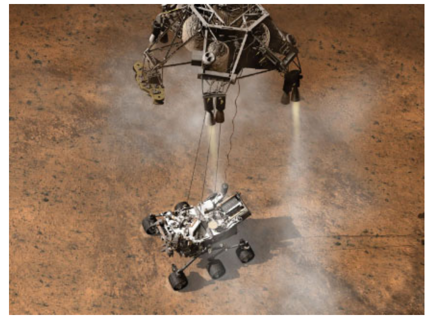
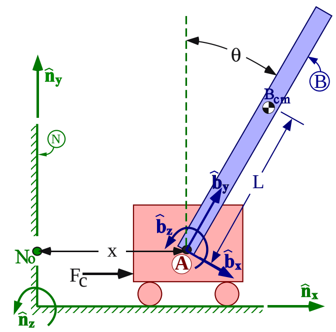


Chapter 30

Example: Inverted pendulum on cart



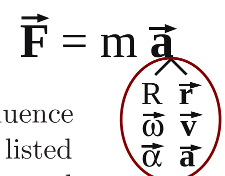
The figure to the right shows a rigid body B attached by an frictionless pin (revolute) joint to a cart A (modeled as a particle). The cart A slides on a horizontal frictionless track. The track is fixed in a Newtonian frame N.

Right-handed orthogonal unit vectors $\widehat{\mathbf{n}}_x$, $\widehat{\mathbf{n}}_y$, $\widehat{\mathbf{n}}_z$ and $\widehat{\mathbf{b}}_x$, $\widehat{\mathbf{b}}_y$, $\widehat{\mathbf{b}}_z$ are fixed in N and B respectively, with:

- $\widehat{\mathbf{n}}_x$ horizontally-right and $\widehat{\mathbf{n}}_y$ vertically-upward
- $\widehat{\mathbf{n}}_z = \widehat{\mathbf{b}}_z$ parallel to B's axis of rotation in N
- $\widehat{\mathbf{b}}_y$ directed from A to the distal end of B

Quantity	Symbol	Value
Mass of A	m^A	10.0 kg
Mass of B	m^B	1.0 kg
Distance between A and B_{cm} (B's center of mass)	L	0.5 m
B's moment of inertia about B_{cm} for $\widehat{\mathbf{b}}_z$	$\mathrm{I_{zz}}$	0.08333 kg∗m^2
Earth's gravitational constant	g	9.8 m/s^2
$\widehat{\mathbf{n}}_x$ measure of feedback-control force applied to A	F_c	**Specified**
$\widehat{\mathbf{n}}_x$ measure of A's position vector from N_o (a point fixed in N)	x	Variable
Angle from $\widehat{\mathbf{n}}_y$ to $\widehat{\mathbf{b}}_y$ with $-\widehat{\mathbf{n}}_z$ sense	θ	Variable

30.1 Kinematics (space and time)

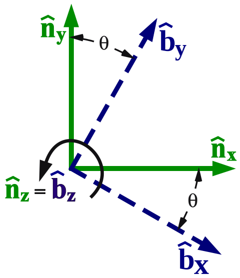


Kinematics is the study of the relationship between space and time, independent of the influence of mass or forces. The kinematic quantities normally needed for motion analysis are listed below. In many circumstances, it is efficient to form rotation matrices, angular velocities, and angular accelerations **before** position vectors, velocities, and accelerations.

Kinematic Quantity	Quantities needed for analyzing the inverted pendulum on a cart
Rotation matrix	${}^{\mathrm{b}}R^{\mathrm{n}}$, the rotation matrix relating $\widehat{\mathbf{b}}_x$, $\widehat{\mathbf{b}}_y$, $\widehat{\mathbf{b}}_z$ and $\widehat{\mathbf{n}}_x$, $\widehat{\mathbf{n}}_y$, $\widehat{\mathbf{n}}_z$
Angular velocity	${}^{N}\vec{\boldsymbol{\omega}}^{B}$, B's angular velocity in N
Angular acceleration	${}^{N}\vec{\boldsymbol{\alpha}}^{B}$, B's angular acceleration in N
Position vectors	$\vec{\mathbf{r}}^{A/N_o}$ and $\vec{\mathbf{r}}^{B_{cm}/A}$, the position vector of A from N_o and of B_{cm} from A
Velocity	${}^{N}\vec{\mathbf{v}}^{A}$ and ${}^{N}\vec{\mathbf{v}}^{B_{cm}}$, A's velocity in N and B_{cm}'s velocity in N
Acceleration	${}^{N}\vec{\mathbf{a}}^{A}$ and ${}^{N}\vec{\mathbf{a}}^{B_{cm}}$, A's acceleration in N and B_{cm}'s acceleration in N

30.2 Rotation matrix, angular velocity, angular acceleration

To relate $\widehat{\mathbf{b}}_x$, $\widehat{\mathbf{b}}_y$, $\widehat{\mathbf{b}}_z$ and $\widehat{\mathbf{n}}_x$, $\widehat{\mathbf{n}}_y$, $\widehat{\mathbf{n}}_z$, **redraw** these unit vectors in the geometrically-suggestive way shown below. To determine the 1^{st} row of the ${}^{b}R^{n}$ rotation matrix, $\widehat{\mathbf{b}}_x$ is expressed in terms of $\widehat{\mathbf{n}}_x$, $\widehat{\mathbf{n}}_y$, $\widehat{\mathbf{n}}_z$ as shown below. Similarly, the 2^{nd} and 3^{rd} rows of ${}^{b}R^{n}$ are found by expressing $\widehat{\mathbf{b}}_y$ and $\widehat{\mathbf{b}}_z$ in terms of $\widehat{\mathbf{n}}_x$, $\widehat{\mathbf{n}}_y$, $\widehat{\mathbf{n}}_z$.



$$\widehat{\mathbf{b}}_x = \cos(\theta)\,\widehat{\mathbf{n}}_x - \sin(\theta)\,\widehat{\mathbf{n}}_y$$
$$\widehat{\mathbf{b}}_y = \sin(\theta)\,\widehat{\mathbf{n}}_x + \cos(\theta)\,\widehat{\mathbf{n}}_y$$
$$\widehat{\mathbf{b}}_z = \widehat{\mathbf{n}}_z$$

