

Show work – except for ♣ fill-in-blanks.

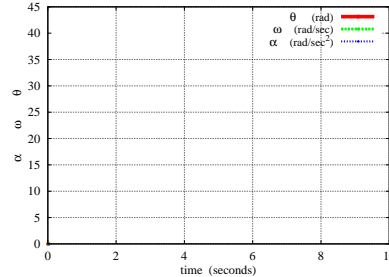
Angular velocity and angular acceleration

6.1 FE/EIT Review – Motion graph: $T \Rightarrow \alpha \Rightarrow \omega \Rightarrow \theta$

A wind turbine generates electricity from wind forces that create a time-dependent torque modeled as $T = 20 \frac{N \cdot m}{sec} * t$. Measures of the wind turbine’s angular acceleration α , angular velocity ω , and angle θ are related as shown below, where $I = 80 \text{ kg} \cdot \text{m}^2$ is the relevant moment of inertia.

$$T = I \alpha \quad \alpha = \frac{d\omega}{dt} \quad \omega = \frac{d\theta}{dt}$$

Graph α in $\frac{rad}{sec^2}$, ω in $\frac{rad}{sec}$, and θ in rad for $0 \leq t \leq 10 \text{ sec}$. Use initial values (i.e., values at $t = 0$) of $\omega = 0$ and $\theta = 0$.



6.2 Drawing a reference frame and unit vector bases. (Section 7.2)

- Draw** a reference frame or rigid body B , shaped like a uniform-density doughnut (having a hole).
- Draw** a right-handed orthogonal bases fixed in B having unit vectors $\hat{b}_x, \hat{b}_y, \hat{b}_z$.
- Draw** a different right-handed orthogonal bases fixed in B with unit vectors $\hat{b}_1, \hat{b}_2, \hat{b}_3$.
- Draw** a properly located center of mass symbol \ominus and label this point as B_{cm} .
- Draw** a point B_o fixed on B , at a location different than B_{cm} .

6.3 ♣ Words and pictures for ${}^bR^a$, ${}^{N\vec{\omega}}B$, ${}^{N\vec{\alpha}}B$. (Chapters 5 and 7)

${}^bR^a$ – Description (words) <div style="background-color: yellow; height: 20px; width: 80%; margin: 5px;"></div> <div style="background-color: yellow; height: 20px; width: 90%; margin: 5px;"></div> <div style="background-color: yellow; height: 20px; width: 60%; margin: 5px;"></div>	${}^{N\vec{\omega}}B$ – Description (words) <div style="background-color: yellow; height: 20px; width: 80%; margin: 5px;"></div> <div style="background-color: yellow; height: 20px; width: 70%; margin: 5px;"></div> <div style="background-color: yellow; height: 20px; width: 50%; margin: 5px;"></div>	${}^{N\vec{\alpha}}B$ – Description (words) <div style="background-color: yellow; height: 20px; width: 80%; margin: 5px;"></div> <div style="background-color: yellow; height: 20px; width: 90%; margin: 5px;"></div> <div style="background-color: yellow; height: 20px; width: 60%; margin: 5px;"></div>
Draw b and a <div style="border: 1px dashed black; width: 80%; height: 80%; margin: 10px auto;"></div>	Draw B and N <div style="border: 1px dashed black; width: 80%; height: 80%; margin: 10px auto;"></div>	

6.4 ♣ Definition of angular velocity? (Section 7.3.3).

The definition of angular velocity of $\vec{\omega} \triangleq \dot{\theta} \vec{k}$ is a functional operational definition, i.e., in general, it is useful for calculating angular velocity and proving its properties (2D or 3D). **True/False**

6.5 ♣ Optional: Textbook and Internet definitions of 3D angular velocity (Section 7.3).

Famed dynamicist Thomas Kane called angular velocity “**one of the most misunderstood concepts in kinematics.**” Report two definitions of angular velocity and determine if the quantities appearing in the definition are **rigorously defined** - and whether they are generally applicable for 3D kinematics or only apply for simple angular velocity (described in Section 7.3.3).

Note: A definition should be able to **prove** important theorems [such as the angular velocity addition theorem of equation (7.4) and the golden rule for vector differentiation in equation (7.1)] and allow for angular velocity **calculations**.

	Source	Definition	Rigorously defined	Works for 3D kinematics?
Textbook:	List textbook	Record equation/definition	Yes/No	Yes/No
Internet:	Provide .html	Record equation/definition	Yes/No	Yes/No

6.6 ♣ **Concept: What objects have a unique angular velocity/acceleration?** (Sections 7.3, 7.4).

${}^N\vec{\omega}^S$, the angular velocity of an object S in a reference frame N is to be determined.

