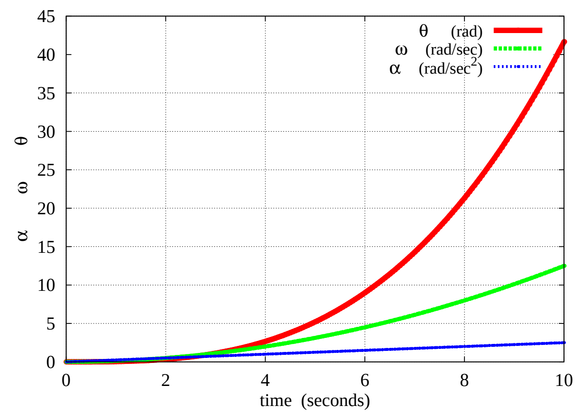
6.1 FE/EIT Review – Motion graph: $\mathrm{T} \Rightarrow \alpha \Rightarrow \omega \Rightarrow \theta$

A wind turbine generates electricity from time-dependent aerodynamic wind forces. The wind creates a torque modeled as $\mathrm{T} = 20 \, \frac{\mathrm{N\,m}}{\mathrm{sec}} * t$.

Measures of the wind turbine's angular acceleration α, angular velocity ω, and angle θ are related by

$$\mathrm{T} \underset{(\mathbf{2D})}{=} \mathrm{I}\,\alpha \qquad \alpha = \frac{d\omega}{dt} \qquad \omega \underset{(\mathbf{2D})}{=} \frac{d\theta}{dt}$$

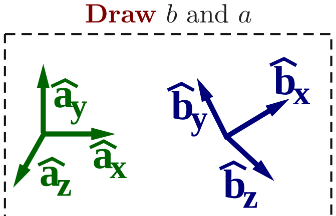
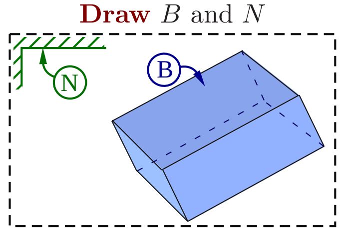
where $\mathrm{I} = 80 \,\mathrm{kg\,m^2}$ is the relevant moment of inertia. Graph α in $\frac{\mathrm{rad}}{\mathrm{sec}^2}$, ω in $\frac{\mathrm{rad}}{\mathrm{sec}}$, and θ in rad for $0 \le t \le 10$ sec. Use initial values (i.e, values at $t = 0$) of $\omega = 0$ and $\theta = 0$.



6.2 Drawing a reference frame and unit vector bases. (Section 7.2)

Draw a reference frame or rigid body B, shaped like a uniform-density doughnut (having a hole).
Draw a right-handed orthogonal bases fixed in B having unit vectors $\widehat{\mathbf{b}}_{\mathrm{x}}$, $\widehat{\mathbf{b}}_{\mathrm{y}}$, $\widehat{\mathbf{b}}_{\mathrm{z}}$.
Draw a different right-handed orthogonal bases fixed in B with unit vectors $\widehat{\mathbf{b}}_1$, $\widehat{\mathbf{b}}_2$, $\widehat{\mathbf{b}}_3$.
Draw a properly located center of mass symbol ◑ and label this point as B_{cm}.
Draw a point B_{o} fixed on B, at a location different than B_{cm}.

6.3 ♣ Words and pictures for ${}^{b}R^{a}$, ${}^{N}\vec{\boldsymbol{\omega}}^{B}$, ${}^{N}\vec{\boldsymbol{\alpha}}^{B}$. (Chapters 5 and 7)

${}^{b}R^{a}$ – Description (**words**)	${}^{N}\vec{\boldsymbol{\omega}}^{B}$ – Description (**words**)	${}^{N}\vec{\boldsymbol{\alpha}}^{B}$ – Description (**words**)
Rotation matrix relating right-handed orthogonal unit vectors $\widehat{\mathbf{b}}_{\mathrm{x}}$, $\widehat{\mathbf{b}}_{\mathrm{y}}$, $\widehat{\mathbf{b}}_{\mathrm{z}}$ to $\widehat{\mathbf{a}}_{\mathrm{x}}$, $\widehat{\mathbf{a}}_{\mathrm{y}}$, $\widehat{\mathbf{a}}_{\mathrm{z}}$	Reference frame B's angular velocity in reference frame N	Reference frame B's angular acceleration in reference frame N
Draw b and a 	**Draw** B and N 	

6.4 ♣ Definitions of angular velocity. (Section 7.3.3).