${}^{b}R^{n}$	$\widehat{\mathbf{n}}_x$	$\widehat{\mathbf{n}}_y$	$\widehat{\mathbf{n}}_z$
$\widehat{\mathbf{b}}_x$	$\cos(\theta)$	$-\sin(\theta)$	0
$\widehat{\mathbf{b}}_y$	$\sin(\theta)$	$\cos(\theta)$	0
$\widehat{\mathbf{b}}_z$	0	0	1

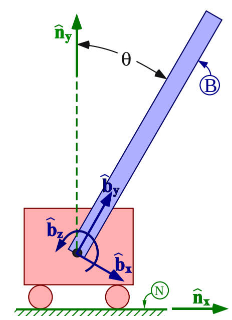
Angular velocity (special 2D case)

When a unit vector $\widehat{\boldsymbol{\lambda}}$ is **fixed** in both reference frames B and N, B has a ***simple angular velocity*** in N that can be calculated via equation (1).

$$\boxed{{}^{N}\vec{\boldsymbol{\omega}}^{B} \underset{\text{(simple)}}{=} \pm\,\dot{\theta}\,\widehat{\boldsymbol{\lambda}}} \qquad (1)$$

Due to the pin joint, $\widehat{\mathbf{b}}_z$ is **fixed**[a] in both B and N, so B has a ***simple angular velocity*** in N.

- $\widehat{\mathbf{b}}_z$ is a unit vector **fixed** in both N and B (parallel to the pin joint)
- $\widehat{\mathbf{n}}_y$ is fixed in N and perpendicular to $\widehat{\mathbf{b}}_z$
- $\widehat{\mathbf{b}}_y$ is fixed in B and perpendicular to $\widehat{\mathbf{b}}_z$
- θ is the angle between $\widehat{\mathbf{n}}_y$ and $\widehat{\mathbf{b}}_y$, and $\dot{\theta}$ is its time-derivative
- After pointing the four fingers of your **right** hand in the direction of $\widehat{\mathbf{n}}_y$ and curling them in the direction of $\widehat{\mathbf{b}}_y$, your thumb points in the $-\widehat{\mathbf{b}}_z$ direction.



Since the right-hand rule produces a sign of $\widehat{\mathbf{b}}_z$ that is negative: ${}^{N}\vec{\boldsymbol{\omega}}^{B} =$

[a] A vector is said to be **fixed** in reference frame B if its magnitude is constant and its direction does not change in B.

Angular acceleration

Equation (2) defines the angular acceleration of a reference frame B in a reference frame N.

${}^{N}\vec{\boldsymbol{\alpha}}^{B}$ also ***happens*** to be equal to the time-derivative in ***B*** of ${}^{N}\vec{\boldsymbol{\omega}}^{B}$.

$$\boxed{{}^{N}\vec{\boldsymbol{\alpha}}^{B} \triangleq \frac{{}^{N}d\,{}^{N}\vec{\boldsymbol{\omega}}^{B}}{dt} = \frac{{}^{B}d\,{}^{N}\vec{\boldsymbol{\omega}}^{B}}{dt}} \qquad (2)$$

Note: Calculate with $\frac{{}^{\mathbf{B}}d\,{}^{N}\vec{\boldsymbol{\omega}}^{B}}{dt}$ if it is easier to compute than $\frac{{}^{\mathbf{N}}d\,{}^{N}\vec{\boldsymbol{\omega}}^{B}}{dt}$.

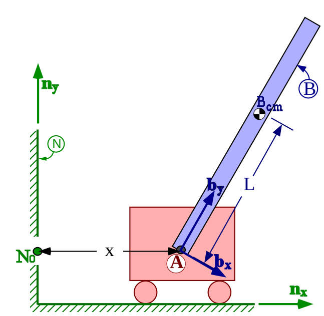
B's angular acceleration in N is most easily calculated with its alternate definition, i.e.,

$${}^{N}\vec{\boldsymbol{\alpha}}^{B} = \frac{{}^{B}d\,{}^{N}\vec{\boldsymbol{\omega}}^{B}}{dt} = \frac{{}^{B}d\,(-\dot{\theta}\,\widehat{\mathbf{b}}_z)}{dt} =$$

30.3 Position vectors, velocity, acceleration

Position vectors are usually formed by inspection and vector addition.

- Inspection of the figure: (A's position vector from N).
- Inspection of the figure: (B_{cm}'s position vector from A).
- Vector addition: $\vec{\mathbf{r}}^{\,B_{cm}/N_o} = \vec{\mathbf{r}}^{\,B_{cm}/A} + \vec{\mathbf{r}}^{\,A/N_o} =$
  (B_{cm}'s position vector from N_o)

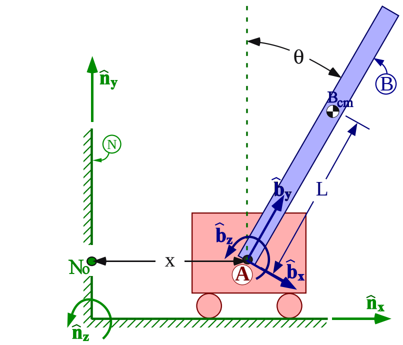


Velocity and acceleration

${}^{N}\vec{\mathbf{v}}^{B_{cm}}$ (the velocity of a point B_{cm} in a reference frame N) is defined as the time-derivative in N of $\vec{\mathbf{r}}^{\,B_{cm}/N_o}$ (B_{cm}'s position vector from N_o).

$$\boxed{{}^{N}\vec{\mathbf{v}}^{B_{cm}} \triangleq \frac{{}^{N}d\,\vec{\mathbf{r}}^{\,B_{cm}/N_o}}{dt}} \qquad (3)$$

