5.1 ♣ Notations for derivatives. (Section 1.7.1).

Date	Person	Symbols for 1^{st}, 2^{nd}, and 3^{rd} derivatives		
1675	Leibniz	$\frac{dy}{dt}$	$\frac{d^2y}{dt^2}$	$\frac{d^3y}{dt^3}$
1675	Newton	$\dot{y}$	$\ddot{y}$	$\dddot{y}$
1797	Lagrange (trained by Euler)	y'	y''	y'''
1850	Cauchy/Weierstrauss	$\lim\limits_{h \to 0} \frac{y(t+h) - y(t)}{h}$	?	?
1786	Legendre (introduced partials then abandoned)	$\frac{\partial y}{\partial x}$	$\frac{\partial^2 y}{\partial x^2}$	$\frac{\partial^3 y}{\partial x^3}$
1841	Jacobi (re-introduced partials again)			

There was bitter rivalry between Newton and Leibniz, and the notations of Leibniz and Newton are not entangled.
For example, $\frac{d\dot{y}}{dt}$ is written in Leibniz's notation as $\frac{d^2y}{dt^2}$ or Newton's as $\ddot{y}$.

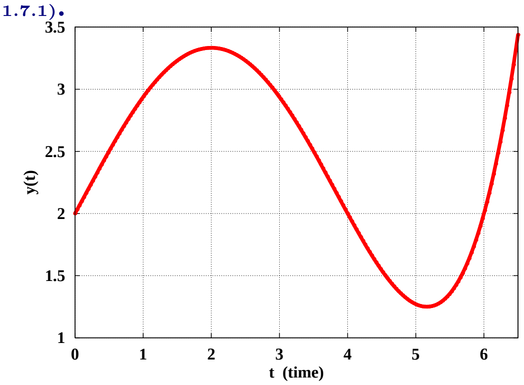
5.2 ♣ Leibniz's shorthand notation for 3^{rd} derivatives. (Section 1.7.1).

Write the explicit expression for the following 3^{rd} derivative (so it contains three 1^{st} derivatives).

Result:

$$\frac{d^3y}{dt^3} \triangleq \frac{d}{dt}\left[\frac{d}{dt}\left(\frac{dy}{dt}\right)\right]$$

5.3 ♣ Geometric interpretation of a derivative. (Section 1.7.1).



Estimate the 1^{st}-derivative of the function $y(t)$ shown to the right at $t = 0,\ 2,\ 4,\ 6$.

Pick your answers from: **-1**, **0**, **1**, **2**.

Result:
$\left.\frac{dy}{dt}\right|_{t=0} = 1$ $\left.\frac{dy}{dt}\right|_{t=2} = 0$
$\left.\frac{dy}{dt}\right|_{t=4} = -1$ $\left.\frac{dy}{dt}\right|_{t=6} = 2$

Estimate the **sign** of the 2^{nd}-derivative of $y(t)$ from the answers $-$, **0**, or $+$.
Answer **0** when the absolute value of the 2^{nd}-derivative is estimated to be less than 0.5.

Result: $\left.\frac{d^2y}{dt^2}\right|_{t=0}$ is **0** $\left.\frac{d^2y}{dt^2}\right|_{t=2}$ is **−** $\left.\frac{d^2y}{dt^2}\right|_{t=4}$ is **0** $\left.\frac{d^2y}{dt^2}\right|_{t=6}$ is **+**

5.4 ♣ Derivatives of commonly-encountered functions. (Section 1.7.5).

Differentiate the following functions that depend on t (time). Ensure answers involving x are valid when x is either constant or depends on time, e.g., when $x = t^3$.

Result:
$\frac{d}{dt}\,t^2 = 2\,t$ $\frac{d}{dt}\,t^3 = 3\,t^2$ $\frac{d}{dt}\,t^{47} = 47\,t^{46}$
$\frac{d}{dt}\sin(t) = \cos(t)$ $\frac{d}{dt}\cos(t) = -\sin(t)$ $\frac{d}{dt}\cos(x) = -\sin(x)\,\frac{dx}{dt}$
$\frac{d}{dt}\,e^t = e^t$ $\frac{d}{dt}\ln(t) = \frac{1}{t}$ $\frac{d}{dt}\ln(x) = \frac{1}{x}\,\frac{dx}{dt}$

5.5 ♣ Good product rule for differentiation (for scalars, $\vec{v}$ectors, matrices, ...). (Section 1.7.7).

