

12.1 ♣ Concepts: What objects have a moment of inertia? (Section 14.1).

Consider the *moment of inertia* $I_{\hat{u}\hat{u}}^{S/O}$ of an object S about a point O for the unit vector \hat{u} . In general, for $I_{\hat{u}\hat{u}}^{S/O}$ to be a positive real number, S should be a (circle **all** appropriate objects):

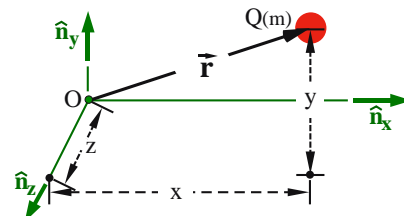
Complex number	Point	Reference Frame	Center of mass of a set of particles
Vector	Set of Points	Rigid Body	Center of mass of a rigid body
Matrix	Particle	Flexible Body	Set of flexible bodies
Orthogonal unit basis	Set of Particles	Set of Rigid bodies	System of particles and bodies

12.2 ♣ Formulas for a particle's moments and products of inertia (Sections 14.1 and 14.2).

The figure shows a particle Q of mass m and right-handed orthogonal unit vectors $\hat{n}_x, \hat{n}_y, \hat{n}_z$. Q 's position vector from a point O is $x\hat{n}_x + y\hat{n}_y + z\hat{n}_z$.

Express I_{xx} (Q 's *moment of inertia* about O for \hat{n}_x) in terms of some or all of m, x, y, z . Similarly for I_{yy} and I_{zz} .

Express I_{xy} (Q 's *product of inertia* about O for \hat{n}_x and \hat{n}_y) in terms of some or all of m, x, y, z . Similarly for I_{xz} and I_{yz} .



Result:

$$I_{xx} = m(y^2 + z^2) \quad I_{yy} = m(x^2 + z^2) \quad I_{zz} = m(x^2 + y^2)$$

$$I_{xy} = -m x y \quad I_{xz} = -m x z \quad I_{yz} = -m y z$$

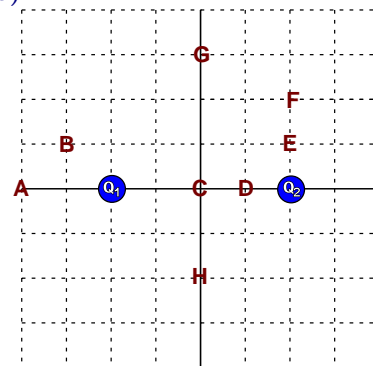
Circa 1895, Gibbs invented the *inertia dyadic* as a **convenient "suitcase"** for holding moments and products of inertia. Write Q 's inertia dyadic about O in terms of $\hat{n}_x, \hat{n}_y, \hat{n}_z$ and I_{ij} ($i, j = x, y, z$). If needed, refer to Section 16.1.

$$\overset{\equiv}{\mathbf{I}} = I_{xx} \hat{n}_x \hat{n}_x + I_{xy} \hat{n}_x \hat{n}_y + I_{xz} \hat{n}_x \hat{n}_z + I_{xy} \hat{n}_y \hat{n}_x + I_{yy} \hat{n}_y \hat{n}_y + I_{yz} \hat{n}_y \hat{n}_z + I_{xz} \hat{n}_z \hat{n}_x + I_{yz} \hat{n}_z \hat{n}_y + I_{zz} \hat{n}_z \hat{n}_z$$

12.3 ♣ Parallel axis theorem and moments of inertia (Section 14.1.5).

The system S shown to the right consists of particles Q_1 and Q_2 , each of mass m , in a plane perpendicular to the unit vector \hat{n}_z .

The *shift theorem* (also called the *parallel axis theorem*) shifts S 's moment of inertia about S_{cm} (the *mass center* of S) for the unit vector \hat{n}_z to an arbitrary point P in the plane using



$$I_{zz}^{S/P} = I_{zz}^{S/S_{cm}} + m^S * d^2$$

where $I_{zz}^{S/S_{cm}}$ is the system's moment of inertia about S_{cm} for \hat{n}_z , m^S is the mass of S , and d is the distance from S_{cm} to P .