Point N_o is ***any*** point ***fixed*** in N



$$\begin{aligned}
{}^{N}\vec{\mathbf{v}}^{B_{cm}} &\triangleq \frac{{}^{N}d\,\vec{\mathbf{r}}^{\,B_{cm}/N_o}}{dt} = \frac{{}^{N}d\,(x\,\widehat{\mathbf{n}}_x \;+\; L\,\widehat{\mathbf{b}}_y)}{dt} \\
&= \frac{{}^{N}d\,(x\,\widehat{\mathbf{n}}_x)}{dt} + \frac{{}^{N}d\,(L\,\widehat{\mathbf{b}}_y)}{dt} \\
&= \dot{x}\,\widehat{\mathbf{n}}_x + \frac{{}^{B}d\,(L\,\widehat{\mathbf{b}}_y)}{dt} + {}^{N}\vec{\boldsymbol{\omega}}^{B} \times L\,\widehat{\mathbf{b}}_y \\
&= \dot{x}\,\widehat{\mathbf{n}}_x + \vec{\mathbf{0}} + \text{-}\dot{\theta}\,\widehat{\mathbf{b}}_z \times L\,\widehat{\mathbf{b}}_y \\
&= \boxed{}
\end{aligned}$$

${}^{N}\vec{\mathbf{a}}^{B_{cm}}$ (the acceleration of point B_{cm} in reference frame N) is defined as the time-derivative in N of ${}^{N}\vec{\mathbf{v}}^{B_{cm}}$ (B_{cm}'s velocity in N).

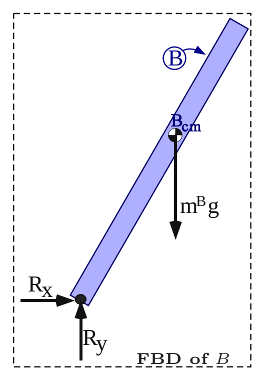
$$\boxed{{}^{N}\vec{\mathbf{a}}^{B_{cm}} \triangleq \frac{{}^{N}d\;{}^{N}\vec{\mathbf{v}}^{B_{cm}}}{dt}} \qquad (4)$$

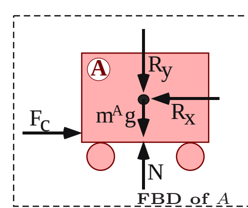
$$\begin{aligned}
{}^{N}\vec{\mathbf{a}}^{B_{cm}} &\triangleq \frac{{}^{N}d\;{}^{N}\vec{\mathbf{v}}^{B_{cm}}}{dt} = \frac{{}^{N}d\,(\dot{x}\,\widehat{\mathbf{n}}_x \;+\; \dot{\theta}\,L\,\widehat{\mathbf{b}}_x)}{dt} = \frac{{}^{N}d\,(\dot{x}\,\widehat{\mathbf{n}}_x)}{dt} + \frac{{}^{N}d\,(\dot{\theta}\,L\,\widehat{\mathbf{b}}_x)}{dt} \\
&= \ddot{x}\,\widehat{\mathbf{n}}_x + \frac{{}^{B}d\,(\dot{\theta}\,L\widehat{\mathbf{b}}_x)}{dt} + {}^{N}\vec{\boldsymbol{\omega}}^{B} \times (\dot{\theta}\,L\,\widehat{\mathbf{b}}_x) = \boxed{}
\end{aligned}$$

30.4 Forces, moments, and free-body diagrams (2D)

$\vec{\mathbf{F}} = \mathrm{m}\,\vec{\mathbf{a}}$

To draw a ***free-body diagram (FBD)***, isolate a single body (or system S of A and B) and draw all the external contact and distance forces that act on it. Shown right are FBDs with all the external forces on the cart A and pendulum B.[a]





Quantity	Description	Type
F_c	$\widehat{\mathbf{n}}_x$ measure of control force applied to A	Contact
N	$\widehat{\mathbf{n}}_y$ measure of the resultant normal force on A from N	Contact
R_x	$\widehat{\mathbf{n}}_x$ measure of the force on B from A across the revolute joint	Contact
R_y	$\widehat{\mathbf{n}}_y$ measure of the force on B from A across the revolute joint	Contact
$m^A\,g$	-$\widehat{\mathbf{n}}_y$ measure of Earth's gravitational force on A	Distance
$m^B\,g$	-$\widehat{\mathbf{n}}_y$ measure of Earth's gravitational force on B	Distance

$$\begin{aligned}
\text{Resultant force on } A\text{:}\quad & \vec{\mathbf{F}}^{A} = (F_c - R_x)\;\widehat{\mathbf{n}}_x \;+\; (N - m^A\,g - R_y)\;\widehat{\mathbf{n}}_y \\
\text{Resultant force on } B\text{:}\quad & \vec{\mathbf{F}}^{B} = R_x\;\widehat{\mathbf{n}}_x \;+\; (R_y - m^B\,g)\;\widehat{\mathbf{n}}_y \\
\text{Resultant force on } S\text{:}\quad & \vec{\mathbf{F}}^{S} = F_c\;\widehat{\mathbf{n}}_x \;+\; [N \;-\; (m^A + m^B)\,g]\;\widehat{\mathbf{n}}_y
\end{aligned}$$

[a] Alternately, to use the efficient ***MG road-map/D'Alembert method*** (Section 30.7) to eliminate "constraint forces" R_x and R_y, **draw** a FBD of the system S consisting of A and B (no need to draw A alone). Since the revolute joint between A and B is ideal, action/reaction is used to minimize the number of unknowns.

The $\widehat{\mathbf{b}}_z$ component of the moment of all forces on B about B_{cm} is[a]

$$\begin{aligned}
\vec{\mathbf{M}}_z^{B/B_{cm}} &= \vec{\mathbf{r}}^{\,A/B_{cm}} \times (R_x\,\widehat{\mathbf{n}}_x + R_y\,\widehat{\mathbf{n}}_y) \;+\; \cancelto{\vec{0}}{\vec{\mathbf{r}}^{\,B_{cm}/B_{cm}}} \times (\text{-}m^B\,g\,\widehat{\mathbf{n}}_y) \\
&= \text{-}L\,\widehat{\mathbf{b}}_y \times (R_x\,\widehat{\mathbf{n}}_x + R_y\,\widehat{\mathbf{n}}_y) = [L\cos(\theta)R_x \;-\; L\sin(\theta)R_y]\;\widehat{\mathbf{b}}_z
\end{aligned}$$

[a] Note: The rotation table to useful for calculating the cross-products $(\widehat{\mathbf{b}}_y \times \widehat{\mathbf{n}}_x)$ and $(\widehat{\mathbf{b}}_y \times \widehat{\mathbf{n}}_y)$.