The ***good product rule for differentiation*** that works when u and v are scalars, $\vec{v}$ectors, or matrices is (circle the correct answer – and update your Calculus teacher):

$$\boxed{\frac{d(u*v)}{dt} = \frac{du}{dt}*\mathrm{v} + \mathrm{u}*\frac{dv}{dt}} \qquad \frac{d(u*v)}{dt} = u*\frac{dv}{dt} + v*\frac{du}{dt} \qquad \frac{d(u*v)}{dt} = v*\frac{du}{dt} + u*\frac{dv}{dt}$$

Knowing u, v, w are scalars or **matrices** that depend on time t, use the ***good product rule for differentiation*** to form the 1^{st} ordinary time-derivative of $y(t) = u*v*w$.

Good product rule: $\frac{dy}{dt} = \frac{d(u*v*w)}{dt} = \frac{du}{dt}*v*w + u*\frac{dv}{dt}*w + u*v*\frac{dw}{dt}$

5.6 Derivative quotient rule? No, just use product rule and exponent. (Section 1.7.8).

Although the "***quotient rule***" can be used to calculate the derivative with respect to t of the ratio of two functions $\frac{f(t)}{g(t)}$, it can be easier to rewrite the ratio as $f(t) * g(t)^{-1}$ then use the ***product rule***. Use this idea to first rewrite the following ratio of two functions as a product and then use the ***product rule*** to calculate its derivative.

Result: $\frac{\ln(t)}{t^2} = \ln(t) * t^{-2}$ $\qquad \frac{d}{dt}[\ln(t)/t^2] = t^{-3} + -2\ln(t)\,t^{-3}$

5.7 ♣ Example of the "good product rule" for differentiation. (Takes less than 2 minutes).

The "good" product rule is easy-to-use for ***very quickly*** differentiating complex expressions. Knowing x and y are variables that depend on the independent variable t (time), determine the ordinary time-derivative of the function f when[1]

$$f(t) = \sin(t) * \cos(x+y) * (\dot{x})^2 * e^t * \ln(y) / x$$

Result:

$$\begin{aligned}\frac{df}{dt} &= \cos(t) * \cos(x+y) * (\dot{x})^2 * e^t * \ln(y)/x \\ &- \sin(t) * \sin(x+y) * (\dot{x}+\dot{y}) * (\dot{x})^2 * e^t * \ln(y)/x \\ &+ \sin(t) * \cos(x+y) * 2*\dot{x}*\ddot{x} * e^t * \ln(y)/x \\ &+ \sin(t) * \cos(x+y) * (\dot{x})^2 * e^t * \ln(y)/x \\ &+ \sin(t) * \cos(x+y) * (\dot{x})^2 * e^t * \tfrac{1}{y} * \dot{y}/x \\ &- \sin(t) * \cos(x+y) * (\dot{x})^3 * e^t * \ln(y)/x^2\end{aligned}$$

5.8 Differentiation concepts. (Section 1.7.10).

Shown right is an equation relating the dependent variable y to the independent variable t.

$$y^4 - 8\,y = 3\,t^2 + \sin(t)$$

Find a general expression for the ordinary derivative $\frac{dy}{dt}$ in terms of t and y.

Find a **numerical** value for $\frac{dy}{dt}$ at $t=0$ when y is **positive**.

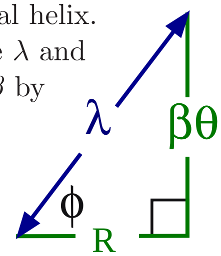
Hint: The value of y is not arbitrary. If you encounter difficulty, consider ***implicit differentiation***..

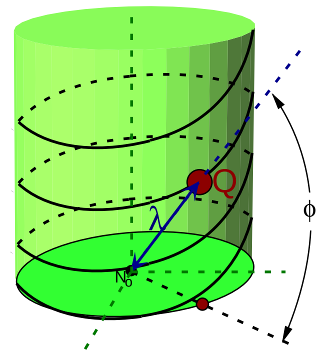
Result: $\frac{dy}{dt} = \frac{6\,t + \cos(t)}{4\,y^3 - 8}$ $\qquad \left.\frac{dy}{dt}\right|_{t=0} = \frac{1}{24}$

[1]Symbols for the 1^{st} and 2^{nd} ordinary time-derivatives of x include $\frac{dx}{dt}$ and $\frac{d^2x}{dt^2}$ (introduced by ***Leibniz***), $\dot{x}$ and $\ddot{x}$ (introduced by ***Newton***), and `x'` and `x''` (introduced by ***Lagrange*** and used by **M**otion**G**enesis).