Use the shift theorem to estimate the order of S 's moment of inertia for the lines parallel to \hat{n}_z that pass through points A, B, C, D, E, F, G , and H , respectively. Note: Grid lines are equally spaced.

Result:

Smallest	C	D	H	E	F	G	B	A	Largest
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Knowing each particle has mass $m = 1$ kg and the grid lines are spaced 1 m apart, calculate S 's moment of inertia about A, B, C, D, E, F, G , and H , respectively.

Result:

$I_{zz}^{S/A}$	$I_{zz}^{S/B}$	$I_{zz}^{S/C}$	$I_{zz}^{S/D}$	$I_{zz}^{S/E}$	$I_{zz}^{S/F}$	$I_{zz}^{S/G}$	$I_{zz}^{S/H}$
40	28	8	10	18	24	26	16

(in kg m²)

12.4 ♣ **Calculations: Product of inertia for a single particle** (Section 14.2).

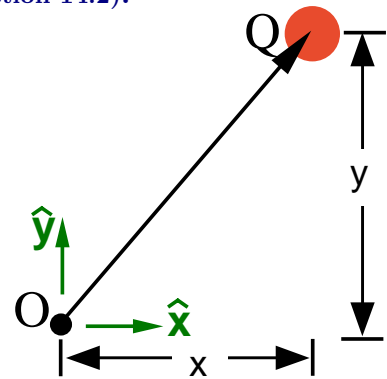
The *product of inertia* of a single particle Q about a point O for the \hat{x} and \hat{y} directions is calculated by the formula

$$I_{xy} = -mxy$$

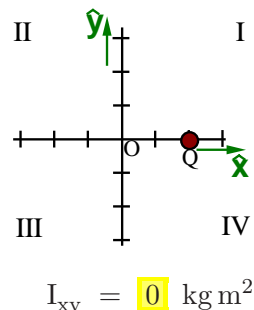
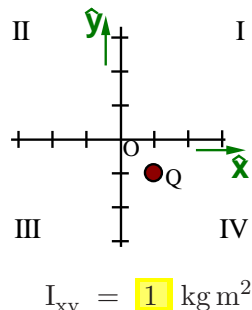
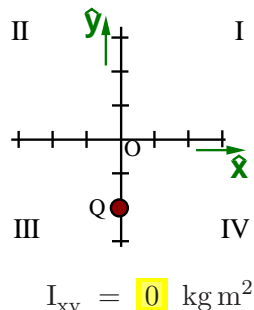
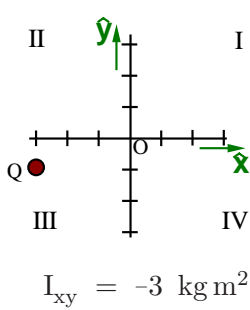
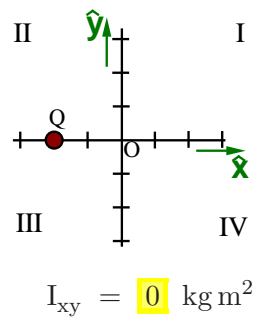
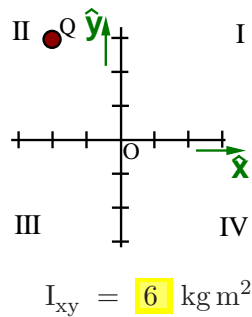
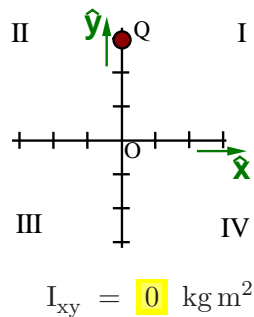
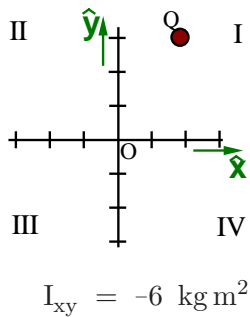
- m is the mass of Q
- x is the \hat{x} measure of Q 's position vector from O
- y is the \hat{y} measure of Q 's position vector from O

For example, if $m = 1$ kg, $x = 2$ m, and $y = 3$ m,

$$I_{xy} = -(1 \text{ kg})(2 \text{ m})(3 \text{ m}) = -6 \text{ kg m}^2$$



Knowing particle Q has a mass of 1 kg and each tick-mark represents 1 m, calculate Q 's *product of inertia* I_{xy} about point O for each figure below.