The definition of angular velocity of $\vec{\boldsymbol{\omega}} \triangleq \dot{\theta}\,\vec{\mathbf{k}}$ is a functional operational definition, i.e., in general, it is useful for calculating angular velocity and proving its properties (2D or 3D). **True**/**False**

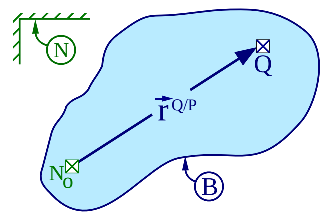
6.5 ♣ Concept: What objects have a unique angular velocity/acceleration? (Sections 7.3, 7.4).

${}^{N}\vec{\boldsymbol{\omega}}^{S}$, the angular velocity of an object S in a reference frame N is to be determined.
In general and **without ambiguity**, S could be a (circle ***all*** appropriate objects):

Real number	Point	**Reference Frame**	Mass center of a set of particles
Vector	Set of Points	**Rigid Body**	Mass center of a rigid body
Matrix	Particle	Flexible Body	Set of flexible bodies
3D Orthogonal unit basis	Set of Particles	Set of Rigid bodies	System of particles and bodies

Repeat for ${}^{N}\vec{\boldsymbol{\alpha}}^{S}$, the angular acceleration of an object S in a reference frame N box appropriate objects.

6.6 ♣ Vector differentiation concepts "$v = \omega\, r$". (Section 7.3).



Point Q is fixed on a rigid body B. Point N_o is fixed in a reference frame N and does not move on B. Complete the following proof that shows how $\vec{\mathbf{v}}$ (Q's velocity in N) can be written in terms of ${}^{N}\vec{\boldsymbol{\omega}}^{B}$ (B's angular velocity in N) and $\vec{\mathbf{r}}$ (Q's position vector from N_o).

Mathematical statement	Reasoning (explain each step in the proof with a brief phrase)
$\vec{\mathbf{v}} \triangleq \frac{{}^{N}d\vec{\mathbf{r}}}{dt}$	Definition of Q's velocity in N
$= \frac{{}^{B}d\vec{\mathbf{r}}}{dt} + {}^{N}\vec{\boldsymbol{\omega}}^{B} \times \vec{\mathbf{r}}$	Golden rule for vector differentiation
$= \vec{\mathbf{0}} + {}^{N}\vec{\boldsymbol{\omega}}^{B} \times \vec{\mathbf{r}}$	$\vec{\mathbf{r}}$ does not change in magnitude or direction in B

6.7 ♣ Concepts: What objects have a uniquely-defined angular velocity? (Section 7.3).

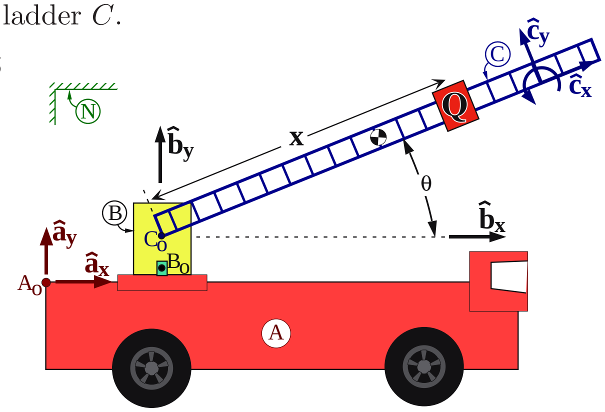
#	For: ${}^{A}\vec{\boldsymbol{\omega}}^{B}$ (B's angular velocity in A)	Object B		Object A	
a.	It is possible to find the angular velocity of a	point	in a	reference frame.	True/**False**
b.	It is possible to find the angular velocity of a	rigid body	in a	particle.	True/**False**
c.	It is possible to find the angular velocity of a	rigid body	in a	reference frame.	**True**/False
d.	It is possible to find the angular velocity of a	reference frame	in a	rigid body.	**True**/False
e.	It is possible to find the angular velocity of a	reference frame	in a	flexible body.	True/**False**
f.	It is possible to find the angular velocity of a	flexible body	in a	reference frame.	True/**False**

6.8 ♣ Rotational kinematics of a fire ladder. (Sections 7.3.3, 7.3.5, 7.3.6).

