

5.1 ♣ Notations for derivatives. (Section 1.6.1).

Date	Person	Symbols for 1 <sup>st</sup> , 2 <sup>nd</sup> , and 3 <sup>rd</sup> derivatives		
1675	<input type="text"/>	$\frac{dy}{dt}$	$\frac{d^2y}{dt^2}$	$\frac{d^3y}{dt^3}$
1675	<input type="text"/>	$\dot{y}$	$\ddot{y}$	$\dddot{y}$
1797	<input type="text"/> (trained by Euler)	$y'$	$y''$	$y'''$
1850	Cauchy (trained by Lagrange)	$\lim_{h \rightarrow 0} \frac{y(t+h) - y(t)}{h}$	?	?
1786	Legendre (introduced partials then abandoned)	$\frac{\partial y}{\partial x}$	$\frac{\partial^2 y}{\partial x^2}$	$\frac{\partial^3 y}{\partial x^3}$
1841	Jacobi (re-introduced partials again)			

There was bitter rivalry between Newton and Leibniz, and the notations of Leibniz and Newton are not entangled.

For example,  $\frac{dy}{dt}$  is written in Leibniz's notation as  or Newton's as .

5.2 ♣ Leibniz's shorthand notation for 3<sup>rd</sup> derivatives. (Section 1.6.1).

Write the explicit expression for the following 3<sup>rd</sup> derivative (so it contains three 1<sup>st</sup> derivatives).

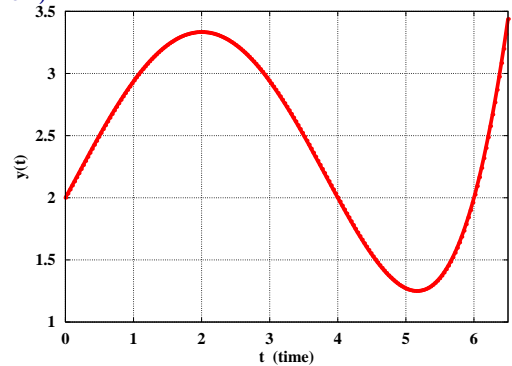
Result:  $\frac{d^3y}{dt^3} \triangleq \frac{d}{dt} \left( \left( \left( \right) \right) \right)$

5.3 ♣ Geometric interpretation of a derivative. (Section 1.6.1).

Estimate the 1<sup>st</sup>-derivative of the function  $y(t)$  shown to the right at  $t = 0, 2, 4, 6$ .

Pick your answers from: **-1, 0, 1, 2**.

Result:  $\frac{dy}{dt} \Big|_{t=0} = \text{$        $\frac{dy}{dt} \Big|_{t=2} = \text{$   
 $\frac{dy}{dt} \Big|_{t=4} = \text{$        $\frac{dy}{dt} \Big|_{t=6} = \text{$



Estimate the sign of the 2<sup>nd</sup>-derivative of  $y(t)$  from the answers **-**, **0**, or **+**.

Answer **0** when the absolute value of the 2<sup>nd</sup>-derivative is estimated to be less than 0.5.

Result:  $\frac{d^2y}{dt^2} \Big|_{t=0}$  is        $\frac{d^2y}{dt^2} \Big|_{t=2}$  is        $\frac{d^2y}{dt^2} \Big|_{t=4}$  is        $\frac{d^2y}{dt^2} \Big|_{t=6}$  is

5.4 ♣ Derivatives of commonly-encountered functions. (Section 1.6.5).

Differentiate the following functions that depend on  $t$  (time). Express results in terms of  $x, \dot{x}, t$  so the results are valid when  $x$  is constant or depends on time (e.g., when  $x = t^3$ ).

