $\theta$  in radians over the range  $0 \leq \theta \leq 2\pi$ .

The mathematician  was first to regard sine and cosine as functions (not just ratios of sides of a triangle).

Result:



#### 4.4 ♣ Ranges for arguments and return values for inverse trigonometric functions.

Determine all real return values and argument values for the following **real** trigonometric and inverse-trigonometric functions in computer languages such as Java, C++, MotionGenesis, and MATLAB®.

Possible return values	Function	Possible argument values	Note
<input type="text"/> ≤ z ≤ <input type="text"/>	$z = \cos(x)$	<input type="text"/> < x < <input type="text"/>	
<input type="text"/> ≤ z ≤ <input type="text"/>	$z = \sin(x)$	<input type="text"/> < x < <input type="text"/>	
<input type="text"/> < z < <input type="text"/>	$z = \tan(x)$	<input type="text"/> < x < <input type="text"/>	$x \neq \frac{-\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \dots$
<input type="text"/> ≤ z ≤ <input type="text"/>	$z = \text{acos}(x)$	<input type="text"/> ≤ x ≤ <input type="text"/>	
<input type="text"/> ≤ z ≤ <input type="text"/>	$z = \text{asin}(x)$	<input type="text"/> ≤ x ≤ <input type="text"/>	
<input type="text"/> < z < <input type="text"/>	$z = \text{atan}(x)$	<input type="text"/> < x < <input type="text"/>	
<input type="text"/> ≤ z ≤ <input type="text"/>	$z = \text{atan2}(y, x)$	<input type="text"/> < y < <input type="text"/> <input type="text"/> < x < <input type="text"/>	$\text{atan2}(0, 0)$ is undefined

#### 4.5 ♣ What is an angle? (Section 5.6).

**Draw** the “geometry equipment” listed in the 1<sup>st</sup> column of the following table. Complete the 2<sup>nd</sup> column with appropriate ranges for the angle  $\theta$  (in degrees).

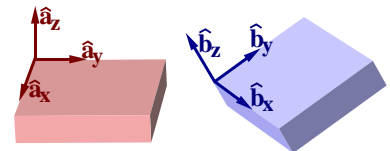


“Geometry equipment”	Draw	Appropriate range for $\theta$
2 lines	<input type="text"/>	$0^\circ \leq \theta \leq$ <input type="text"/>
Vector and line	<input type="text"/>	<input type="text"/> ≤ $\theta$ ≤ <input type="text"/>
2 vectors	<input type="text"/>	<input type="text"/> ≤ $\theta$ ≤ <input type="text"/>
2 vectors and a sense of positive rotation	<input type="text"/>	<input type="text"/> < $\theta$ ≤ <input type="text"/>
2 vectors, a sense of $\pm$ rotation, and time-history/continuity	Not applicable	<input type="text"/> < $\theta$ < <input type="text"/>

#### 4.6 Calculating dot-products, cross-products, and angles between vectors. (Section 5.4.3).

The  ${}^aR^b$  rotation table relates two sets of right-handed, orthogonal, unit vectors, namely  $\hat{\mathbf{a}}_x, \hat{\mathbf{a}}_y, \hat{\mathbf{a}}_z$  and  $\hat{\mathbf{b}}_x, \hat{\mathbf{b}}_y, \hat{\mathbf{b}}_z$ .

${}^aR^b$	$\hat{\mathbf{b}}_x$	$\hat{\mathbf{b}}_y$	$\hat{\mathbf{b}}_z$
$\hat{\mathbf{a}}_x$	0.9623	-0.0842	0.2588
$\hat{\mathbf{a}}_y$	0.1701	0.9284	-0.3304
$\hat{\mathbf{a}}_z$	-0.2125	0.3619	0.9077



- (a) Efficiently determine the following dot-products and angles between vectors (2<sup>+</sup> significant digits). Then perform the following calculations involving  $\vec{\mathbf{v}}_1 = 2\hat{\mathbf{a}}_x$  and  $\vec{\mathbf{v}}_2 = \hat{\mathbf{a}}_x + \hat{\mathbf{b}}_x$ .