The following figure shows a fire truck chassis A traveling at constant speed in straight-line motion on Earth (A does not rotate relative to Earth). Earth is a ***Newtonian reference frame*** N.
A rigid hub B is connected to fire truck A by a revolute motor at point B_o of B.
A rigid ladder C is connected to hub B by a revolute motor at point C_o of C.
A fire-fighter Q (modeled as a particle of mass m) climbs ladder C.

Right-handed orthogonal unit vectors $\hat{\mathbf{a}}_x$, $\hat{\mathbf{a}}_y$, $\hat{\mathbf{a}}_z$; $\hat{\mathbf{b}}_x$, $\hat{\mathbf{b}}_y$, $\hat{\mathbf{b}}_z$; $\hat{\mathbf{c}}_x$, $\hat{\mathbf{c}}_y$, $\hat{\mathbf{c}}_z$; are fixed in A, B, C, with:

- $\hat{\mathbf{a}}_x$ pointing forward on the fire truck
- $\hat{\mathbf{a}}_y$ vertically-upward and from B_o to C_o
- $\hat{\mathbf{b}}_y = \hat{\mathbf{a}}_y$ parallel to the axis of the revolute motor that connects B and A
- $\hat{\mathbf{b}}_z = \hat{\mathbf{c}}_z$ parallel to the axis of the revolute motor that connects B and C
- $\hat{\mathbf{c}}_x$ directed from C_o to Q (along C's long axis)



Note: **Visualize** C's "**Body yz**" (or "**Space zy**") rotation sequence in N (e.g., with a ruler).

Quantity	Symbol	Type
$\hat{\mathbf{b}}_y$ measure of B's angular velocity in A	ω_B	Constant
Angle from $\hat{\mathbf{b}}_x$ to $\hat{\mathbf{c}}_x$ with $^+\hat{\mathbf{c}}_z$ sense	θ	Variable

${}^cR^b$	$\hat{\mathbf{b}}_x$	$\hat{\mathbf{b}}_y$	$\hat{\mathbf{b}}_z$
$\hat{\mathbf{c}}_x$	$\cos(\theta)$	$\sin(\theta)$	0
$\hat{\mathbf{c}}_y$	$-\sin(\theta)$	$\cos(\theta)$	0
$\hat{\mathbf{c}}_z$	0	0	1

(a) Complete the previous ${}^cR^b$ rotation table (to the right).
Note: ${}^cR^b$ is unnecessary for the remainder of this problem.

(b) Clarify the process to determine ${}^B\vec{\boldsymbol{\omega}}^C$, then express it in terms of $\hat{\mathbf{b}}_x$, $\hat{\mathbf{b}}_y$, $\hat{\mathbf{b}}_z$. **(Section 7.3.3)**.

- C's angular velocity in B is **simple** since $\hat{\mathbf{c}}_z = \hat{\mathbf{b}}_z$ is fixed in **both** C and B.
- $\dot{\theta}$ is the time-derivative of the angle between $\hat{\mathbf{b}}_x$ and $\hat{\mathbf{c}}_x$.
- The sign ($\pm$) was determined using the right-hand rule (sweep from $\hat{\mathbf{b}}_x$ to $\hat{\mathbf{c}}_x$).
- ${}^B\vec{\boldsymbol{\omega}}^C = \dot{\theta}\,\hat{\mathbf{b}}_z$

(c) B's angular velocity in A is known to be a **simple** angular velocity of ${}^A\vec{\boldsymbol{\omega}}^B = \omega_B\,\hat{\mathbf{b}}_y$ because $\hat{\mathbf{b}}_y$ is a vector fixed in **both** B and A.

(d) Form C's angular velocity in N and express it in terms of $\hat{\mathbf{b}}_x$, $\hat{\mathbf{b}}_y$, $\hat{\mathbf{b}}_z$.
Result:

$$ {}^N\vec{\boldsymbol{\omega}}^C \underset{(7.4)}{=} {}^N\vec{\boldsymbol{\omega}}^A + {}^A\vec{\boldsymbol{\omega}}^B + {}^B\vec{\boldsymbol{\omega}}^C = \vec{0} + \omega_B\,\hat{\mathbf{b}}_y + \dot{\theta}\,\hat{\mathbf{b}}_z $$

(e) When both ω_B and $\dot{\theta}$ are ***constant***, ${}^N\vec{\boldsymbol{\alpha}}^C = \vec{0}$. **True/[False]**.

