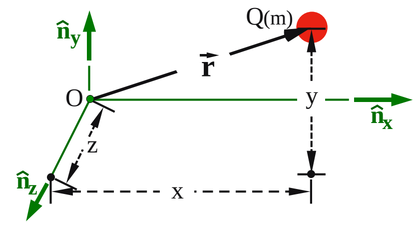
10.1 ♣ Concepts: What objects have a moment of inertia? (Section 12.1).

Consider the ***moment of inertia*** $\mathrm{I}_{\hat{\mathbf{u}}\,\hat{\mathbf{u}}}^{S/O}$ of an object S about a point O for the unit vector $\hat{\mathbf{u}}$.
In general, for $\mathrm{I}_{\hat{\mathbf{u}}\,\hat{\mathbf{u}}}^{S/O}$ to be a positive real number, S should be a (circle ***all*** appropriate objects):

Complex number	Point	Reference Frame	Center of mass of a set of particles
Vector	Set of Points	Rigid Body	Center of mass of a rigid body
Matrix	Particle	Flexible Body	Set of flexible bodies
Orthogonal unit basis	Set of Particles	Set of Rigid bodies	System of particles and bodies

10.2 ♣ Formulas for a particle's moments and products of inertia (Sections 12.1 and 12.2).

The figure shows a particle Q of mass m and right-handed orthogonal unit vectors $\hat{\mathbf{n}}_x$, $\hat{\mathbf{n}}_y$, $\hat{\mathbf{n}}_z$.
Q's position vector from a point O is $x\,\hat{\mathbf{n}}_x + y\,\hat{\mathbf{n}}_y + z\,\hat{\mathbf{n}}_z$.



Express I_{xx} (Q's ***moment of inertia*** about O for $\hat{\mathbf{n}}_x$) in terms of some or all of m, x, y z. Similarly for I_{yy} and I_{zz}.

Express I_{xy} (Q's ***product of inertia*** about O for $\hat{\mathbf{n}}_x$ and $\hat{\mathbf{n}}_y$) in terms of some or all of m, x, y z. Similarly for I_{xz} and I_{yz}.

Result:

$\mathrm{I}_{xx} = \_\_ (\_\_ + \_\_)$ $\quad$ $\mathrm{I}_{yy} = \_\_$ $\quad$ $\mathrm{I}_{zz} = \_\_$

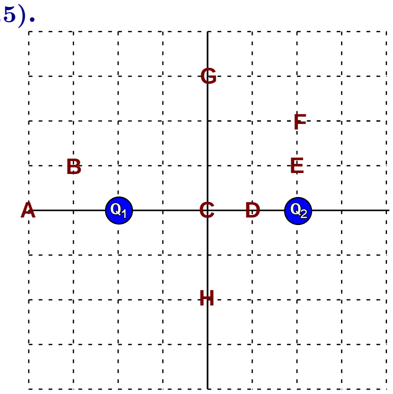
$\mathrm{I}_{xy} = -\_\_\,\_\_\,\_\_$ $\quad$ $\mathrm{I}_{xz} = \_\_$ $\quad$ $\mathrm{I}_{yz} = \_\_$

Circa 1895, Gibbs invented the ***inertia dyadic*** as a **convenient "suitcase"** for holding moments and products of inertia. Write Q's inertia dyadic about O in terms of $\hat{\mathbf{n}}_x$, $\hat{\mathbf{n}}_y$, $\hat{\mathbf{n}}_z$ and I_{ij} $(i, j = \mathrm{x}, \mathrm{y}, \mathrm{z})$. If needed, refer to Section 14.1.

$$\begin{aligned}\vec{\vec{\mathbf{I}}} \;=\; & \mathrm{I}_{xx}\,\hat{\mathbf{n}}_x\,\hat{\mathbf{n}}_x + \mathrm{I}_{xy}\,\hat{\mathbf{n}}_x\,\hat{\mathbf{n}}_y + \_\_\,\hat{\mathbf{n}}_x\,\hat{\mathbf{n}}_z \\ & + \mathrm{I}_{xy}\,\hat{\mathbf{n}}_y\,\hat{\mathbf{n}}_x + \_\_\,\hat{\mathbf{n}}_y\,\hat{\mathbf{n}}_y + \_\_ \\ & + \_\_ + \_\_ + \_\_\end{aligned}$$

10.3 ♣ Parallel axis theorem and moments of inertia (Section 12.1.5).