$\hat{\mathbf{a}}_x \cdot \hat{\mathbf{a}}_x =$ <input type="text"/>	$\hat{\mathbf{a}}_y \cdot \hat{\mathbf{a}}_z =$ <input type="text"/>	$\hat{\mathbf{b}}_z \cdot \hat{\mathbf{b}}_y =$ <input type="text"/>
$\hat{\mathbf{a}}_x \cdot \hat{\mathbf{b}}_x =$ <input type="text"/>	$\hat{\mathbf{a}}_x \cdot \hat{\mathbf{b}}_y =$ <input type="text"/>	$\hat{\mathbf{b}}_z \cdot \hat{\mathbf{a}}_y =$ <input type="text"/>
$\angle(\hat{\mathbf{a}}_y, \hat{\mathbf{a}}_y) =$ <input type="text"/> °	$\angle(\hat{\mathbf{b}}_z, \hat{\mathbf{b}}_x) =$ <input type="text"/> °	
$\angle(\hat{\mathbf{a}}_y, \hat{\mathbf{b}}_y) =$ <input type="text"/> °	$\angle(\hat{\mathbf{b}}_y, \hat{\mathbf{a}}_z) =$ <input type="text"/> °	

**Result:**  $\vec{\mathbf{v}}_1 \cdot \vec{\mathbf{v}}_2 =$         $\angle(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2) =$  °  
 $\vec{\mathbf{v}}_1 \times \vec{\mathbf{v}}_2 =$    $\hat{\mathbf{b}}_y +$    $\hat{\mathbf{b}}_z =$    $\hat{\mathbf{a}}_y +$    $\hat{\mathbf{a}}_z$

- (b) Express the unit vector  $\hat{\mathbf{u}}$  in the direction of  $3\hat{\mathbf{a}}_z + 4\hat{\mathbf{b}}_z$  in terms of  $\hat{\mathbf{a}}_z$  and  $\hat{\mathbf{b}}_z$ . Express  $\vec{\mathbf{v}} = \hat{\mathbf{a}}_y + \hat{\mathbf{b}}_y$  in terms of  $\hat{\mathbf{a}}_x, \hat{\mathbf{a}}_y, \hat{\mathbf{a}}_z$ .

**Result:**  $\hat{\mathbf{u}} =$    $\hat{\mathbf{a}}_z +$    $\hat{\mathbf{b}}_z$   
 $\vec{\mathbf{v}} =$    $\hat{\mathbf{a}}_x +$    $\hat{\mathbf{a}}_y +$    $\hat{\mathbf{a}}_z$

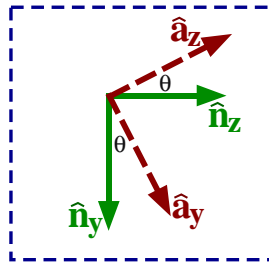
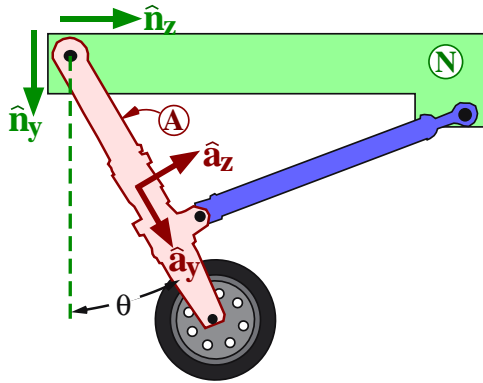
4.7 ♣ **Efficient calculation of the inverse of a rotation matrix.** (Section 5.4.2).

The following rotation matrix  $R$  relates two right-handed, orthogonal, unitary bases. Calculate its inverse by-hand (no calculator) in less than 30 seconds.

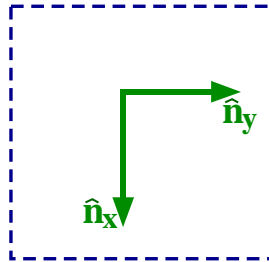
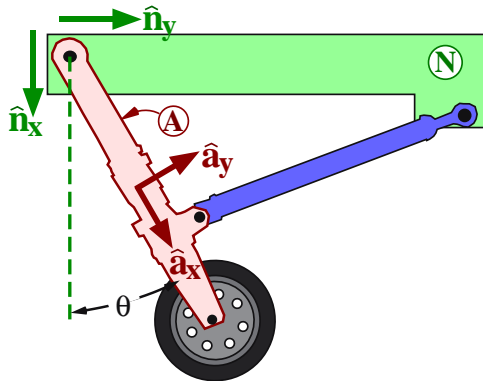
$$R = \begin{bmatrix} 0.3830 & -0.6634 & 0.6428 \\ 0.9237 & 0.2795 & -0.2620 \\ -0.0058 & 0.6941 & 0.7198 \end{bmatrix} \Rightarrow R^{-1} = \begin{bmatrix} \boxed{\phantom{000}} & \boxed{\phantom{000}} & \boxed{\phantom{000}} \\ \boxed{\phantom{000}} & \boxed{\phantom{000}} & \boxed{\phantom{000}} \\ \boxed{\phantom{000}} & \boxed{\phantom{000}} & \boxed{\phantom{000}} \end{bmatrix}$$