Circle the correct answer (negative, zero, or positive) for each statement about particle Q .

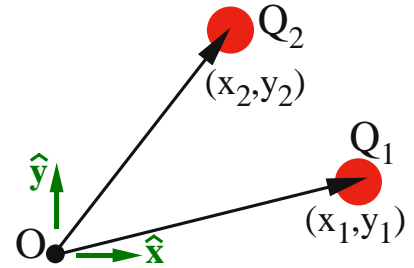
- When Q is in quadrant **I**, I_{xy} is negative/zero/positive.
- When Q is in quadrant **II**, I_{xy} is negative/zero/positive.
- When Q is in quadrant **III**, I_{xy} is negative/zero/positive.
- When Q is in quadrant **IV**, I_{xy} is negative/zero/positive.
- When Q is on a quadrant boundary, I_{xy} is negative/zero/positive.

12.5 ♣ Calculations: Product of inertia for a system of particles (Section 14.2).

The **product of inertia** of a system of particles is simply the sum of the products of inertias of each of the individual particles. For example, the product of inertia of particles Q_1 and Q_2 about point O for the \hat{x} and \hat{y} directions is calculated by the formula

$$I_{xy} = \sum_{i=1}^2 -m_i x_i y_i$$

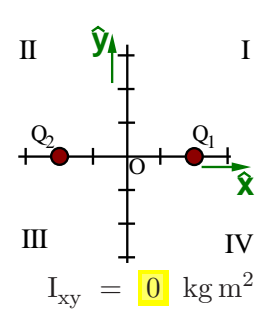
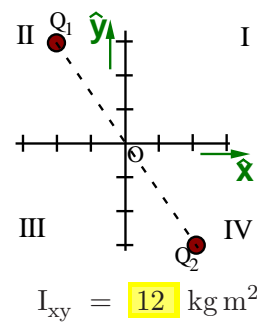
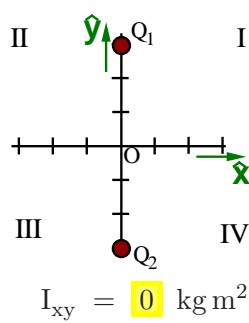
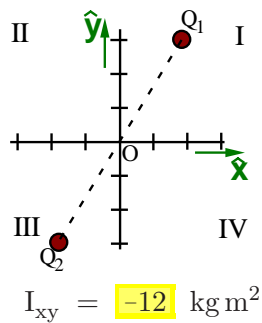
- m_i is the mass of particle Q_i ($i=1, 2$)
- x_i is the \hat{x} measure of Q_i 's position vector from O
- y_i is the \hat{y} measure of Q_i 's position vector from O



For example, if $m_1 = 1$ kg, $x_1 = 3$ m, $y_1 = 1$ m, and $m_2 = 2$ kg, $x_2 = 2$ m, $y_2 = 3$ m,

$$I_{xy} = -m_1 x_1 y_1 + -m_2 x_2 y_2 = -(1 \text{ kg})(3 \text{ m})(1 \text{ m}) + -(2 \text{ kg})(2 \text{ m})(3 \text{ m}) = -15 \text{ kg m}^2$$

Knowing each particle has a mass of 1 kg and each tick-mark represents 1 m, calculate the system's **product of inertia** I_{xy} about point O for each figure below.

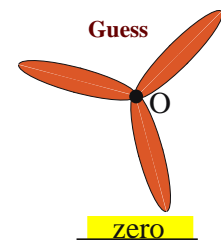
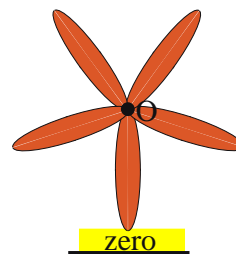
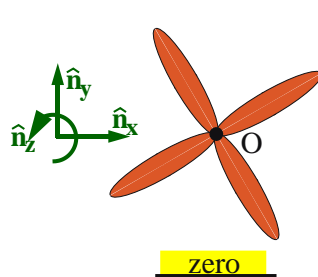
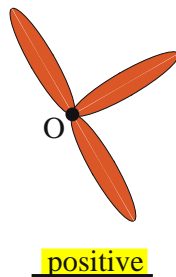


Circle the correct answer (negative, zero, or positive) for each of the following statements.