The system S shown to the right consists of particles Q_1 and Q_2, each of mass m, in a plane perpendicular to the unit vector $\hat{\mathbf{n}}_z$.



The ***shift theorem*** (also called the ***parallel axis theorem***) shifts S's moment of inertia about S_{cm} (the ***mass center*** of S) for the unit vector $\hat{\mathbf{n}}_z$ to an arbitrary point P in the plane using

$$\boxed{\mathrm{I}_{zz}^{S/P} \;=\; \mathrm{I}_{zz}^{S/S_{\mathrm{cm}}} \;+\; m^S * d^2}$$

where $\mathrm{I}_{zz}^{S/S_{\mathrm{cm}}}$ is the system's moment of inertia about S_{cm} for $\hat{\mathbf{n}}_z$, m^S is the mass of S, and d is the distance from S_{cm} to P.

Use the shift theorem to estimate the order of S's moment of inertia for the lines parallel to $\hat{\mathbf{n}}_z$ that pass through points A, B, C, D, E, F, G, and H, respectively. Note: Grid lines are equally spaced.

Result:

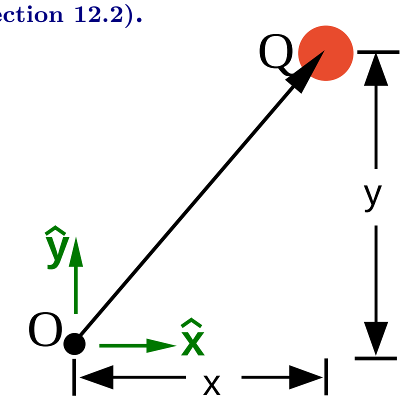
Smallest									Largest

Knowing each particle has mass $m = 1$ kg and the grid lines are spaced 1 m apart, calculate S's moment of inertia about A, B, C, D, E, F, G, and H, respectively.

Result: (in $\mathrm{kg\,m^2}$)

$\mathrm{I}_{zz}^{S/A}$	$\mathrm{I}_{zz}^{S/B}$	$\mathrm{I}_{zz}^{S/C}$	$\mathrm{I}_{zz}^{S/D}$	$\mathrm{I}_{zz}^{S/E}$	$\mathrm{I}_{zz}^{S/F}$	$\mathrm{I}_{zz}^{S/G}$	$\mathrm{I}_{zz}^{S/H}$
		8					

10.4 ♣ Calculations: Product of inertia for a single particle (Section 12.2).



The ***product of inertia*** of a single particle Q about a point O for the $\widehat{\mathbf{x}}$ and $\widehat{\mathbf{y}}$ directions is calculated by the formula

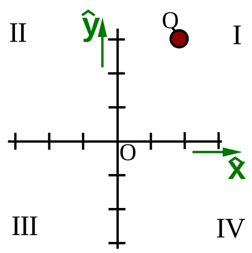
$$\boxed{\text{I}_{\text{xy}} \;=\; -m\,x\,y}$$

- m is the mass of Q
- x is the $\widehat{\mathbf{x}}$ measure of Q's position vector from O
- y is the $\widehat{\mathbf{y}}$ measure of Q's position vector from O

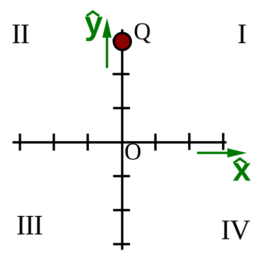
For example, if $m = 1$ kg, $x = 2$ m, and $y = 3$ m,

$$\text{I}_{\text{xy}} \;=\; -(1\text{ kg})\,(2\text{ m})\,(3\text{ m}) \;=\; -6\text{ kg m}^2$$

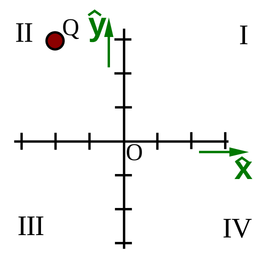
Knowing particle Q has a mass of 1 kg and each tick-mark represents 1 m, calculate Q's ***product of inertia*** I_{xy} about point O for each figure below.