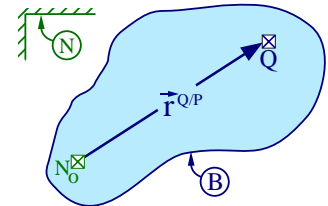
In general and **without ambiguity**, S could be a (circle all appropriate objects):

Real number	Point	Reference frame	Mass center of a set of particles
Vector	Set of points	3D rigid body	Mass center of a rigid body
Matrix	Particle	Flexible body	Set of flexible bodies
3D orthogonal unit basis	Set of particles	Set of rigid bodies	System of particles and bodies

Repeat for ${}^N\vec{\alpha}^S$, the angular acceleration of an object S in a reference frame N box appropriate objects.

6.7 ♣ **Vector differentiation concepts** “ $v = \omega r$ ”. (Section 7.3).

Point Q is fixed on a rigid body B . Point N_o is fixed in a reference frame N and does not move on B . Complete the following proof that shows how \vec{v} (Q 's velocity in N) can be written in terms of ${}^N\vec{\omega}^B$ (B 's angular velocity in N) and \vec{r} (Q 's position vector from N_o).



Mathematical statement	Reasoning (explain each step in the proof with a brief phrase)
$\vec{v} \triangleq \frac{{}^N d\vec{r}}{dt}$	Definition of Q 's velocity in N
$= \square + \square$	
$= \vec{0} + {}^N\vec{\omega}^B \times \vec{r}$	

6.8 ♣ **Concepts: What objects have a uniquely-defined angular velocity?** (Section 7.3).

a.	It is possible to find the angular velocity of a point	in a reference frame.	True/False
b.	It is possible to find the angular velocity of a 3D rigid body	in a particle.	True/False
c.	It is possible to find the angular velocity of a 3D rigid body	in a reference frame.	True/False
d.	It is possible to find the angular velocity of a reference frame	in a 3D rigid body.	True/False
d.	It is possible to find the angular velocity of a 3D rigid basis	in a reference frame.	True/False
e.	It is possible to find the angular velocity of a reference frame	in a flexible body.	True/False
f.	It is possible to find the angular velocity of a flexible body	in a reference frame.	True/False

6.9 ♣ **Rotational kinematics of a fire ladder.** (Sections 7.3.3, 7.3.5, 7.3.6).

The following figure shows a fire truck chassis A traveling at constant speed in straight-line motion on Earth (A does not rotate relative to Earth). Earth is a **Newtonian reference frame** N .

A rigid hub B is connected to fire truck A by a revolute motor at point B_o of B .

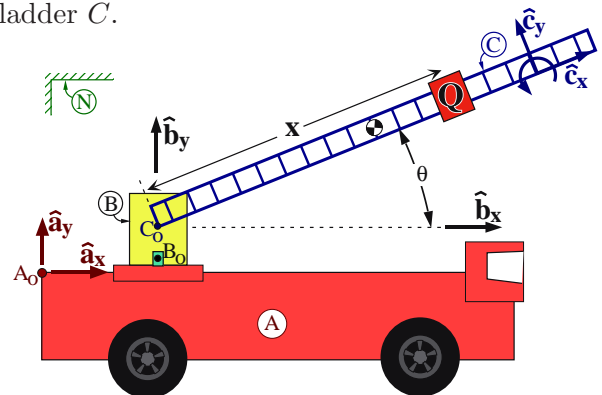
A rigid ladder C is connected to hub B by a revolute motor at point C_o of C .

A fire-fighter Q (modeled as a particle of mass m) climbs ladder C .

Right-handed orthogonal unit vectors $\hat{a}_x, \hat{a}_y, \hat{a}_z$;

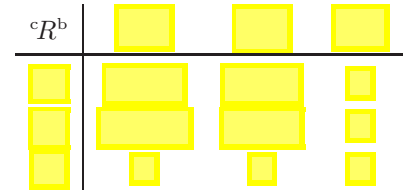
$\hat{b}_x, \hat{b}_y, \hat{b}_z$; $\hat{c}_x, \hat{c}_y, \hat{c}_z$; are fixed in A, B, C , with:

- \hat{a}_x pointing forward on the fire truck
- \hat{a}_y vertically-upward and from B_o to C_o
- $\hat{b}_y = \hat{a}_y$ parallel to the axis of the revolute motor that connects B and A
- $\hat{b}_z = \hat{c}_z$ parallel to the axis of the revolute motor that connects B and C
- \hat{c}_x directed from C_o to Q (along C 's long axis)



Note: **Visualize** C 's “**Body yz**” (or “**Space zy**”) rotation sequence in N (e.g., with a ruler).