30.5 Mass, center of mass, inertia (required by dynamics)

- Mass of each particle and body, e.g., m^A (mass of particle A) and m^B (mass of body B).
- Location of each particle and body center of mass, e.g., $\vec{\mathbf{r}}^{\,A/N_o}$ and $\vec{\mathbf{r}}^{\,B_{cm}/A}$.
- Inertia dyadic of each rigid body about a point fixed on the body. Since B's angular velocity in N is **simple**, I_{zz} (B's moment of inertia about B_{cm} for $\widehat{\mathbf{b}}_z$) suffices for this analyses.

$$\vec{\mathbf{F}} = m\,\vec{\mathbf{a}}$$

30.6 Newton/Euler laws of motion for A and B separately (inefficient)

An inefficient way to form this system's equations of motion is with separate analyses of A and B. Using $\vec{\mathbf{F}} = m\,\vec{\mathbf{a}}$ for particle A and body B [in conjunction with the previous ***free-body diagrams (FBDs)***] yields,

$$\vec{\mathbf{F}}^{A} = m^A * {}^{N}\vec{\mathbf{a}}^{A} \quad \Rightarrow \quad (F_c - R_x)\,\widehat{\mathbf{n}}_x + (N - m^A g - R_y)\,\widehat{\mathbf{n}}_y = m^A\,\ddot{x}\,\widehat{\mathbf{n}}_x$$

Dot-multiply with $\widehat{\mathbf{n}}_x$: 　　　　Dot-multiply with $\widehat{\mathbf{n}}_y$:

$$\vec{\mathbf{F}}^{B} = m^B * {}^{N}\vec{\mathbf{a}}^{B_{cm}} \quad \Rightarrow \quad R_x\,\widehat{\mathbf{n}}_x + (R_y - m^B g)\,\widehat{\mathbf{n}}_y = m^B\,(\ddot{x}\,\widehat{\mathbf{n}}_x + \ddot{\theta}\,L\,\widehat{\mathbf{b}}_x - \dot{\theta}^2\,L\,\widehat{\mathbf{b}}_y)$$

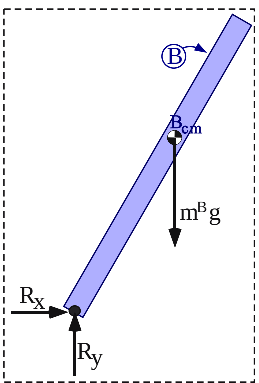
Dot-multiplication with $\widehat{\mathbf{n}}_x$ and $\widehat{\mathbf{n}}_y$ (use the rotation table to calculate dot-products) gives

$$R_x = m^B\,[\ddot{x} + \ddot{\theta}\,L\,(\widehat{\mathbf{b}}_x \cdot \widehat{\mathbf{n}}_x) - \dot{\theta}^2\,L\,(\widehat{\mathbf{b}}_y \cdot \widehat{\mathbf{n}}_x)] \qquad (R_y - m^B g) = m^B\,[\ddot{\theta}\,L\,(\widehat{\mathbf{b}}_x \cdot \widehat{\mathbf{n}}_y) - \dot{\theta}^2\,L\,(\widehat{\mathbf{b}}_y \cdot \widehat{\mathbf{n}}_y)]$$

$$\boxed{R_x = m^B\,[\ddot{x} + \ddot{\theta}\,L\,\cos(\theta) - \dot{\theta}^2\,L\,\sin(\theta)]}$$

Note: Separate analyses of A and B is less efficient than the ***MG road-map/D'Alembert method*** of Section 23.1.3 and Homework 19.8.

30.6.1 Dynamics for a rigid body with a simple angular velocity (special 2D case)



Euler's equation for a rigid body B with a ***simple angular velocity*** in a Newtonian reference frame N is:

$$\boxed{\vec{\mathrm{M}}_z^{B/B_{cm}} \underset{(22.6)}{=} \mathrm{I}_{zz}\;{}^{N}\vec{\boldsymbol{\alpha}}^{B}}$$

- $\vec{\mathrm{M}}_z^{B/B_{cm}}$ is the $\widehat{\mathbf{b}}_z = \widehat{\mathbf{n}}_z$ component of the moment of all forces on B about B_{cm}.
- I_{zz} is B's moment of inertia about the line passing through B_{cm} and parallel to $\widehat{\mathbf{b}}_z$.
- ${}^{N}\vec{\boldsymbol{\alpha}}^{B}$ is B's angular acceleration in N.

Assembling these terms and subsequent dot-multiplication with $\widehat{\mathbf{b}}_z$ produces

$$\boxed{L\,\cos(\theta)\,R_x - L\,\sin(\theta)\,R_y = -\mathrm{I}_{zz}\,\ddot{\theta}}$$

30.6.2 Optional: Angular momentum principle (2D alternative to Section 30.6.1)

The ***angular momentum principle*** for ***any*** system S in a Newtonian reference frame N relates the moment of all forces on S about S_{cm} to the time-derivative of S's angular momentum about S_{cm} in N.

$$\boxed{\vec{\mathbf{M}}^{S/S_{cm}} \underset{(22.4)}{=} \frac{{}^{N}d\;{}^{N}\vec{\mathrm{H}}^{S/S_{cm}}}{dt}}$$

When S is a rigid body B, ${}^{N}\vec{\mathrm{H}}_z^{B/B_{cm}}$ (the $\widehat{\mathbf{b}}_z$ component of ${}^{N}\vec{\mathrm{H}}_z^{B/B_{cm}}$) can be written in terms of $\mathrm{I}_{zz}^{B/B_{cm}}$ (B's moment of inertia about B_{cm} for $\widehat{\mathbf{b}}_z$) and ${}^{N}\vec{\boldsymbol{\omega}}^{B}$.