Result:  $\frac{d}{dt} t^2 = \text{$        $\frac{d}{dt} t^3 = \text{$        $\frac{d}{dt} t^{47} = \text{$   
 $\frac{d}{dt} \sin(t) = \text{$        $\frac{d}{dt} \cos(t) = \text{$        $\frac{d}{dt} \cos(x) = \text{$   
 $\frac{d}{dt} e^t = \text{$        $\frac{d}{dt} \ln(t) = \frac{\text{}}{\text{$        $\frac{d}{dt} \ln(x) = \frac{1}{\text{$

**5.5 ♣ Good product rule for differentiation (for scalars, vectors, matrices, ...).** (Section 1.6.7).

The *good product rule for differentiation* that works when  $u$  and  $v$  are scalars, vectors, or matrices is (circle the correct answer – and update your Calculus teacher):

$$\frac{d(u * v)}{dt} = \frac{du}{dt} * v + u * \frac{dv}{dt} \quad \frac{d(u * v)}{dt} = u * \frac{dv}{dt} + v * \frac{du}{dt} \quad \frac{d(u * v)}{dt} = v * \frac{du}{dt} + u * \frac{dv}{dt}$$

Knowing  $u, v, w$  are scalars or **matrices** that depend on time  $t$ , use the *good product rule for differentiation* to form the 1<sup>st</sup> ordinary time-derivative of  $y(t) = u * v * w$ .

**Good product rule:**  $\frac{dy}{dt} = \frac{d(u * v * w)}{dt} =$     $+$     $+$   

**5.6 Derivative quotient rule? No, just use product rule and exponent.** (Section 1.6.8).

Although the “*quotient rule*” can be used to calculate the derivative with respect to  $t$  of the ratio of two functions  $\frac{f(t)}{g(t)}$ , it can be easier to rewrite the ratio as  $f(t) * g(t)^{-1}$  then use the *product rule*. Use this idea to first rewrite the following ratio of two functions as a product and then use the *product rule* to calculate its derivative.

**Result:**  $\frac{\ln(t)}{t^2} = \ln(t) * t$     $\frac{d}{dt} [\ln(t) / t^2] =$   

**5.7 ♣ Example of the “good product rule” for differentiation.** (Takes less than 2 minutes).

The “good” product rule is easy-to-use for *very quickly* differentiating complex expressions. Knowing  $x$  and  $y$  are variables that depend on the independent variable  $t$  (time), determine the ordinary time-derivative of the function  $f$  when<sup>1</sup>

$$f(t) = \sin(t) * \cos(x + y) * (\dot{x})^2 * e^t * \ln(y) / x$$

**Result:**  $\frac{df}{dt} = \cos(t) * \cos(x + y) * (\dot{x})^2 * e^t * \ln(y) / x$   
 $- \sin(t) * \sin(x + y) * (\dot{x} + \dot{y}) * (\dot{x})^2 * e^t * \ln(y) / x$   
 $+$     
 $+$     
 $+$     
 $-$   

**5.8 ♣ Ordinary derivative of the function  $f(t) = \sin(t) * \cos(xyz)$ .** (Sections 1.6.7 and 1.6.9).

Knowing each of  $x, y, z$  depend on time  $t$ , form the 1<sup>st</sup>-derivative of  $f(t)$  (in terms of  $x, y, z, t$ , etc).

**Result:**  $\frac{d[\sin(t) \cos(xyz)]}{dt} =$   

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<sup>1</sup>Symbols for the 1<sup>st</sup> and 2<sup>nd</sup> ordinary time-derivatives of  $x$  include  $\frac{dx}{dt}$  and  $\frac{d^2x}{dt^2}$  (introduced by *Leibniz*),  $\dot{x}$  and  $\ddot{x}$  (introduced by *Newton*), and  $\mathbf{x}'$  and  $\mathbf{x}''$  (introduced by *Lagrange* and used by MotionGenesis).

**5.9 Differentiation concepts.** (Section 1.6.10).

Shown right is an equation relating the dependent variable  $y$  to the independent variable  $t$ .

$$y^4 - 8y = 3t^2 + \sin(t)$$

Find a general expression for the ordinary derivative  $\frac{dy}{dt}$  in terms of  $t$  and  $y$ .

Find a **numerical** value for  $\frac{dy}{dt}$  at  $t = 0$  when  $y$  is **positive**.