4.8 ♣ **SohCahToa: Rotation tables for a landing gear system.** (Section 5.5).

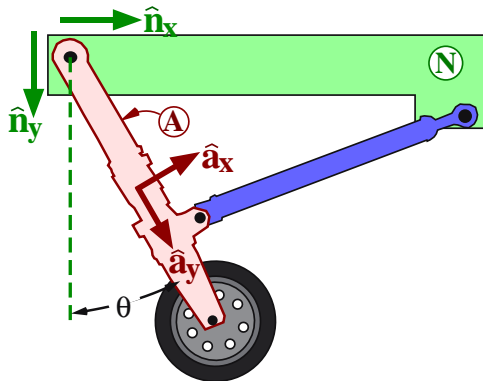
The figures below show three versions of the same landing gear system with a strut  $A$  that has a simple rotation relative to a fuselage  $N$ . Each figure has a different orientation for right-handed orthogonal unit vectors  $\hat{n}_x, \hat{n}_y, \hat{n}_z$  (fixed in  $N$ ) and  $\hat{a}_x, \hat{a}_y, \hat{a}_z$  (fixed in  $A$ ). **Redraw**  $\hat{n}_x, \hat{n}_y, \hat{n}_z$  and  $\hat{a}_x, \hat{a}_y, \hat{a}_z$  so it is **easy to see** sines and cosines. Then, form the  ${}^aR^n$  rotation table for each figure.<sup>1</sup>



${}^aR^n$	$\hat{n}_x$	$\hat{n}_y$	$\hat{n}_z$
$\hat{a}_x$	1	<input type="text"/>	<input type="text"/>
$\hat{a}_y$	0	$\cos(\theta)$	$\sin(\theta)$
$\hat{a}_z$	0	$-\sin(\theta)$	$\cos(\theta)$



${}^aR^n$	$\hat{n}_x$	$\hat{n}_y$	$\hat{n}_z$
$\hat{a}_x$	<input type="text"/>	<input type="text"/>	<input type="text"/>
$\hat{a}_y$	<input type="text"/>	<input type="text"/>	<input type="text"/>
$\hat{a}_z$	<input type="text"/>	<input type="text"/>	<input type="text"/>



${}^aR^n$	$\hat{n}_x$	$\hat{n}_y$	$\hat{n}_z$
$\hat{a}_x$	<input type="text"/>	<input type="text"/>	<input type="text"/>
$\hat{a}_y$	<input type="text"/>	<input type="text"/>	<input type="text"/>
$\hat{a}_z$	<input type="text"/>	<input type="text"/>	<input type="text"/>

<sup>1</sup>Each figure has two missing vectors (e.g.,  $\hat{n}_x$  and  $\hat{a}_x$  are missing from the first figure). Use the fact that each set of vectors is **right-handed** to add the missing vectors to each figure.

4.9 ♣ **Rotation table concepts: What is an angle.** (Section 5.6)

**Given:** Two sets of right-handed, orthogonal, unitary bases  $\hat{a}_x, \hat{a}_y, \hat{a}_z$  and  $\hat{b}_x, \hat{b}_y, \hat{b}_z$ .

**Question:** Determine a numerical expression for each element of the  $3 \times 3$  rotation table  ${}^bR^a$  below so  $\hat{b}_z = \hat{a}_z$  and the angle between  $\hat{b}_x$  and  $\hat{a}_x$  is  $30^\circ$ .

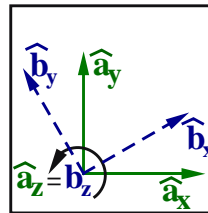
**Draw**  $\hat{b}_x, \hat{b}_y, \hat{b}_z$ , clearly showing the relative orientation of the two bases.