$$\boxed{{}^{N}\vec{\mathrm{H}}_z^{B/B_{cm}} \underset{(17.3)}{=} \mathrm{I}_{zz}^{B/B_{cm}} * {}^{N}\vec{\boldsymbol{\omega}}^{B}}$$

Assembling terms in the ***angular momentum principle*** and dot-multiplication with $\widehat{\mathbf{b}}_z$ produces

$$\left[\vec{\mathbf{M}}^{B/B_{cm}} = \frac{{}^{N}d\;{}^{N}\vec{\mathrm{H}}^{B/B_{cm}}}{dt}\right] \cdot \widehat{\mathbf{b}}_z \quad \Rightarrow \quad \boxed{L\,\cos(\theta)\,R_x - L\,\sin(\theta)\,R_y = -\mathrm{I}_{zz}\,\ddot{\theta}}$$

30.6.3 Optional: Angular momentum principle (3D alternative to Section 30.6.1)

For general 3D motion of rigid body B, ${}^N\vec{\mathrm{H}}^{B/B_{cm}}$ can be written in terms of $\vec{\vec{\mathrm{I}}}^{B/B_{cm}}$ (B's inertia dyadic about B_{cm}) and ${}^N\vec{\boldsymbol{\omega}}^B$ as:

$$\boxed{{}^N\vec{\mathrm{H}}^{B/B_{cm}} \underset{(17.1)}{=} \vec{\vec{\mathrm{I}}}^{B/B_{cm}} \cdot {}^N\vec{\boldsymbol{\omega}}^B}$$

Using the general form of $\vec{\vec{\mathrm{I}}}^{B/B_{cm}}$ with ${}^N\vec{\boldsymbol{\omega}}^B = \text{-}\dot{\theta}\,\widehat{\mathbf{b}}_z$, yields ${}^N\vec{\mathrm{H}}^{B/B_{cm}}$ (shown right) whose time-derivative in N is given below.

$$
\begin{aligned}
{}^N\vec{\mathrm{H}}^{B/B_{cm}} &= \begin{pmatrix} \mathrm{I}_{xx}\,\widehat{\mathbf{b}}_x\,\widehat{\mathbf{b}}_x + \mathrm{I}_{xy}\,\widehat{\mathbf{b}}_x\,\widehat{\mathbf{b}}_y + \mathrm{I}_{xz}\,\widehat{\mathbf{b}}_x\,\widehat{\mathbf{b}}_z \\ \mathrm{I}_{xy}\,\widehat{\mathbf{b}}_y\,\widehat{\mathbf{b}}_x + \mathrm{I}_{yy}\,\widehat{\mathbf{b}}_y\,\widehat{\mathbf{b}}_y + \mathrm{I}_{yz}\,\widehat{\mathbf{b}}_y\,\widehat{\mathbf{b}}_z \\ \mathrm{I}_{xz}\,\widehat{\mathbf{b}}_z\,\widehat{\mathbf{b}}_x + \mathrm{I}_{yz}\,\widehat{\mathbf{b}}_z\,\widehat{\mathbf{b}}_y + \mathrm{I}_{zz}\,\widehat{\mathbf{b}}_z\,\widehat{\mathbf{b}}_z \end{pmatrix} \cdot (\text{-}\dot{\theta}\,\widehat{\mathbf{b}}_z) \\
&= \text{-}\mathrm{I}_{xz}\,\dot{\theta}\,\widehat{\mathbf{b}}_x - \mathrm{I}_{yz}\,\dot{\theta}\,\widehat{\mathbf{b}}_y - \mathrm{I}_{zz}\,\dot{\theta}\,\widehat{\mathbf{b}}_z \\
\frac{{}^N d\;{}^N\vec{\mathrm{H}}^{B/B_{cm}}}{dt} &= \text{-}(\mathrm{I}_{yz}\,\dot{\theta}^2 + \mathrm{I}_{xz}\,\ddot{\theta})\,\widehat{\mathbf{b}}_x + (\mathrm{I}_{xz}\,\dot{\theta}^2 - \mathrm{I}_{yz}\,\ddot{\theta})\,\widehat{\mathbf{b}}_y - \mathrm{I}_{zz}\,\ddot{\theta}\,\widehat{\mathbf{b}}_z
\end{aligned}
$$

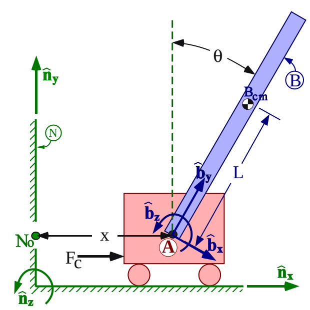
Assembling terms in the ***angular momentum principle*** and dot-multiplication with $\widehat{\mathbf{b}}_z$ produces

$$\left[\vec{\mathrm{M}}^{B/B_{cm}} \underset{(17.1)}{=} \frac{{}^N d\;{}^N\vec{\mathrm{H}}^{B/B_{cm}}}{dt}\right] \cdot \widehat{\mathbf{b}}_z \quad \Rightarrow \quad \boxed{L\cos(\theta)\,R_x - L\sin(\theta)\,R_y = \text{-}\mathrm{I}_{zz}\,\ddot{\theta}}$$

Note: $\frac{{}^N d\;{}^N\vec{\mathrm{H}}^{B/B_{cm}}}{dt}$ has non-zero $\widehat{\mathbf{b}}_x$ and $\widehat{\mathbf{b}}_y$ terms. Dot-multiplication of $[\vec{\mathrm{M}}^{B/B_{cm}} \underset{(17.1)}{=} \frac{{}^N d\;{}^N\vec{\mathrm{H}}^{B/B_{cm}}}{dt}]$ with $\widehat{\mathbf{b}}_x$ or $\widehat{\mathbf{b}}_y$ requires a three-dimensional (3D) free-body diagram (FBD). This problem's FBDs in Section 30.4 were two-dimensional (2D) and ignored the $\widehat{\mathbf{b}}_x$ and $\widehat{\mathbf{b}}_y$ measures of the torque on B from A that constrain B's rotational motion to solely about $\widehat{\mathbf{b}}_z$. The FBDs also ignored the $\widehat{\mathbf{n}}_z$ measure of the force on B from A across the revolute joint. Non-zero $\widehat{\mathbf{b}}_x$ and $\widehat{\mathbf{b}}_y$ constraint torque measures arise due to terms associated with $\dot{\theta}^2$, $\ddot{\theta}$, and products of inertia I_{xz} and I_{yz}.