5.9 Review of explicit and implicit differentiation. (Section 1.7.10).

The figure to the right shows a point Q on a cylindrical helix. Two geometrically significant quantities are a distance λ and an angle ϕ that are related to two **constants** R and β by





$$\lambda^2 \;=\; R^2 \,+\, (\beta\,\theta)^2 \qquad\qquad \tan(\phi) \;=\; \frac{\beta\,\theta}{R}$$

Determine $\dot{\lambda}$ and $\dot{\phi}$ (time-derivatives of λ and ϕ) in terms of θ, $\dot{\theta}$, R, β, etc., using the two methods described below. Note: $\dfrac{\partial\,\text{atan}(x)}{\partial x} = \dfrac{1}{1+x^2}$

(a) ***Explicit differentiation***
Solve explicitly for λ and ϕ and then differentiate the resulting expression.
Result: $\lambda = \sqrt{R^2 + (\beta\,\theta)^2}$ $\qquad \phi = \text{atan}(\frac{\beta\,\theta}{R})$

$\dot{\lambda} = \boxed{\dfrac{\beta^2\,\theta}{\sqrt{R^2+(\beta\,\theta)^2}}\,\dot{\theta}}$ $\qquad \dot{\phi} = \boxed{\dfrac{\beta/R}{1+(\frac{\beta\,\theta}{R})^2}\,\dot{\theta}}$

(b) ***Implicit differentiation***
Differentiate the equations involving λ^2 and $\tan(\phi)$ and then solve for $\dot{\lambda}$ and $\dot{\phi}$.
Result: $\dot{\lambda} = \boxed{\frac{\beta^2\,\theta}{\lambda}\,\dot{\theta}}$ $\qquad \dot{\phi} = \boxed{\cos^2(\phi)\,\frac{\beta}{R}\,\dot{\theta}} = \frac{\beta\,R}{\lambda^2}\,\dot{\theta}$

(c) **Explicit/Implicit** differentiation of λ is easier and computationally more efficient.

5.10 ♣ Review of partial and ordinary differentiation. (Section 1.7.2).

The mathematical process below is used in *Lagrange's equations of motion*.

The kinetic energy K of a bridge-crane (shown right) can be written in terms of constants M, m, L and variables x, $\dot{x}$, θ, $\dot{\theta}$, as

$$K \;=\; \tfrac{1}{2}\,M\,\dot{x}^2 \;+\; \tfrac{1}{2}\,m\,[L^2\,\dot{\theta}^2 + 2\,L\,\cos(\theta)\,\dot{x}\,\dot{\theta}]$$

- First, regard x, $\dot{x}$, θ, $\dot{\theta}$ as independent variables [so K depends separately on each, i.e., $K(x,\,\dot{x},\,\theta,\,\dot{\theta})$], form the **partial derivatives** below (left).
- Next, regard x, $\dot{x}$, θ, $\dot{\theta}$ as time-dependent variables and form the **ordinary derivatives** below (right).

$\dfrac{\partial K}{\partial \theta} = \boxed{-m\,L\,\dot{x}\,\sin(\theta)\,\dot{\theta}}$ $\quad \dfrac{\partial K}{\partial \dot{\theta}} = \boxed{m\,L\,[\dot{x}\,\cos(\theta) + L\,\dot{\theta}]}$ $\quad \dfrac{d}{dt}\left(\dfrac{\partial K}{\partial \dot{\theta}}\right) = \boxed{m\,L\,[\ddot{x}\,\cos(\theta) - \dot{x}\,\dot{\theta}\,\sin(\theta) + L\,\ddot{\theta}]}$

$\dfrac{\partial K}{\partial x} = \boxed{0}$ $\quad \dfrac{\partial K}{\partial \dot{x}} = \boxed{M\,\dot{x} \,+\, m\,L\,\cos(\theta)\,\dot{\theta}}$ $\quad \dfrac{d}{dt}\left(\dfrac{\partial K}{\partial \dot{x}}\right) = \boxed{M\,\ddot{x} \,+\, m\,L\,[\sin(\theta)\,\dot{\theta}^2 \,+\, \cos(\theta)\,\ddot{\theta}]}$

5.11 Differentiation concepts: What is dt? (Section 1.7.1).

A continuous function $z(t)$ depends on $x(t)$, $y(t)$, and time t as $\qquad z \;=\; x \;+\; y^2\,\sin(t)$

At a certain instant of time, $y = 1$ and z simplifies to $\qquad z \;=\; x \;+\; \sin(t)$

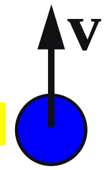
Find the time-derivative of z at the instant when $y = 1$.