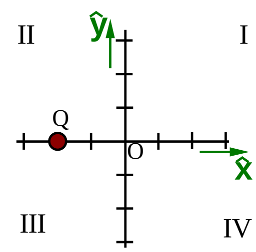
$\text{I}_{\text{xy}} = -6\ \text{kg m}^2$



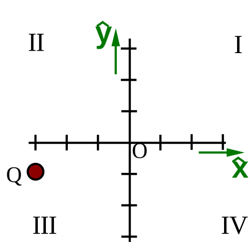
$\text{I}_{\text{xy}} = \square\ \text{kg m}^2$



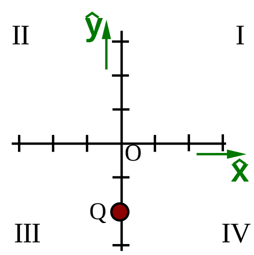
$\text{I}_{\text{xy}} = \square\ \text{kg m}^2$



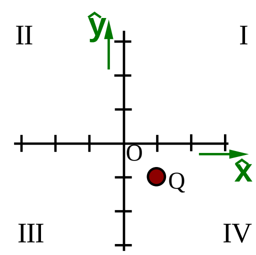
$\text{I}_{\text{xy}} = \square\ \text{kg m}^2$



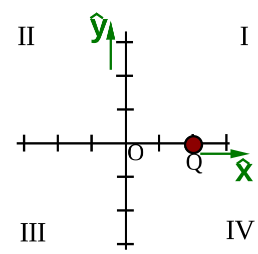
$\text{I}_{\text{xy}} = -3\ \text{kg m}^2$



$\text{I}_{\text{xy}} = \square\ \text{kg m}^2$



$\text{I}_{\text{xy}} = \square\ \text{kg m}^2$



$\text{I}_{\text{xy}} = \square\ \text{kg m}^2$

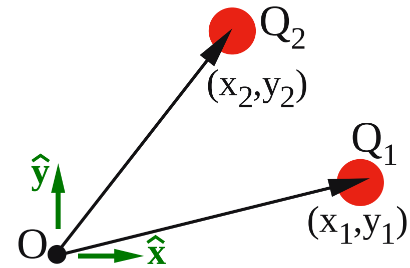
Circle the correct answer (negative, zero, or positive) for each statement about particle Q.

- When Q is in quadrant **I**, I_{xy} is **negative/zero/positive**.
- When Q is in quadrant **II**, I_{xy} is **negative/zero/positive**.
- When Q is in quadrant **III**, I_{xy} is **negative/zero/positive**.
- When Q is in quadrant **IV**, I_{xy} is **negative/zero/positive**.
- When Q is on a quadrant boundary, I_{xy} is **negative/zero/positive**.

10.5 ♣ Calculations: Product of inertia for a system of particles (Section 12.2).

The ***product of inertia*** of a system of particles is simply the sum of the products of inertias of each of the individual particles. For example, the product of inertia of particles Q_1 and Q_2 about point O for the $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ directions is calculated by the formula

$$\boxed{\mathrm{I}_{\mathrm{xy}} \;=\; \sum_{i=1}^{2} -m_i \, x_i \, y_i}$$

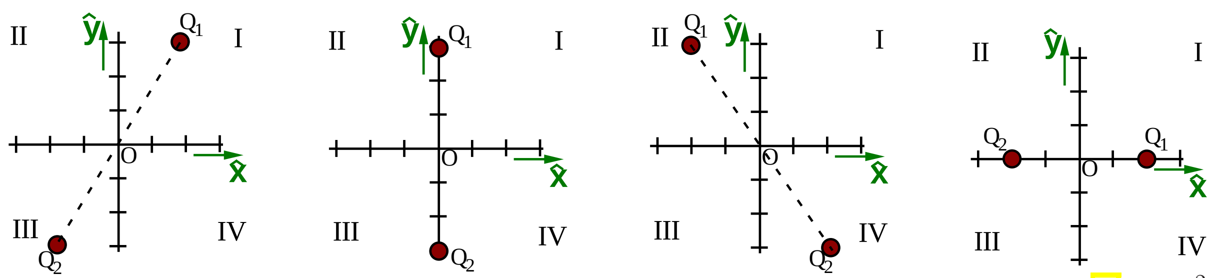


- m_i is the mass of particle Q_i (i=1, 2)
- x_i is the $\hat{\mathbf{x}}$ measure of Q_i's position vector from O
- y_i is the $\hat{\mathbf{y}}$ measure of Q_i's position vector from O