- When the particles are in quadrants **I** and **III**, I_{xy} is **negative**/zero/positive.
- When the particles are in quadrants **II** and **IV**, I_{xy} is negative/zero/**positive**.
- When the particles are on quadrant boundaries, I_{xy} is negative/**zero**/positive.

12.6 ♣ Concepts: Products of inertia of propellers (Section 14.2.2).

The following shows four uniform-density objects. For each object, consider I_{xy} the product of inertia of the object for lines that pass through point O and are parallel to \hat{n}_x and \hat{n}_y . For each object, **visually determine** whether the product of inertia is **negative**, **zero**, or **positive**.

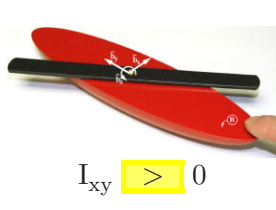
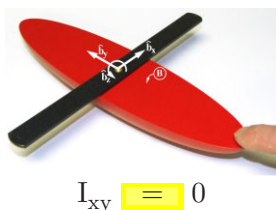
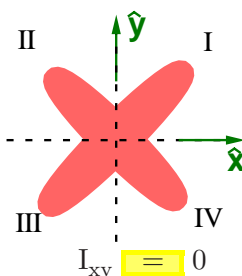
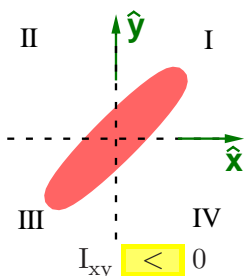
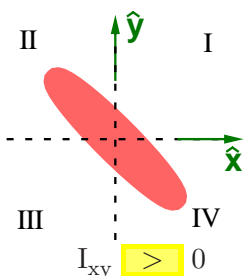
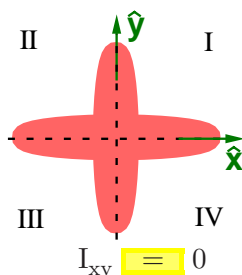
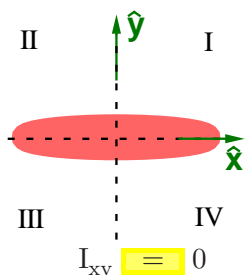
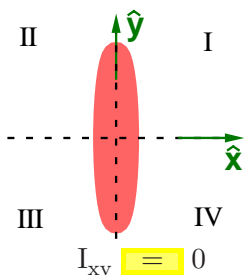
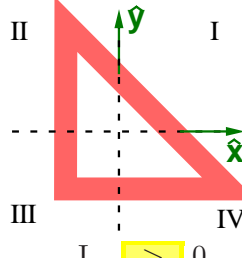
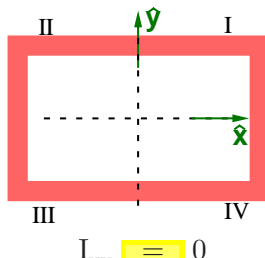
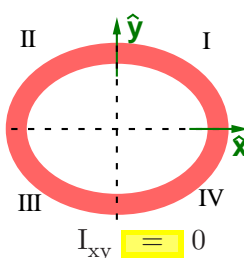
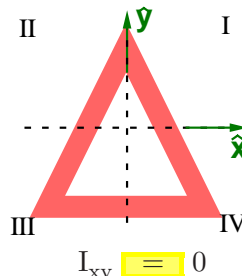
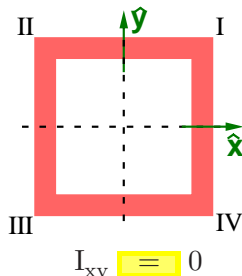
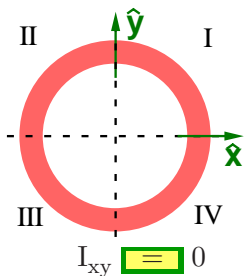


12.7 ♣ **Concepts: Products of inertia – what is I_{xy} ?** (Section 14.2.2).