Hint: The value of  $y$  is not arbitrary. If you encounter difficulty, consider *implicit differentiation*.

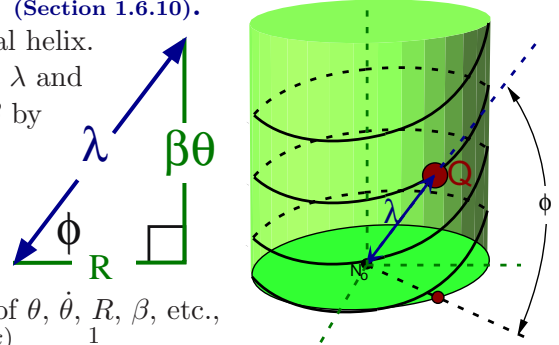
**Result:**

$$\frac{dy}{dt} = \frac{\text{[ ]}}{\text{[ ]}} \qquad \frac{dy}{dt} \Big|_{t=0} = \frac{1}{\text{[ ]}}$$

**5.10 Review of explicit and implicit differentiation.** (Section 1.6.10).

The figure to the right shows a point  $Q$  on a cylindrical helix. Two geometrically significant quantities are a distance  $\lambda$  and an angle  $\phi$  that are related to two **constants**  $R$  and  $\beta$  by

$$\lambda^2 = R^2 + (\beta\theta)^2 \qquad \tan(\phi) = \frac{\beta\theta}{R}$$



Determine  $\dot{\lambda}$  and  $\dot{\phi}$  (time-derivatives of  $\lambda$  and  $\phi$ ) in terms of  $\theta$ ,  $\dot{\theta}$ ,  $R$ ,  $\beta$ , etc., using the two methods described below. Note:  $\frac{\partial \text{atan}(x)}{\partial x} = \frac{1}{1+x^2}$

(a) **Explicit differentiation**

Solve explicitly for  $\lambda$  and  $\phi$  and then differentiate the resulting expression.

**Result:**

$$\lambda = \sqrt{R^2 + (\beta\theta)^2} \qquad \phi = \text{atan}\left(\frac{\beta\theta}{R}\right)$$

$$\dot{\lambda} = \frac{\text{[ ]}}{\text{[ ]}} \dot{\theta} \qquad \dot{\phi} = \frac{\text{[ ]}}{\text{[ ]}} \dot{\theta}$$

(b) **Implicit differentiation**

Differentiate the equations involving  $\lambda^2$  and  $\tan(\phi)$  and then solve for  $\dot{\lambda}$  and  $\dot{\phi}$ .

**Result:**

$$\dot{\lambda} = \text{[ ]} \dot{\theta} \qquad \dot{\phi} = \text{[ ]} \dot{\theta} = \frac{\beta R}{\lambda^2} \dot{\theta}$$

(c) **Explicit/Implicit** differentiation of  $\lambda$  is easier and computationally more efficient.

**5.11 ♣ Review of partial and ordinary differentiation.** (Section 1.6.2).

The kinetic energy  $K$  of a bridge-crane (shown right) can be written in terms of constants  $M$ ,  $m$ ,  $L$  and variables  $x$ ,  $\dot{x}$ ,  $\theta$ ,  $\dot{\theta}$ , as

$$K = \frac{1}{2} M \dot{x}^2 + \frac{1}{2} m [L^2 \dot{\theta}^2 + 2L \cos(\theta) \dot{x} \dot{\theta}]$$

- First, regard  $x$ ,  $\dot{x}$ ,  $\theta$ ,  $\dot{\theta}$  as independent variables [so  $K$  depends separately on each, i.e.,  $K(x, \dot{x}, \theta, \dot{\theta})$ ], form the **partial derivatives** below (left).
- Next, regard  $x$ ,  $\dot{x}$ ,  $\theta$ ,  $\dot{\theta}$  as time-dependent variables and form the **ordinary derivatives** below (right).