For example, if $m_1 = 1$ kg, $x_1 = 3$ m, $y_1 = 1$ m, and $m_2 = 2$ kg, $x_2 = 2$ m, $y_2 = 3$ m,

$$\mathrm{I}_{\mathrm{xy}} \;=\; -m_1\, x_1\, y_1 \;+\; -m_2\, x_2\, y_2 \;=\; -(1\text{ kg})\,(3\text{ m})\,(1\text{ m}) \;+\; -(2\text{ kg})\,(2\text{ m})\,(3\text{ m}) \;=\; -15\text{ kg m}^2$$

Knowing each particle has a mass of 1 kg and each tick-mark represents 1 m, calculate the system's ***product of inertia*** I_{xy} about point O for each figure below.



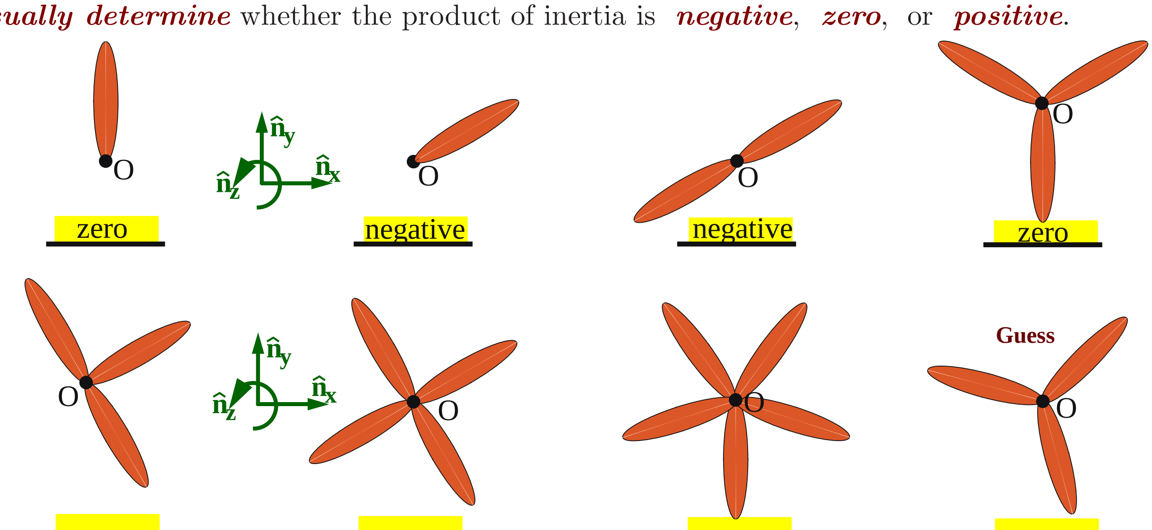
$\mathrm{I}_{\mathrm{xy}} =$ ___ kg m^2 $\quad$ $\mathrm{I}_{\mathrm{xy}} =$ ___ kg m^2 $\quad$ $\mathrm{I}_{\mathrm{xy}} =$ ___ kg m^2 $\quad$ $\mathrm{I}_{\mathrm{xy}} =$ ___ kg m^2

Circle the correct answer (negative, zero, or positive) for each of the following statements.

- When the particles are in quadrants **I** and **III**, I_{xy} is **negative/zero/positive**.
- When the particles are in quadrants **II** and **IV**, I_{xy} is **negative/zero/positive**.
- When the particles are on quadrant boundaries, I_{xy} is **negative/zero/positive**.