Product of inertia is a measure of the symmetry of mass distribution in two directions about a point. To investigate this concept, use your geometrical insights (not equations) to determine which of the following uniform-density objects have a negative, zero, or positive **product of inertia** I_{xy} .

Visually sum the mass distribution for I_{xy} in quadrants **II** and **IV** and compare that to the mass distribution in quadrants **I** and **III**. Complete each blank below with **<** or **=** or **>**.

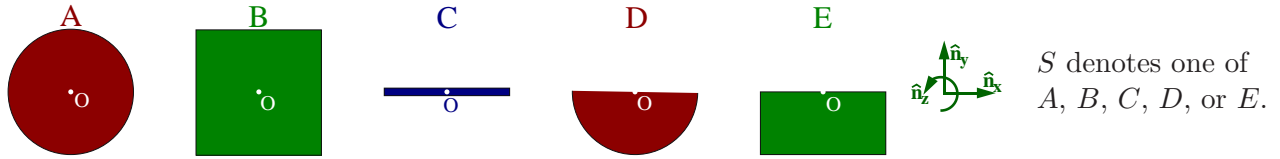
- If mass distribution in quadrants **II** + **IV** is greater than in quadrants **I** + **III**, $I_{xy} > 0$.
- If mass distribution in quadrants **II** + **IV** is smaller than in quadrants **I** + **III**, $I_{xy} < 0$.
- If mass distribution in quadrants **II** + **IV** is equal to quadrants **I** + **III**, $I_{xy} = 0$.



Purchase rattleback at www.arbor-sci.com. Explained: <http://www.youtube.com/watch?v=0RyLV-Fs14A>

12.8 ♣ Conceptual understanding of moments and products of inertia (Sections 14.1.2 and 14.2.2).

Objects *A*, *B*, *C*, *D*, and *E* are all flat planar objects with uniform density and the *same* mass. The circle and semi-circle's diameter, square and rectangle's width, and thin rod's length are *equal*.



36% Consider $I_{zz}^{S/O}$, *S*'s moment of inertia about the line passing through point *O* and parallel to \hat{n}_z . Knowing moment of inertia is mass * distance², use **visual estimates** to list the objects in ascending order of $I_{zz}^{S/O}$. If two objects have the same value of $I_{zz}^{S/O}$, group them together.

Result:

Smallest	C		D, A		E, B	Largest
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28% Consider $I_{zz}^{S/S_{cm}}$, *S*'s moment of inertia about the line passing through S_{cm} (the mass center of *S*) and parallel to \hat{n}_z . Use visual estimates to list the objects in ascending order of $I_{zz}^{S/S_{cm}}$. Note: *A* and *E* have nearly equal $I_{zz}^{S/S_{cm}}$. The textbook's inertia appendix helps resolve their difference.

Result:

Smallest	D (given)	C	E	A	B	Largest
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23% Consider $I_{xy}^{S/O}$, *S*'s product of inertia for point *O* and unit vectors \hat{n}_x and \hat{n}_y . For each object, visually determine if $I_{xy}^{S/O}$ is negative (-), zero (0), or positive (+).

Result:

A	B	C	D	E
- 0 +	- 0 +	- 0 +	- 0 +	- 0 +

12.9 ♣ Assembling inertia dyadics from the textbook's inertia appendix (Sections 16.1 and 16.2).

Referring to the following figures and the textbook's inertia appendix, assemble $\overset{\rightarrow}{\mathbf{I}}^{B/B_{cm}}$ (*B*'s inertia dyadic about its center of mass B_{cm}) for the unit vectors \hat{b}_x , \hat{b}_y , \hat{b}_z . Express results in matrix form.