**Question:** Are there other orientations of  $\hat{b}_x, \hat{b}_y, \hat{b}_z$  and  $\hat{a}_x, \hat{a}_y, \hat{a}_z$  such that  $\hat{b}_z = \hat{a}_z$  and the angle between  $\hat{b}_x$  and  $\hat{a}_x$  is  $30^\circ$ ? **Yes/No.**

**Question:** Is  ${}^bR^a$  unique when  $\hat{b}_z = \hat{a}_z$  and  $\hat{b}_x \cdot \hat{a}_x = \frac{\sqrt{3}}{2}$ ? **Yes/No.**

If your answer is **No**, **draw** an alternative orientation for  $\hat{b}_x, \hat{b}_y, \hat{b}_z$ .

${}^bR^a$	$\hat{a}_x$	$\hat{a}_y$	$\hat{a}_z$
$\hat{b}_x$			
$\hat{b}_y$			
$\hat{b}_z$			

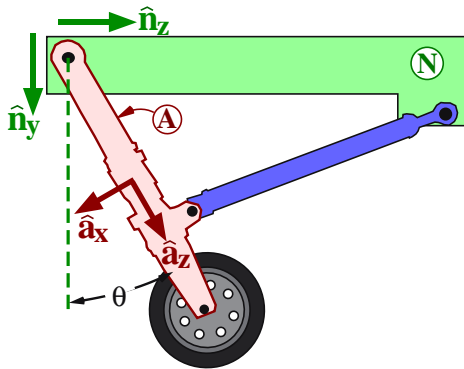


If "no", alternative

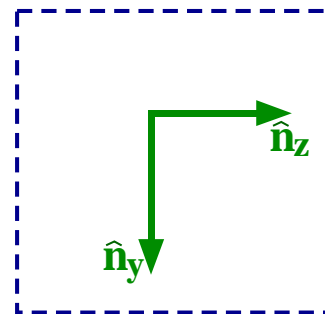


4.10 ♣ **SohCahToa: Rotation table with disorderly unit vectors.** (Section 5.5)

The following figure shows a landing gear system with a strut A that has a simple rotation relative to a fuselage N. Right-handed sets of orthogonal unit vectors  $\hat{a}_x, \hat{a}_y, \hat{a}_z$  and  $\hat{n}_x, \hat{n}_y, \hat{n}_z$  are fixed in A and N, respectively.  $\theta$  is the angle from  $\hat{n}_y$  to  $\hat{a}_z$  with  $+\hat{n}_x$  sense.



**Redraw**  $\hat{a}_x, \hat{a}_y, \hat{a}_z$  in a geometrically suggestive way for forming the  ${}^aR^n$  rotation matrix with sine and cosine.



Note: When  $\theta = 0$ ,  $\hat{a}_x \neq \hat{n}_x$  and  $\hat{a}_y \neq \hat{n}_y$  and  $\hat{a}_z \neq \hat{n}_z$ . Thus,  $\hat{a}_x, \hat{a}_y, \hat{a}_z$  are "**disordered**" with  $\hat{n}_x, \hat{n}_y, \hat{n}_z$ .

Complete the blanks in the equations relating  $\hat{a}_x, \hat{a}_y, \hat{a}_z$  to  $\hat{n}_x, \hat{n}_y, \hat{n}_z$  and in the  ${}^aR^n$  rotation table.

$$\hat{a}_x = \underline{\hspace{2cm}} \hat{n}_x + \underline{\hspace{2cm}} \hat{n}_y + \underline{\hspace{2cm}} \hat{n}_z$$

$$\hat{a}_y = \underline{\hspace{2cm}} \hat{n}_x + \underline{\hspace{2cm}} \hat{n}_y + \underline{\hspace{2cm}} \hat{n}_z$$

$$\hat{a}_z = \underline{\hspace{2cm}} \hat{n}_x + \underline{\hspace{2cm}} \hat{n}_y + \underline{\hspace{2cm}} \hat{n}_z$$

${}^aR^n$	$\hat{n}_x$	$\hat{n}_y$	$\hat{n}_z$
$\hat{a}_x$			
$\hat{a}_y$			
$\hat{a}_z$			