The mathematical process below is used in *Lagrange's equations of motion*.

$$\frac{\partial K}{\partial \theta} = \text{[ ]} \qquad \frac{\partial K}{\partial \dot{\theta}} = \text{[ ]} \qquad \frac{d}{dt} \left( \frac{\partial K}{\partial \dot{\theta}} \right) = \text{[ ]}$$

$$\frac{\partial K}{\partial x} = \text{[ ]} \qquad \frac{\partial K}{\partial \dot{x}} = \text{[ ]} \qquad \frac{d}{dt} \left( \frac{\partial K}{\partial \dot{x}} \right) = \text{[ ]}$$

**5.12 Differentiation concepts: What is  $dt$ ?** (Section 1.6.1).

A continuous function  $z(t)$  depends on  $x(t)$ ,  $y(t)$ , and time  $t$  as  $z = x + y^2 \sin(t)$   
 At a certain instant of time,  $y = 1$  and  $z$  simplifies to  $z = x + \sin(t)$   
 Find the time-derivative of  $z$  at the instant when  $y = 1$ .

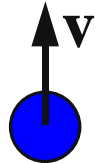
**Result:**  $\frac{dz}{dt}\Big|_{y=1} =$

**5.13 ♣ Differentiation concepts – what is wrong?** (Section 1.6.1 and previous problem).

The scalar  $v$  measures a baseball's upward-velocity. Knowing  $v = 0$  only when the ball reaches maximum height, explain what is wrong with the following statement about  $v$ 's time derivative.

$\frac{dv}{dt} = \frac{d(0)}{dt} = 0$  is **wrong**. You know the correct answer is:  $\frac{dv}{dt} = g \approx 9.8 \frac{\text{m}}{\text{s}^2}$ .

**Explain what is wrong:** It is incorrect to time-differentiate as shown above because:



**5.14 ♣ Integrals of commonly-encountered functions.** (Section 1.7).

Calculate the following indefinite integrals in terms of an indefinite constant  $C$  (regard  $t$  as positive).

**Result:**

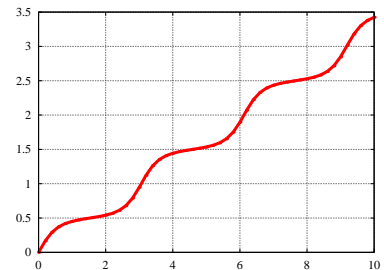
$\int t^2 dt =$ <span style="background-color: yellow; border: 1px solid black; display: inline-block; width: 40px; height: 20px;"></span> $+ C$	$\int t^3 dt =$ <span style="background-color: yellow; border: 1px solid black; display: inline-block; width: 60px; height: 20px;"></span>	$\int t^8 dt =$ <span style="background-color: yellow; border: 1px solid black; display: inline-block; width: 70px; height: 20px;"></span>
$\int t^{-3} dt =$ <span style="background-color: yellow; border: 1px solid black; display: inline-block; width: 80px; height: 20px;"></span>	$\int t^{-2} dt =$ <span style="background-color: yellow; border: 1px solid black; display: inline-block; width: 80px; height: 20px;"></span>	$\int t^{-1} dt =$ <span style="background-color: yellow; border: 1px solid black; display: inline-block; width: 80px; height: 20px;"></span>
$\int \sin(t) dt =$ <span style="background-color: yellow; border: 1px solid black; display: inline-block; width: 100px; height: 20px;"></span>	$\int \cos(t) dt =$ <span style="background-color: yellow; border: 1px solid black; display: inline-block; width: 100px; height: 20px;"></span>	$\int e^t dt =$ <span style="background-color: yellow; border: 1px solid black; display: inline-block; width: 60px; height: 20px;"></span>
$\int 5 dt =$ <span style="background-color: yellow; border: 1px solid black; display: inline-block; width: 60px; height: 20px;"></span>	$\int 5/t dt =$ <span style="background-color: yellow; border: 1px solid black; display: inline-block; width: 100px; height: 20px;"></span>	$\int (5 + \frac{1}{t}) dt =$ <span style="background-color: yellow; border: 1px solid black; display: inline-block; width: 130px; height: 20px;"></span>