$$\begin{bmatrix} \frac{1}{4} m b^2 & 0 & 0 \\ 0 & \frac{1}{4} m a^2 & 0 \\ 0 & 0 & \frac{1}{4} m (a^2 + b^2) \end{bmatrix}_{\hat{b}_{xyz}}$$

$$\begin{bmatrix} \frac{1}{12} m (b^2 + c^2) & 0 & 0 \\ 0 & \frac{1}{12} m (a^2 + c^2) & 0 \\ 0 & 0 & \frac{1}{12} m (a^2 + b^2) \end{bmatrix}_{\hat{b}_{xyz}}$$

$$\frac{2}{5} m r^2 * \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{\hat{b}_{xyz}} = \frac{2}{5} m r^2 \overset{\rightarrow}{\mathbf{1}}$$

$$\begin{bmatrix} \frac{1}{12} m (3r^2 + h^2) & 0 & 0 \\ 0 & \frac{1}{12} m (3r^2 + h^2) & 0 \\ 0 & 0 & \frac{1}{2} m r^2 \end{bmatrix}_{\hat{b}_{xyz}}$$

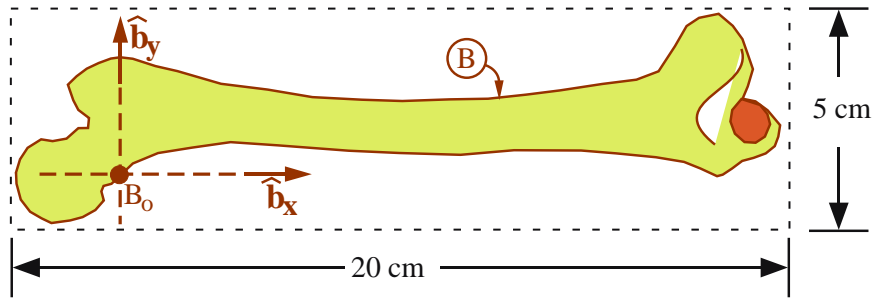
12.10 ♣ **Biomechanics: Estimating mass distribution properties of a bone** (Sections 14.1 and 14.2).

The following figure shows a point B_o fixed on a relatively **thin**, uniform-density, rigid bone B . Right-handed orthogonal unit vectors $\hat{\mathbf{b}}_x, \hat{\mathbf{b}}_y, \hat{\mathbf{b}}_z$ are fixed in B .

Estimate and **draw** the location of B_{cm} (B 's center of mass) on the figure.

Hint: Draw horizontal (parallel to $\hat{\mathbf{b}}_x$) and vertical (parallel to $\hat{\mathbf{b}}_y$) lines passing through B_o . Similarly for B_{cm} .

Note: B_o is midway through the bone so $\vec{\mathbf{r}}^{B_{cm}/B_o} \cdot \hat{\mathbf{n}}_z = 0$.



Knowing the 1 kg bone fits snugly in a 20 cm by 5 cm by 1 cm box, **visually estimate** the following values. Note: Other than 0.0 which is used 4 times, use each of the following values **once** in your table.

-13 -2 0.0 1.0 3.0 30 31 86 89

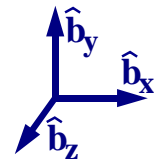
Description	Symbol	Approximate value
$\hat{\mathbf{b}}_x$ measure of B_{cm} 's position vector from B_o	x	7.5 cm
$\hat{\mathbf{b}}_y$ measure of B_{cm} 's position vector from B_o	y	1.5 cm
B 's moment of inertia about B_{cm} for $\hat{\mathbf{b}}_x$	$I_{xx}^{B/B_{cm}}$	1 kg cm ²
B 's moment of inertia about B_{cm} for $\hat{\mathbf{b}}_y$	$I_{yy}^{B/B_{cm}}$	30 kg cm ²
B 's moment of inertia about B_{cm} for $\hat{\mathbf{b}}_z$	$I_{zz}^{B/B_{cm}}$	31 kg cm ²
B 's product of inertia about B_{cm} for $\hat{\mathbf{b}}_x$ and $\hat{\mathbf{b}}_y$	$I_{xy}^{B/B_{cm}}$	-2 kg cm ²
B 's product of inertia about B_{cm} for $\hat{\mathbf{b}}_x$ and $\hat{\mathbf{b}}_z$	$I_{xz}^{B/B_{cm}}$	0 kg cm ²
B 's product of inertia about B_{cm} for $\hat{\mathbf{b}}_y$ and $\hat{\mathbf{b}}_z$	$I_{yz}^{B/B_{cm}}$	0 kg cm ²
B 's moment of inertia about B_o for $\hat{\mathbf{b}}_x$	I_{xx}^{B/B_o}	3 kg cm ²
B 's moment of inertia about B_o for $\hat{\mathbf{b}}_y$	I_{yy}^{B/B_o}	86 kg cm ²
B 's moment of inertia about B_o for $\hat{\mathbf{b}}_z$	I_{zz}^{B/B_o}	89 kg cm ²
B 's product of inertia about B_o for $\hat{\mathbf{b}}_x$ and $\hat{\mathbf{b}}_y$	I_{xy}^{B/B_o}	-13 kg cm ²
B 's product of inertia about B_o for $\hat{\mathbf{b}}_x$ and $\hat{\mathbf{b}}_z$	I_{xz}^{B/B_o}	0 kg cm ²
B 's product of inertia about B_o for $\hat{\mathbf{b}}_y$ and $\hat{\mathbf{b}}_z$	I_{yz}^{B/B_o}	0 kg cm ²