Summary of Newton/Euler equations of motion (inefficient)

$$
\begin{aligned}
F_c - R_x &= m^A\,\ddot{x} \\
N - m^A\,g - R_y &= 0 \\
R_x &= m^B\,[\ddot{x} + \ddot{\theta}\,L\cos(\theta) - \dot{\theta}^2\,L\sin(\theta)] \\
(R_y - m^B g) &= m^B\,[\text{-}\ddot{\theta}\,L\sin(\theta) - \dot{\theta}^2\,L\cos(\theta)] \\
L\cos(\theta)\,R_x - L\sin(\theta)\,R_y &= \text{-}\mathrm{I}_{zz}\,\ddot{\theta}
\end{aligned}
$$



There are **5** unknown variables in the previous set of equations, namely R_x, R_y, N, x, θ.
Note: Once $\theta(t)$ is known, $\dot{\theta}(t)$ and $\ddot{\theta}(t)$ are known. Similarly, once $x(t)$ is known, $\dot{x}(t)$ and $\ddot{x}(t)$ are known.

30.7 Equations of motion via MG road-maps/D'Alembert (efficient)

For various purposes (e.g., control system design), it is useful to eliminate the unknown "constraint forces" R_x, R_y, N. Although tedious linear-algebra can reduce the previous set of 5 equations in 5 unknowns to 2 equations in 2 unknowns ($\ddot{x}$, $\ddot{\theta}$), it is **more efficient** to use ***MG road-maps*** (Section 23.1.3) or the methods of Lagrange (Section 30.9) or Kane (Section 30.10) as they automatically eliminate R_x, R_y, N.

Variable	Translate/ Rotate	Direction (unit vector)	System S	FBD of S	About point	***MG road-map equation***
x				**Draw**	Not applicable	$\cdot\;(\quad \underset{(22.1)}{=} \quad)$
θ				**Draw**		$\cdot\;(\quad \underset{(22.4)}{=} \quad)$

Homework 19.8 and Chapter 30 complete these calculations.

Note: $m^S * {}^N\vec{\mathbf{a}}^{S_{cm}} \underset{(13.3)}{=} m^A * {}^N\vec{\mathbf{a}}^A + m^B * {}^N\vec{\mathbf{a}}^{B_{cm}}$ and $\frac{{}^N d\;{}^N\vec{\mathrm{H}}^{B/A}}{dt} + \ldots \underset{(22.6)}{=} \mathrm{I}_{zz}^{B/A} * {}^N\vec{\boldsymbol{\alpha}}^B + m^B * \vec{\mathbf{r}}^{B_{cm}/A} \times {}^N\vec{\mathbf{a}}^A$.

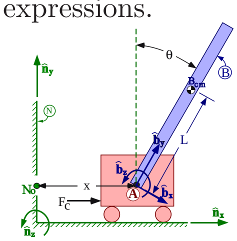
MG road-map for x $\Rightarrow$ $\boxed{F_c = (m^A + m^B)\,\ddot{x} + m^B\,L\cos(\theta)\,\ddot{\theta} - m^B\,L\sin(\theta)\,\dot{\theta}^2}$

MG road-map for θ $\Rightarrow$ $\boxed{m^B\,g\,L\sin(\theta) = m^B\,L\cos(\theta)\,\ddot{x} + (\mathrm{I}_{zz} + m^B\,L^2)\,\ddot{\theta}}$

30.8 Generalized forces for Lagrange/Kane's equations of motion

As described in Section 26.4, ***generalized forces*** depend on ***forces*** and ***partial velocities***. Partial velocities are calculated for each ***generalized coordinate/speed***. For ***Lagrange's method***, a natural choice of generalized coordinates is $q_r = x,\ \theta$. ***Kane's method*** uses analogous generalized speeds $u_r = \dot{x},\ \dot{\theta}$. The relevant partial velocities are **determined by inspection** of velocity expressions.

Velocity	Partial velocities for $u_r = \dot{q}_r$	Symbol	$\dot{q}_r = \dot{x}$	$\dot{q}_r = \dot{\theta}$
${}^{N}\vec{\mathbf{v}}^{A} \;=\; \dot{x}\,\widehat{\mathbf{n}}_{\mathrm{x}}$	A's partial velocity in N for $\dot{q}_r$	$\dfrac{\partial\, {}^{N}\vec{\mathbf{v}}^{A}}{\partial\, \dot{q}_r}$	$\widehat{\mathbf{n}}_{\mathrm{x}}$	$\vec{\mathbf{0}}$
${}^{N}\vec{\mathbf{v}}^{B_{\mathrm{cm}}} = \dot{x}\,\widehat{\mathbf{n}}_{\mathrm{x}} + L\,\dot{\theta}\,\widehat{\mathbf{b}}_{\mathrm{x}}$	B_{cm}'s partial velocity in N for $\dot{q}_r$	$\dfrac{\partial\, {}^{N}\vec{\mathbf{v}}^{B_{\mathrm{cm}}}}{\partial\, \dot{q}_r}$	$\widehat{\mathbf{n}}_{\mathrm{x}}$	$L\,\widehat{\mathbf{b}}_{\mathrm{x}}$