**5.15 Optional: † Continuous numerical solution of a nonlinear ODE.**

Plot the continuous solution  $x(t)$  to the following ordinary differential equation for  $0 \leq t \leq 10$  with data every 0.2 sec. Use an initial value  $x(0) = 0$  and use the initial value of  $\dot{x}$  that is closest to 1.

$$\sin(\dot{x}) + 4\dot{x}^2 - 1.9 \cos(2\pi x) - 2 = 0$$

$x(t)$  vs.  $t$



Hint: A “clever” way to solve this **nonlinear** ODE for  $x(t)$  is

- Use the given equation and initial value  $x(0) = 0$  to solve for  $\dot{x}$  at  $t=0$ . For example, the technique in Section 1.10 finds  $\dot{x}(t=0) \approx 0.8841161$  when  $x(t=0) = 0$ .
- Time-differentiate the 1<sup>st</sup>-order ODE that is **nonlinear** in  $\dot{x}$  to form a 2<sup>nd</sup>-order ODE that is **linear** in  $\ddot{x}$ . Then, solve the 2<sup>nd</sup>-order ODE for  $\ddot{x}$ .

$$\cos(\dot{x}) \ddot{x} + 8\dot{x} \ddot{x} + 3.8\pi \sin(2\pi x) \dot{x} = 0 \quad \Rightarrow \quad \ddot{x} = \frac{-3.8\pi \sin(2\pi x) \dot{x}}{\cos(\dot{x}) + 8\dot{x}}$$

- Numerically integrate the 2<sup>nd</sup>-order ODE with the initial values of  $x(0)$  and  $\dot{x}(0)$

### 5.16 Physics and calculus: Graphing $\vec{F} = m\vec{a}$ for a sky-diver and rocket-sled.

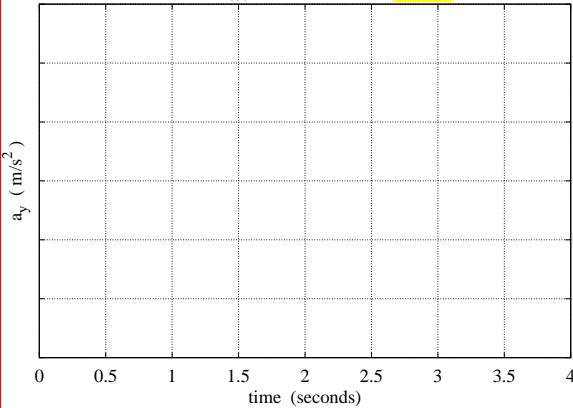
Complete the missing statements, axes values, and graphs. Use Earth's gravitational acceleration  $g \approx 10 \frac{m}{s^2}$ .

A sky-diver free-falls for 4 seconds after leaving a stationary helicopter from a height  $y = 200$  m above Earth ( $y$  is positive-upward).

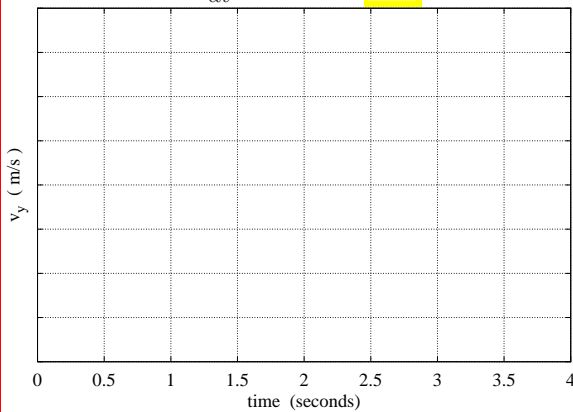


The only relevant force is Earth's gravity.