10.6 ♣ Concepts: Products of inertia of propellers (Section 12.2.2).

The following shows four uniform-density objects. For each object, consider I_{xy} the product of inertia of the object for lines that pass through point O and are parallel to $\hat{\mathbf{n}}_{\mathrm{x}}$ and $\hat{\mathbf{n}}_{\mathrm{y}}$. For each object, ***visually determine*** whether the product of inertia is ***negative***, ***zero***, or ***positive***.

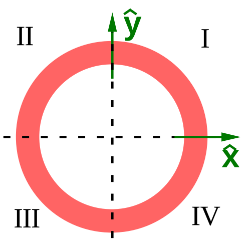


10.7 ♣ Concepts: Products of inertia – what is I_{xy}? (Section 12.2.2).

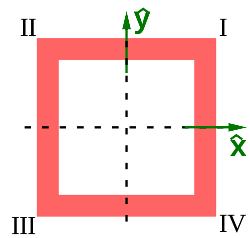
Product of inertia is a measure of the symmetry of mass distribution in two directions about a point. To investigate this concept, use your geometrical insights (not equations) to determine which of the following uniform-density objects have a negative, zero, or positive ***product of inertia*** I_{xy}.

Visually sum the mass distribution for I_{xy} in quadrants **II** and **IV** and compare that to the mass distribution in quadrants **I** and **III**. Complete each blank below with $<$ or $=$ or $>$.

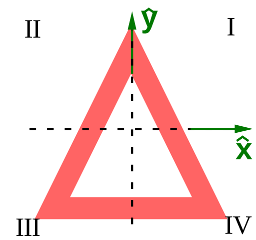
- If mass distribution in quadrants **II** + **IV** is greater than in quadrants **I** + **III**, $I_{xy} > 0$.
- If mass distribution in quadrants **II** + **IV** is smaller than in quadrants **I** + **III**, $I_{xy} < 0$.
- If mass distribution in quadrants **II** + **IV** is equal to quadrants **I** + **III**, $I_{xy} = 0$.