Generalized forces are calculated for a system (not with free-body diagrams). It can be seen that certain forces do not contribute to generalized forces. For this system, the ground's normal force (N), gravity force on A ($m^A g$), and pin force on B from A ($R_x\,\widehat{\mathbf{n}}_{\mathrm{x}} + R_y\,\widehat{\mathbf{n}}_{\mathrm{y}}$) **do not** contribute to generalized forces.

$$\text{Generalized force for } \dot{x} \qquad \mathcal{F}_{\dot{x}} \underset{(26.8)}{=} F_c\,\widehat{\mathbf{n}}_{\mathrm{x}} \cdot \frac{\partial\, {}^{N}\vec{\mathbf{v}}^{A}}{\partial\, \dot{x}} \;+\; -m^B\, g\,\widehat{\mathbf{n}}_{\mathrm{y}} \cdot \frac{\partial\, {}^{N}\vec{\mathbf{v}}^{B_{\mathrm{cm}}}}{\partial\, \dot{x}} \;=\; \boxed{}$$

$$\text{Generalized force for } \dot{\theta} \qquad \mathcal{F}_{\dot{\theta}} \underset{(26.8)}{=} F_c\,\widehat{\mathbf{n}}_{\mathrm{x}} \cdot \frac{\partial\, {}^{N}\vec{\mathbf{v}}^{A}}{\partial\, \dot{\theta}} \;+\; -m^B\, g\,\widehat{\mathbf{n}}_{\mathrm{y}} \cdot \frac{\partial\, {}^{N}\vec{\mathbf{v}}^{B_{\mathrm{cm}}}}{\partial\, \dot{\theta}} \;=\; \boxed{}$$

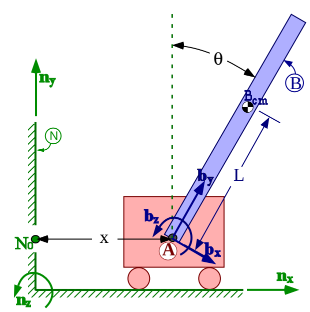
Note: Section 30.6 analyzed this system using separate ***free-body diagrams*** of A and B and generated 5 equations and 5 unknowns ($\ddot{x},\ \ddot{\theta},\ R_x,\ R_y,\ N$). The ***Lagrange/Kane generalized forces*** automatically eliminate the "***constraint forces***" $R_x,\ R_y,\ N$, reducing the unknowns to only $x(t)$ and $\theta(t)$.

30.9 Lagrange's equations of motion (described in Chapter 27)

Lagrange's equations of motion for the ***generalized coordinates*** x and θ are calculated using the ***generalized forces*** $\mathcal{F}_r$ in Section 30.8 and the kinetic energy below.

$$\boxed{\mathcal{F}_r \underset{(27.1)}{=} \frac{d}{dt}\frac{\partial K}{\partial \dot{q}_r} - \frac{\partial K}{\partial q_r}} \quad \Rightarrow \quad \boxed{\begin{aligned} F_c \;&=\; (m^A + m^B)\,\ddot{x} \;+\; m^B\,[L\,\cos(\theta)\,\ddot{\theta} - L\,\sin(\theta)\,\dot{\theta}^2] \\ m^B\,g\,L\,\sin(\theta) \;&=\; m^B\,L\,\cos(\theta)\,\ddot{x} \;+\; (\mathrm{I_{zz}} + m^B\,L^2)\,\ddot{\theta} \end{aligned}}$$

Kinetic energy of inverted pendulum on cart



Since A is a particle, its kinetic energy is defined in equation (12.6) as

$${}^{N}K^{A} \underset{(12.6)}{\triangleq} \frac{1}{2}\,m^A * {}^{N}\vec{\mathbf{v}}^{A} \cdot {}^{N}\vec{\mathbf{v}}^{A} \;=\; \frac{1}{2}\,m^A\,(\dot{x}\,\widehat{\mathbf{n}}_{\mathrm{x}}) \cdot (\dot{x}\,\widehat{\mathbf{n}}_{\mathrm{x}}) \;=\; \frac{1}{2}\,m^A\,\dot{x}^2$$

Since B is a rigid body with a simple angular velocity, its kinetic energy can be calculated with equation (17.12) as

$$\begin{aligned} {}^{N}K^{B} &\underset{(17.12)}{=} \frac{1}{2}\,m\,\vec{\mathbf{v}}^2 \;+\; \frac{1}{2}\,\mathrm{I}\,\vec{\boldsymbol{\omega}}^2 \;=\; \frac{1}{2}\,m^B\,(\dot{x}\,\widehat{\mathbf{n}}_{\mathrm{x}} + \dot{\theta}\,L\,\widehat{\mathbf{b}}_{\mathrm{x}}) \cdot (\dot{x}\,\widehat{\mathbf{n}}_{\mathrm{x}} + \dot{\theta}\,L\,\widehat{\mathbf{b}}_{\mathrm{x}}) \;+\; \frac{1}{2}\,\mathrm{I_{zz}}\,\dot{\theta}^2 \\ &= \frac{1}{2}\,m^B\,\left[\dot{x}^2 \;+\; 2\,L\,\dot{x}\,\dot{\theta}\,\cos(\theta) \;+\; (\dot{\theta}\,L)^2\right] \;+\; \frac{1}{2}\,\mathrm{I_{zz}}\,\dot{\theta}^2 \end{aligned}$$

S's kinetic energy in N is the sum of A and B's ***kinetic energy***, i.e.,

$${}^{N}K^{S} \;=\; {}^{N}K^{A} \;+\; {}^{N}K^{B} \;=\; \frac{1}{2}\,m^A\,\dot{x}^2 \;+\; \frac{1}{2}\,m^B\,\left[\dot{x}^2 \;+\; 2\,L\,\dot{x}\,\dot{\theta}\,\cos(\theta) \;+\; (\dot{\theta}\,L)^2\right] \;+\; \frac{1}{2}\,\mathrm{I_{zz}}\,\dot{\theta}^2$$

30.10 Kane's equations of motion (described in Chapter 26)

Kane's equation of motion for the ***generalized speeds*** $\dot{x}$ and $\dot{\theta}$ are calculated using the ***generalized forces*** $\mathcal{F}_r$ in Section 30.8 and the ***generalized effective forces*** ${}^N\mathcal{F}_r^S$ $(r = \dot{x},\, \dot{\theta})$ below.