Result: $\left.\dfrac{dz}{dt}\right|_{y\,=\,1} \;=\; \boxed{\dot{x} \;+\; 2\,\dot{y}\,\sin(t) \;+\; \cos(t)}$

5.12 ♣ Differentiation concepts – what is wrong? **(Section 1.7.1 and previous problem).**

The scalar v measures a baseball's upward-velocity. Knowing $v = 0$ only when the ball reaches maximum height, explain what is wrong with the following statement about v's time derivative.

$\frac{dv}{dt} = \frac{d\,(0)}{dt} = 0$ is **wrong**. You know the correct answer is: $\frac{dv}{dt} = g \approx 9.8\,\frac{\text{m}}{\text{s}^2}$.



Explain what is wrong: It is incorrect to time-differentiate as shown above because:
$v = 0$ is an ***instantaneous*** value of v. Differentiation must occur over a ***interval*** of time.
dt is defined as a ***non-zero interval*** of t (not at an ***instant***).

5.13 ♣ Integrals of commonly-encountered functions. **(Section 1.8).**

Calculate the following indefinite integrals in terms of an indefinite constant C (regard t as positive).

Result:

$\int t^2\,dt = \frac{t^3}{3} + C$ $\quad \int t^3\,dt = \frac{t^4}{4} + C$ $\quad \int t^8\,dt = \frac{t^9}{9} + C$

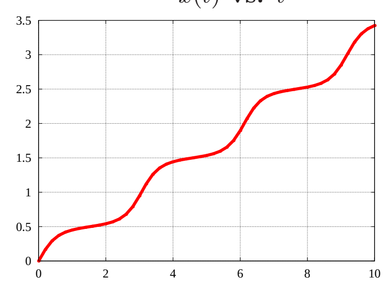
$\int t^{-3}\,dt = \frac{t^{-2}}{-2} + C$ $\quad \int t^{-2}\,dt = \frac{t^{-1}}{-1} + C$ $\quad \int t^{-1}\,dt = \ln(t) + C$

$\int \sin(t)\,dt = -\cos(t) + C$ $\quad \int \cos(t)\,dt = \sin(t) + C$ $\quad \int e^t\,dt = e^t + C$

$\int 5\,dt = 5\,t + C$ $\quad \int 5/t\,dt = 5\,\ln(t) + C$ $\quad \int (5 + \frac{1}{t})\,dt = 5\,t + \ln(t) + C$

5.14 Optional: † Continuous numerical solution of a nonlinear ODE.

$x(t)$ **vs.** t



Plot the continuous solution $x(t)$ to the following ordinary differential equation for $0 \le t \le 10$ with data every 0.2 sec. Use an initial value $x(0) = 0$ and use the initial value of $\dot{x}$ that is closest to 1.

$$\sin(\dot{x}) \;+\; 4\,\dot{x}^2 \;-\; 1.9\,\cos(2\,\pi\,x) \;-\; 2 \;=\; 0$$

Hint: A "clever" way to solve this **nonlinear** ODE for $x(t)$ is

- Use the given equation and initial value $x(0) = 0$ to solve for $\dot{x}$ at $t\!=\!0$.
  For example, the technique in Section 1.11 finds $\dot{x}(t\!=\!0) \approx 0.8841161$ when $x(t\!=\!0) = 0$.
- Time-differentiate the 1^{st}-order ODE that is **nonlinear** in $\dot{x}$ to form a 2^{nd}-order ODE that is **linear** in $\ddot{x}$.
  Then, solve the 2^{nd}-order ODE for $\ddot{x}$.

$$\cos(\dot{x})\,\ddot{x} \;+\; 8\,\dot{x}\,\ddot{x} \;+\; 3.8\,\pi\,\sin(2\,\pi\,x)\,\dot{x} \;=\; 0 \qquad \Rightarrow \qquad \ddot{x} \;=\; \frac{-3.8\,\pi\,\sin(2\,\pi\,x)\,\dot{x}}{\cos(\dot{x}) \;+\; 8\,\dot{x}}$$

- Numerically integrate the 2^{nd}-order ODE with the initial values of $x(0)$ and $\dot{x}(0)$