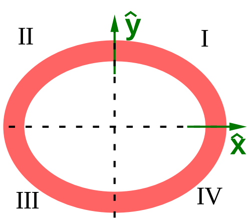
I_{xy} $=$ 0



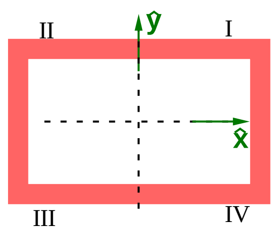
I_{xy} ___ 0



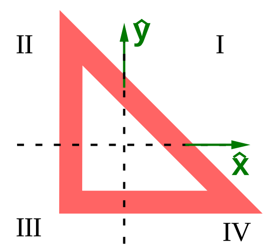
I_{xy} ___ 0



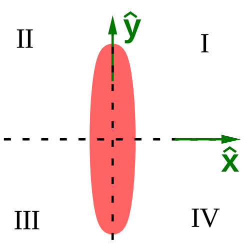
I_{xy} ___ 0



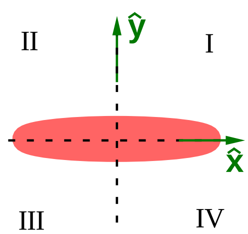
I_{xy} ___ 0



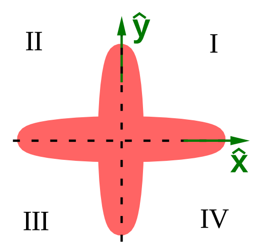
I_{xy} ___ 0



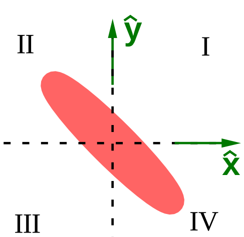
I_{xy} ___ 0



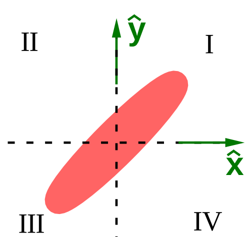
I_{xy} ___ 0



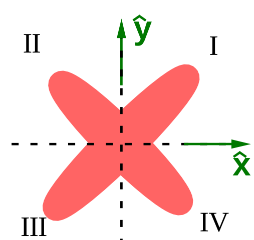
I_{xy} ___ 0



I_{xy} ___ 0



I_{xy} ___ 0



I_{xy} ___ 0

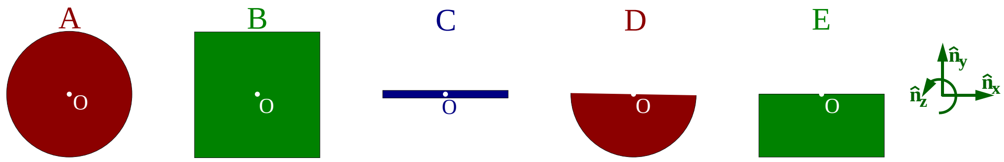

I_{xy} ___ 0


I_{xy} ___ 0


I_{xy} ___ 0

Purchase rattleback at www.arbor-sci.com. Explained: http://www.youtube.com/watch?v=0RyLV-Fsl4A

10.8 ♣ Conceptual understanding of moments and products of inertia (Sections 12.1.2 and 12.2.2).
Objects A, B, C, D, and E are all flat planar objects with uniform density and the ***same*** mass. The circle and semi-circle's diameter, square and rectangle's width, and thin rod's length are ***equal***.



S denotes one of A, B, C, D, or E.

36% Consider $I_{zz}^{S/O}$, S's moment of inertia about the line passing through point O and parallel to $\hat{\mathbf{n}}_z$. Knowing moment of inertia is mass $*$ distance^2, use **visual estimates** to list the objects in ascending order of $I_{zz}^{S/O}$. If two objects have the same value of $I_{zz}^{S/O}$, group them together.

Result:

Smallest						Largest

28% Consider $I_{zz}^{S/S_{cm}}$, S's moment of inertia about the line passing through S_{cm} (the mass center of S) and parallel to $\hat{\mathbf{n}}_z$. Use visual estimates to list the objects in ascending order of $I_{zz}^{S/S_{cm}}$.
Note: A and E have nearly equal $I_{zz}^{S/S_{cm}}$. The textbook's inertia appendix helps resolve their difference.

Result:

Smallest	D (**given**)					Largest

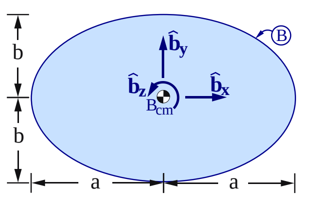
23% Consider $I_{xy}^{S/O}$, S's product of inertia for point O and unit vectors $\hat{\mathbf{n}}_x$ and $\hat{\mathbf{n}}_y$. For each object, visually determine if $I_{xy}^{S/O}$ is negative $(-)$, zero (0), or positive $(+)$.

Result:

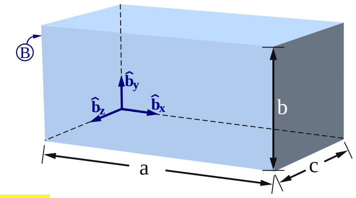
A	B	C	D	E
$-\ 0\ +$	$-\ 0\ +$	$-\ 0\ +$	$-\ 0\ +$	$-\ 0\ +$

10.9 ♣ Assembling inertia dyadics from the textbook's inertia appendix (Sections 14.1 and 14.4).

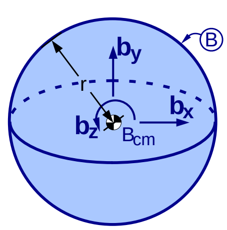
Referring to the following figures and the textbook's inertia appendix, assemble $\vec{\vec{I}}^{B/B_{cm}}$ (B's inertia dyadic about its center of mass B_{cm}) for the unit vectors $\hat{\mathbf{b}}_x$, $\hat{\mathbf{b}}_y$, $\hat{\mathbf{b}}_z$. Express results in matrix form.