Hint: Start with products of inertia, then smallest moments of inertia, then largest moments of inertia.
Hint: Draw a side-view of the bone to help estimate the two I_{yz} . Draw a top-view to estimate the two I_{xz} .

12.11 ♣ **Dyadics and dot-products with orthogonal unit vectors** $\hat{\mathbf{b}}_x, \hat{\mathbf{b}}_y, \hat{\mathbf{b}}_z$ (Section 5.1).

Non-symmetric dyadic: $\vec{\vec{\mathbf{N}}} = \hat{\mathbf{b}}_x \hat{\mathbf{b}}_x + 2 \hat{\mathbf{b}}_y \hat{\mathbf{b}}_y + 6 \hat{\mathbf{b}}_y \hat{\mathbf{b}}_z + 3 \hat{\mathbf{b}}_z \hat{\mathbf{b}}_z$

Symmetric dyadic: $\vec{\vec{\mathbf{S}}} = \hat{\mathbf{b}}_x \hat{\mathbf{b}}_x + 2 \hat{\mathbf{b}}_y \hat{\mathbf{b}}_y + 6 \hat{\mathbf{b}}_y \hat{\mathbf{b}}_z + 6 \hat{\mathbf{b}}_z \hat{\mathbf{b}}_y + 3 \hat{\mathbf{b}}_z \hat{\mathbf{b}}_z$



Given the previous dyadics $\vec{\vec{\mathbf{N}}}$ and $\vec{\vec{\mathbf{S}}}$, calculate the following dot-products.

Complete the last statements with **=** or **≠** (extrapolate/guess).

<i>Non-symmetric dyadic</i>	<i>Symmetric dyadic</i>
$\vec{\vec{\mathbf{N}}} \cdot (\hat{\mathbf{b}}_x + \hat{\mathbf{b}}_y) = \hat{\mathbf{b}}_x + 2 \hat{\mathbf{b}}_y$	$\vec{\vec{\mathbf{S}}} \cdot (\hat{\mathbf{b}}_x + \hat{\mathbf{b}}_y) = \hat{\mathbf{b}}_x + 2 \hat{\mathbf{b}}_y + 6 \hat{\mathbf{b}}_z$
$(\hat{\mathbf{b}}_x + \hat{\mathbf{b}}_y) \cdot \vec{\vec{\mathbf{N}}} = \hat{\mathbf{b}}_x + 2 \hat{\mathbf{b}}_y + 6 \hat{\mathbf{b}}_z$	$(\hat{\mathbf{b}}_x + \hat{\mathbf{b}}_y) \cdot \vec{\vec{\mathbf{S}}} = \hat{\mathbf{b}}_x + 2 \hat{\mathbf{b}}_y + 6 \hat{\mathbf{b}}_z$
$\vec{\vec{\mathbf{N}}} \cdot \text{any Vector} \neq \text{any Vector} \cdot \vec{\vec{\mathbf{N}}}$	$\vec{\vec{\mathbf{S}}} \cdot \text{any Vector} = \text{any Vector} \cdot \vec{\vec{\mathbf{S}}}$