(f) Write the definition for C's angular acceleration in N and form ${}^N\vec{\boldsymbol{\alpha}}^C$. **(Sections 7.4, 7.3)**.
Result:

$$ {}^N\vec{\boldsymbol{\alpha}}^C \underset{(7.8)}{\triangleq} \frac{{}^N d\, {}^N\vec{\boldsymbol{\omega}}^C}{dt} \qquad {}^N\vec{\boldsymbol{\alpha}}^C \underset{(7.1)}{=} \omega_B\,\dot{\theta}\,\hat{\mathbf{b}}_x + \ddot{\theta}\,\hat{\mathbf{b}}_z $$

6.9 ♣ Theorems: Rotation matrices R, angular velocity $\vec{\boldsymbol{\omega}}$, angular acceleration $\vec{\boldsymbol{\alpha}}$? (Section 7.4).

Determine whether or not each theorem to the right is valid for general 3D motion of reference frames A, B, C, and D.

Theorem	True or false
${}^aR^d = {}^aR^b * {}^bR^c * {}^cR^d$	[True]/False
${}^A\vec{\boldsymbol{\omega}}^D = {}^A\vec{\boldsymbol{\omega}}^B + {}^B\vec{\boldsymbol{\omega}}^C + {}^C\vec{\boldsymbol{\omega}}^D$	[True]/False
${}^A\vec{\boldsymbol{\alpha}}^D = {}^A\vec{\boldsymbol{\alpha}}^B + {}^B\vec{\boldsymbol{\alpha}}^C + {}^C\vec{\boldsymbol{\alpha}}^D$	True/[False]

6.10 Alternate formula for angular acceleration. (Section 7.3).

Prove ${}^N\vec{\boldsymbol{\alpha}}^B \triangleq \dfrac{{}^N d\, {}^N\vec{\boldsymbol{\omega}}^B}{dt}$ can also be calculated as ${}^N\vec{\boldsymbol{\alpha}}^B = \dfrac{{}^B d\, {}^N\vec{\boldsymbol{\omega}}^B}{dt}$.

6.11 ♣ Concepts: Angular acceleration for general 3D motion. (Sections 7.3, 7.4).

Determine whether or not each of the following equations generally apply to the angular acceleration $\vec{\boldsymbol{\alpha}}$ of reference frames A, B, and C in general 3D motion.

${}^A\vec{\boldsymbol{\alpha}}^B = \dfrac{{}^A d\, {}^A\vec{\boldsymbol{\omega}}^B}{dt}$	[True]/False	${}^A\vec{\boldsymbol{\alpha}}^C = {}^A\vec{\boldsymbol{\alpha}}^B + {}^B\vec{\boldsymbol{\alpha}}^C + {}^A\vec{\boldsymbol{\omega}}^B \times {}^B\vec{\boldsymbol{\omega}}^C$	[True]/False
${}^A\vec{\boldsymbol{\alpha}}^B = \dfrac{{}^A d\, {}^B\vec{\boldsymbol{\omega}}^A}{dt}$	True/[False]	${}^A\vec{\boldsymbol{\alpha}}^B = -\dfrac{{}^A d\, {}^B\vec{\boldsymbol{\omega}}^A}{dt}$	[True]/False
${}^A\vec{\boldsymbol{\alpha}}^B = \dfrac{{}^C d\, {}^A\vec{\boldsymbol{\omega}}^B}{dt}$	True/[False]	${}^A\vec{\boldsymbol{\alpha}}^B = \dfrac{{}^C d\, {}^A\vec{\boldsymbol{\omega}}^B}{dt} + {}^A\vec{\boldsymbol{\omega}}^C \times {}^A\vec{\boldsymbol{\omega}}^B$	[True]/False
${}^A\vec{\boldsymbol{\alpha}}^B = \dfrac{{}^B d\, {}^A\vec{\boldsymbol{\omega}}^B}{dt}$	[True]/False	${}^A\vec{\boldsymbol{\alpha}}^B = \dfrac{{}^C d\, {}^A\vec{\boldsymbol{\omega}}^B}{dt} + {}^B\vec{\boldsymbol{\omega}}^C \times {}^A\vec{\boldsymbol{\omega}}^B$	[True]/False
${}^A\vec{\boldsymbol{\alpha}}^B = {}^B\vec{\boldsymbol{\alpha}}^A$	True/[False]	${}^A\vec{\boldsymbol{\alpha}}^B = -{}^B\vec{\boldsymbol{\alpha}}^A$	[True]/False