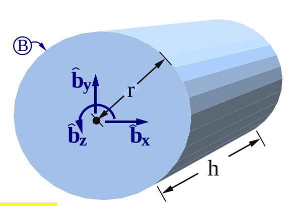
$$\begin{bmatrix} \square & \square & 0 \\ \square & \boxed{\frac{1}{4} m a^2} & 0 \\ 0 & 0 & \square \end{bmatrix}_{\hat{\mathbf{b}}_{xyz}}$$



$$\begin{bmatrix} \square & \square & \square \\ \square & \square & \square \\ \square & \square & \square \end{bmatrix}_{\hat{\mathbf{b}}_{xyz}}$$



$$\square * \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{\hat{\mathbf{b}}_{xyz}} = \square \ \vec{\vec{1}}$$



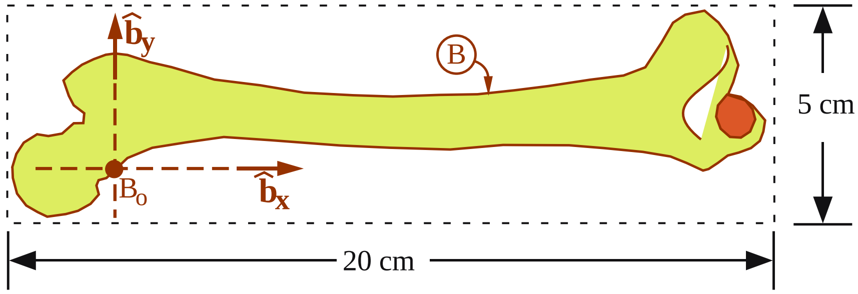
$$\begin{bmatrix} \square & \square & \square \\ \square & \square & \square \\ \square & \square & \square \end{bmatrix}_{\hat{\mathbf{b}}_{xyz}}$$

10.10 ♣ Biomechanics: Estimating mass distribution properties of a bone (Sections 12.1 and 12.2).

The following figure shows a relatively **thin**, uniform-density, rigid bone B. Right-handed orthogonal unit vectors $\widehat{\mathbf{b}}_x$, $\widehat{\mathbf{b}}_y$, $\widehat{\mathbf{b}}_z$ are fixed in B. A point B_o of B is midway though the bone ($\vec{\mathbf{r}}^{\,B_{cm}/B_o} \cdot \widehat{\mathbf{b}}_z = 0$).

Estimate and **draw** the location of B_{cm} (B's center of mass) ◐ on the figure.

Hint: Draw horizontal (parallel to $\widehat{\mathbf{b}}_x$) and vertical (parallel to $\widehat{\mathbf{b}}_y$) lines passing through B_o. Similarly for B_{cm}.



Knowing the 1 kg bone fits snugly in a 20 cm by 5 cm by 1 cm box, ***visually estimate*** the following values. Note: Other than 0.0 which is used 4 times, use each of the following values ***once*** in your table.

−13   −2   0.0   1.0   3.0   30   31   86   89

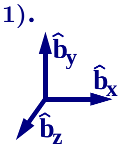
Description	Symbol	Approximate value
$\widehat{\mathbf{b}}_x$ measure of B_{cm}'s position vector from B_o	x	7.5 cm
$\widehat{\mathbf{b}}_y$ measure of B_{cm}'s position vector from B_o	y	1.5 cm
B's moment of inertia about B_{cm} for $\widehat{\mathbf{b}}_x$	$I_{xx}^{B/B_{cm}}$	$\boxed{}$ kg cm^2
B's moment of inertia about B_{cm} for $\widehat{\mathbf{b}}_y$	$I_{yy}^{B/B_{cm}}$	$\boxed{}$ kg cm^2
B's moment of inertia about B_{cm} for $\widehat{\mathbf{b}}_z$	$I_{zz}^{B/B_{cm}}$	$\boxed{}$ kg cm^2
B's product of inertia about B_{cm} for $\widehat{\mathbf{b}}_x$ and $\widehat{\mathbf{b}}_y$	$I_{xy}^{B/B_{cm}}$	$\boxed{}$ kg cm^2
B's product of inertia about B_{cm} for $\widehat{\mathbf{b}}_x$ and $\widehat{\mathbf{b}}_z$	$I_{xz}^{B/B_{cm}}$	$\boxed{}$ kg cm^2
B's product of inertia about B_{cm} for $\widehat{\mathbf{b}}_y$ and $\widehat{\mathbf{b}}_z$	$I_{yz}^{B/B_{cm}}$	$\boxed{}$ kg cm^2
B's moment of inertia about B_o for $\widehat{\mathbf{b}}_x$	I_{xx}^{B/B_o}	$\boxed{}$ kg cm^2
B's moment of inertia about B_o for $\widehat{\mathbf{b}}_y$	I_{yy}^{B/B_o}	$\boxed{}$ kg cm^2
B's moment of inertia about B_o for $\widehat{\mathbf{b}}_z$	I_{zz}^{B/B_o}	$\boxed{}$ kg cm^2
B's product of inertia about B_o for $\widehat{\mathbf{b}}_x$ and $\widehat{\mathbf{b}}_y$	I_{xy}^{B/B_o}	$\boxed{}$ kg cm^2
B's product of inertia about B_o for $\widehat{\mathbf{b}}_x$ and $\widehat{\mathbf{b}}_z$	I_{xz}^{B/B_o}	$\boxed{}$ kg cm^2
B's product of inertia about B_o for $\widehat{\mathbf{b}}_y$ and $\widehat{\mathbf{b}}_z$	I_{yz}^{B/B_o}	$\boxed{}$ kg cm^2