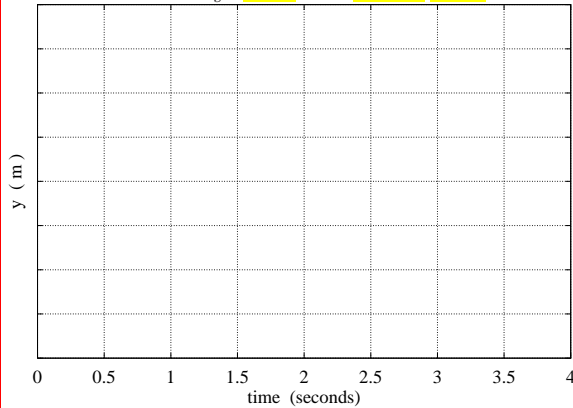
$$a_y \triangleq \frac{d^2 y}{dt^2} = \boxed{\phantom{00}} \boxed{\phantom{00}}$$



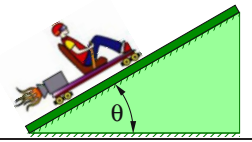
$$v_y \triangleq \frac{dy}{dt} = \boxed{\phantom{00}} \boxed{\phantom{00}} \boxed{\phantom{00}}$$



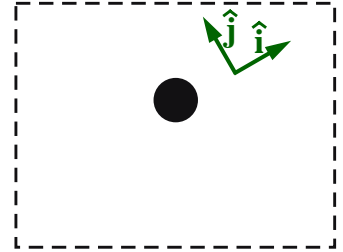
$$y = -5 \frac{m}{s^2} \boxed{\phantom{00}} + \boxed{\phantom{00}} \boxed{\phantom{00}}$$



A rocket-sled of mass  $m$  is thrust along smooth inclined rails with time-varying force  $F_T$ . The variable  $x$  measure's the sled's position along the rails. Initially,  $x = 0$  and  $\dot{x} = 0$ .



**FBD. Draw forces**



Below: Form  $\vec{F}_{Net}$  and then set  $\vec{F}_{Net} = m\vec{a}$ .

Use symbols  $m, g, F_T, F_N, \theta$ .

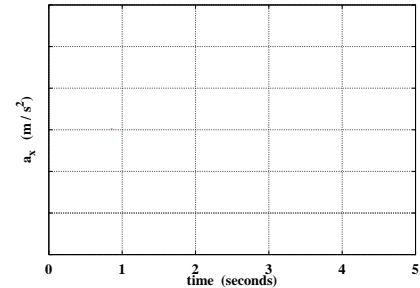
$$\boxed{\phantom{00}} \hat{i} + \boxed{\phantom{00}} \hat{j} = m \boxed{\phantom{00}} \hat{i}$$

$$\hat{i}: \boxed{\phantom{00}} = \boxed{\phantom{00}} \Rightarrow \frac{d^2 x}{dt^2} = \boxed{\phantom{00}} - \boxed{\phantom{00}}$$

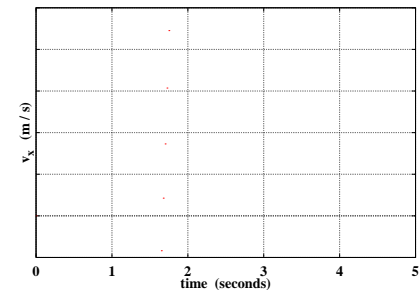
$$\hat{j}: \boxed{\phantom{00}} = \boxed{\phantom{00}} \Rightarrow F_N = \boxed{\phantom{00}}$$

Use  $\theta = 30^\circ, m = 100$  kg,  $F_T = 600 \frac{N}{s} * t$  for the following.

$$a_x \triangleq \frac{dv_x}{dt} = \boxed{\phantom{00}} \frac{m}{s^3} * \boxed{\phantom{00}} - \boxed{\phantom{00}} \frac{m}{s^2}$$



$$v_x \triangleq \frac{dx}{dt} = \boxed{\phantom{00}} \frac{m}{s^3} * \boxed{\phantom{00}} - \boxed{\phantom{00}} \frac{m}{s^2} * \boxed{\phantom{00}}$$



$$x = \boxed{\phantom{00}} \frac{m}{s^3} * \boxed{\phantom{00}} - \boxed{\phantom{00}} \frac{m}{s^2} * \boxed{\phantom{00}}$$