Quantity	Symbol	Type
$\hat{\mathbf{b}}_y$ measure of B 's angular velocity in A	ω_B	Constant
Angle from $\hat{\mathbf{b}}_x$ to $\hat{\mathbf{c}}_x$ with $+\hat{\mathbf{c}}_z$ sense	θ	Variable



- (a) Complete the previous ${}^cR^b$ rotation table (to the right).
Note: ${}^cR^b$ is unnecessary for the remainder of this problem.
- (b) Clarify the process to determine C 's angular velocity in B . (Section 7.3.3).
- ${}^B\vec{\omega}^C$ is a **simple** angular velocity because $\hat{\mathbf{b}}_z$ is a vector fixed in **both** \square and \square .
 - ${}^B\vec{\omega}^C = \square$, where \square is the angle between \square and \square .
 - The sign (\pm) is determined using the \square -hand rule (sweep from \square to \square).
- (c) ${}^A\vec{\omega}^B = \omega_B \hat{\mathbf{b}}_y$ is a **simple** angular velocity because $\hat{\mathbf{b}}_y$ is a vector fixed in **both** \square and \square .
- (d) Form C 's angular velocity in N and express it in terms of $\hat{\mathbf{b}}_x, \hat{\mathbf{b}}_y, \hat{\mathbf{b}}_z$.

Result:
$${}^N\vec{\omega}^C \stackrel{(7.4)}{=} \square \vec{\omega}^{\square} + \square \vec{\omega}^{\square} + \square \vec{\omega}^{\square} = \vec{0} + \square \hat{\mathbf{b}}_y + \square \hat{\mathbf{b}}_z$$

- (e) When both ω_B and $\dot{\theta}$ are **constant**, ${}^N\vec{\alpha}^C = \vec{0}$. **True/False**.
- (f) Write the definition for C 's angular acceleration in N and form ${}^N\vec{\alpha}^C$. (Sections 7.4, 7.3).

Result:
$${}^N\vec{\alpha}^C \stackrel{(7.5)}{\triangleq} \square \quad \quad \quad {}^N\vec{\alpha}^C \stackrel{(7.1)}{=} \omega_B \dot{\theta} \hat{\mathbf{b}}_x + \ddot{\theta} \hat{\mathbf{b}}_z$$

6.10 ♣ Theorems: Rotation matrices R , angular velocity $\vec{\omega}$, angular acceleration $\vec{\alpha}$? (Section 7.4).

Determine whether or not each theorem to the right is valid for general 3D motion of reference frames A, B, C , and D .

Theorem	True or false
${}^aR^d = {}^aR^b * {}^bR^c * {}^cR^d$	True/False
${}^A\vec{\omega}^D = {}^A\vec{\omega}^B + {}^B\vec{\omega}^C + {}^C\vec{\omega}^D$	True/False
${}^A\vec{\alpha}^D = {}^A\vec{\alpha}^B + {}^B\vec{\alpha}^C + {}^C\vec{\alpha}^D$	True/False

6.11 Alternate formula for angular acceleration. (Section 7.3).

Prove ${}^N\vec{\alpha}^B \triangleq \frac{N_d {}^N\vec{\omega}^B}{dt}$ can also be calculated as ${}^N\vec{\alpha}^B = \frac{B_d {}^N\vec{\omega}^B}{dt}$.

6.12 Optional: Angular acceleration addition theorem. (Sections 7.3, 7.3.5, 7.4).

Use the angular velocity addition theorem and the definition of angular acceleration to prove:

$${}^N\vec{\alpha}^B = {}^N\vec{\alpha}^A + {}^A\vec{\alpha}^B + {}^N\vec{\omega}^A \times {}^A\vec{\omega}^B$$

6.13 †♣ Optional: Concepts: Angular acceleration for general 3D motion. (Sections 7.3, 7.4).

Determine whether or not each of the following equations generally apply to the angular acceleration $\vec{\alpha}$ of reference frames A, B , and C in general 3D motion.

${}^A\vec{\alpha}^B = \frac{A_d {}^A\vec{\omega}^B}{dt}$ True/False	${}^A\vec{\alpha}^C = {}^A\vec{\alpha}^B + {}^B\vec{\alpha}^C + {}^A\vec{\omega}^B \times {}^B\vec{\omega}^C$ True/False
${}^A\vec{\alpha}^B = \frac{A_d {}^B\vec{\omega}^A}{dt}$ True/False	${}^A\vec{\alpha}^B = -\frac{A_d {}^B\vec{\omega}^A}{dt}$ True/False
${}^A\vec{\alpha}^B = \frac{C_d {}^A\vec{\omega}^B}{dt}$ True/False	${}^A\vec{\alpha}^B = \frac{C_d {}^A\vec{\omega}^B}{dt} + {}^A\vec{\omega}^C \times {}^A\vec{\omega}^B$ True/False
${}^A\vec{\alpha}^B = \frac{B_d {}^A\vec{\omega}^B}{dt}$ True/False	${}^A\vec{\alpha}^B = \frac{C_d {}^A\vec{\omega}^B}{dt} + {}^B\vec{\omega}^C \times {}^A\vec{\omega}^B$ True/False
${}^A\vec{\alpha}^B = {}^B\vec{\alpha}^A$ True/False	${}^A\vec{\alpha}^B = -{}^B\vec{\alpha}^A$ True/False