$$\boxed{\mathcal{F}_r \underset{(26.1)}{=} {}^N\mathcal{F}_r^S} \qquad \Rightarrow \qquad \boxed{\begin{aligned} F_c \;&=\; (m^A + m^B)\,\ddot{x} \;+\; m^B\,[L\,\cos(\theta)\,\ddot{\theta} - L\,\sin(\theta)\,\dot{\theta}^2] \\ m^B\,g\,L\,\sin(\theta) \;&=\; m^B\,L\,\cos(\theta)\,\ddot{x} \;+\; (\mathrm{I_{zz}} + m^B\,L^2)\,\ddot{\theta} \end{aligned}}$$

Generalized effective forces for inverted pendulum on cart

Generalized effective forces depend on ***effective force*** and ***partial velocities/angular velocities***, and are calculated by summing the contribution of each particle or body in the system.

- Particle A's generalized effective forces for the generalized speeds $u_r = \dot{x},\, \dot{\theta}$ are

$${}^N\mathcal{F}_r^A \underset{(26.17)}{\triangleq} \frac{\partial\, {}^N\vec{\mathbf{v}}^A}{\partial\, u_r} \cdot {}^N\vec{\mathbf{F}}^Q \underset{(12.8)}{=} \frac{\partial\, {}^N\vec{\mathbf{v}}^A}{\partial\, u_r} \cdot (m^A * {}^N\vec{\mathbf{a}}^A) \;=\; \frac{\partial\, (\dot{x}\,\widehat{\mathbf{n}}_\mathrm{x})}{\partial\, u_r} \cdot (m^A\,\ddot{x}\,\widehat{\mathbf{n}}_\mathrm{x}) \quad \Rightarrow \quad \begin{cases} {}^N\mathcal{F}_{\dot{x}}^A \;=\; m^A\,\ddot{x} \\ {}^N\mathcal{F}_{\dot{\theta}}^A \;=\; 0 \end{cases}$$

- Rigid body B's generalized effective forces for the generalized speeds $u_r = \dot{x},\, \dot{\theta}$ are

$$\begin{aligned} {}^N\mathcal{F}_r^B &\underset{(26.19)}{=} \frac{\partial\, {}^N\vec{\mathbf{v}}^{B_{\mathrm{cm}}}}{\partial\, u_r} \cdot {}^N\vec{\mathbf{F}}^B \;+\; \frac{\partial\, {}^N\vec{\boldsymbol{\omega}}^B}{\partial\, u_r} \cdot {}^N\vec{\mathbf{M}}^{B/B_{\mathrm{cm}}} \underset{(17.6,\,17.8)}{=} \frac{\partial\, {}^N\vec{\mathbf{v}}^{B_{\mathrm{cm}}}}{\partial\, u_r} \cdot \left(m^B * {}^N\vec{\mathbf{a}}^{B_{\mathrm{cm}}}\right) \;+\; \frac{\partial\, {}^N\vec{\boldsymbol{\omega}}^B}{\partial\, u_r} \cdot \left(\mathrm{I_{zz}} * {}^N\vec{\boldsymbol{\alpha}}^B\right) \\ &= \frac{\partial\, {}^N\vec{\mathbf{v}}^{B_{\mathrm{cm}}}}{\partial\, u_r} \cdot m^B\left(\ddot{x}\,\widehat{\mathbf{n}}_\mathrm{x} + L\,\ddot{\theta}\,\widehat{\mathbf{b}}_\mathrm{x} - L\,\dot{\theta}^2\,\widehat{\mathbf{b}}_\mathrm{y}\right) \;+\; \frac{\partial\, {}^N\vec{\boldsymbol{\omega}}^B}{\partial\, u_r} \cdot \left(\text{-}\mathrm{I_{zz}}\,\ddot{\theta}\,\widehat{\mathbf{b}}_\mathrm{z}\right) \end{aligned}$$

$${}^N\mathcal{F}_{\dot{x}}^B \;=\; m^B\,[\ddot{x} + L\,\cos(\theta)\,\ddot{\theta} - L\,\sin(\theta)\,\dot{\theta}^2] \qquad\qquad {}^N\mathcal{F}_{\dot{\theta}}^B \;=\; m^B\,L\,\cos(\theta)\,\ddot{x} \;+\; (\mathrm{I_{zz}} + m^B\,L^2)\,\ddot{\theta}$$

- The system S's generalized effective forces for the generalized speeds $u_r = \dot{x},\, \dot{\theta}$ are

$${}^N\mathcal{F}_r^S \underset{(26.18)}{=} {}^N\mathcal{F}_r^A \;+\; {}^N\mathcal{F}_r^B \qquad \Rightarrow \qquad \begin{cases} {}^N\mathcal{F}_{\dot{x}}^S \;=\; \boxed{(m^A + m^B)\,\ddot{x} \;+\; m^B\,[L\,\cos(\theta)\,\ddot{\theta} - L\,\sin(\theta)\,\dot{\theta}^2]} \\ {}^N\mathcal{F}_{\dot{\theta}}^S \;=\; \boxed{m^B\,L\,\cos(\theta)\,\ddot{x} \;+\; (\mathrm{I}_{zz} + m^B\,L^2)\,\ddot{\theta}} \end{cases}$$

30.11 Matrix form of equations of motion (for solution, controls, ...)

For numerical solution and various control-systems techniques, it can be useful to write this system's nonlinear equations of motion in matrix form as

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \left[\, F_c \,\right] \;=\; \begin{bmatrix} m^A + m^B & m^B\,L\,\cos(\theta) \\ m^B\,L\,\cos(\theta) & \mathrm{I_{zz}} + m^B\,L^2 \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} \;+\; \begin{bmatrix} \text{-}m^B\,L\,\sin(\theta)\,\dot{\theta}^2 \\ \text{-}m^B\,g\,L\,\sin(\theta) \end{bmatrix}$$