Hint: Start with products of inertia, then smallest moments of inertia, then largest moments of inertia.
Hint: Draw a side-view of the bone to help estimate the two I_{yz}. Draw a top-view to estimate the two I_{xz}.

10.11 ♣ Dyadics and dot-products with orthogonal unit vectors $\widehat{\mathbf{b}}_x$, $\widehat{\mathbf{b}}_y$, $\widehat{\mathbf{b}}_z$ (Section 13.1).

Non-symmetric dyadic: $\overset{\Rightarrow}{\mathbf{N}} = \widehat{\mathbf{b}}_x\widehat{\mathbf{b}}_x + 2\,\widehat{\mathbf{b}}_y\widehat{\mathbf{b}}_y + 6\,\widehat{\mathbf{b}}_y\widehat{\mathbf{b}}_z + 3\,\widehat{\mathbf{b}}_z\widehat{\mathbf{b}}_z$

Symmetric dyadic: $\overset{\Rightarrow}{\mathbf{S}} = \widehat{\mathbf{b}}_x\widehat{\mathbf{b}}_x + 2\,\widehat{\mathbf{b}}_y\widehat{\mathbf{b}}_y + 6\,\widehat{\mathbf{b}}_y\widehat{\mathbf{b}}_z + 6\,\widehat{\mathbf{b}}_z\widehat{\mathbf{b}}_y + 3\,\widehat{\mathbf{b}}_z\widehat{\mathbf{b}}_z$



Non-symmetric dyadic	***Symmetric dyadic***
$\overset{\Rightarrow}{\mathbf{N}} \cdot (\widehat{\mathbf{b}}_x + \widehat{\mathbf{b}}_y) = \boxed{\widehat{\mathbf{b}}_x} + \boxed{2\,\widehat{\mathbf{b}}_y}$	$\overset{\Rightarrow}{\mathbf{S}} \cdot (\widehat{\mathbf{b}}_x + \widehat{\mathbf{b}}_y) = \boxed{} + \boxed{} + \boxed{}$
$(\widehat{\mathbf{b}}_x + \widehat{\mathbf{b}}_y) \cdot \overset{\Rightarrow}{\mathbf{N}} = \boxed{} + \boxed{} + \boxed{}$	$(\widehat{\mathbf{b}}_x + \widehat{\mathbf{b}}_y) \cdot \overset{\Rightarrow}{\mathbf{S}} = \boxed{} + \boxed{} + \boxed{}$
$\overset{\Rightarrow}{\mathbf{N}} \cdot \text{any}\vec{\text{Vector}} = \text{any}\vec{\text{Vector}} \cdot \overset{\Rightarrow}{\mathbf{N}}$ **True/False**	$\overset{\Rightarrow}{\mathbf{S}} \cdot \text{any}\vec{\text{Vector}} = \text{any}\vec{\text{Vector}} \cdot \overset{\Rightarrow}{\mathbf{S}}$ **True/False**