

10.1 ♣ **Concepts: What objects have a moment of inertia?** (Section 12.1).

Consider the **moment of inertia** $I_{\hat{u}\hat{u}}^{S/O}$ of an object S about a point O for the unit vector \hat{u} .
In general, for $I_{\hat{u}\hat{u}}^{S/O}$ to be a positive real number, S should be a (circle **all** appropriate objects):

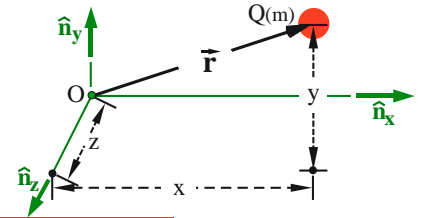
Real number	Matrix	Set of points	Mass center of a rigid body
Vector	Point	Reference frame	Flexible body
3D orthogonal unit basis	Particle	Rigid body	System of particles and bodies

10.2 ♣ **Formulas for a particle's moments and products of inertia** (Sections 12.1.1 and 12.2.1).

The figure shows a particle Q of mass m and right-handed orthogonal unit vectors $\hat{n}_x, \hat{n}_y, \hat{n}_z$.
 Q 's position vector from a point O is $x\hat{n}_x + y\hat{n}_y + z\hat{n}_z$.

Express I_{xx} (Q 's **moment of inertia** about O for \hat{n}_x) in terms of some or all of m, x, y, z . Similarly for I_{yy} and I_{zz} .

Express I_{xy} (Q 's **product of inertia** about O for \hat{n}_x and \hat{n}_y) in terms of some or all of m, x, y, z . Similarly for I_{xz} and I_{yz} .



Result:

$$\begin{aligned}
 I_{xx} &= \square (\square^2 + \square^2) & I_{yy} &= \square & I_{zz} &= \square \\
 I_{xy} &= -\square \square \square & I_{xz} &= \square & I_{yz} &= \square
 \end{aligned}$$

Circa 1895, Gibbs invented the **inertia dyadic** as a **convenient "suitcase"** for holding moments and products of inertia. Write Q 's inertia dyadic about O in terms of $\hat{n}_x, \hat{n}_y, \hat{n}_z$ and I_{ij} ($i, j = x, y, z$).
Hint: Pattern match or refer to Section 14.1.

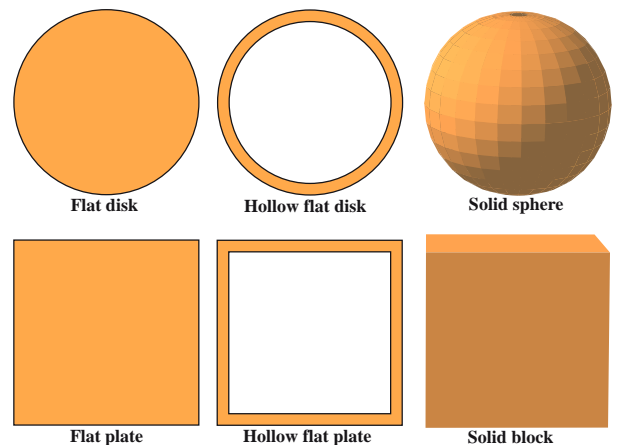
$$\begin{aligned}
 \hat{\mathbb{I}} &= I_{xx} \hat{n}_x \hat{n}_x + I_{xy} \hat{n}_x \hat{n}_y + \square \hat{n}_x \hat{n}_z \\
 &+ I_{xy} \hat{n}_y \hat{n}_x + \square \hat{n}_y \hat{n}_y + \square \\
 &+ \square + \square + \square
 \end{aligned}$$

10.3 ♣ **Moment of inertia concepts** (Section 12.1.2).

Each object below has uniform density and a mass of 1 kg. One can visually determine the relative size of each object's moment of inertia I_{zz} about the line perpendicular to the plane of the paper that passes through its center of mass.

Flat disk	<input type="checkbox"/>	Hollow flat disk
Flat disk	<input type="checkbox"/>	Solid sphere
Hollow flat disk	<input type="checkbox"/>	Solid sphere
Flat plate	<input type="checkbox"/>	Hollow flat plate
Flat plate	<input type="checkbox"/>	Solid block
Hollow flat plate	<input type="checkbox"/>	Solid block
Flat disk	<input type="checkbox"/>	Flat plate
Hollow flat disk	<input type="checkbox"/>	Hollow flat plate
Solid sphere	<input type="checkbox"/>	Solid block

Since I_{zz} depends on mass * distance², objects with a higher mass concentration **further** from the mass center have the **larger** moment of inertia.

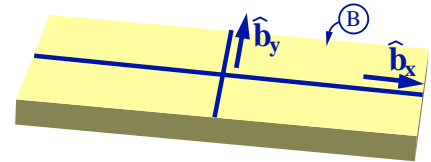


10.4 ♣ **Demo: How moment of inertia affects a spinning book** (Section 12.1.2).

Experiment: Spin a uniform rigid body B (such as a book with a rubber-band to keep it closed) about $\hat{\mathbf{b}}_x$, then $\hat{\mathbf{b}}_y$, then $\hat{\mathbf{b}}_z$. The spin can be “*neutrally stable*” (small perturbations of spin do not grow or decay – the book spins “smoothly”) or “*unstable*” (small perturbations grow exponentially – the book spins crazily).

Consider I_{xx} , I_{yy} , I_{zz} , B 's moments of inertia about B_{cm} (B 's center of mass) for $\hat{\mathbf{b}}_x$, $\hat{\mathbf{b}}_y$, $\hat{\mathbf{b}}_z$, respectively.

Axis to spin book	Moment of inertia & size	Stability
$\hat{\mathbf{b}}_x$	I_{xx} <input type="text"/>	neutrally stable
$\hat{\mathbf{b}}_y$	I_{yy} <input type="text"/>	unstable
$\hat{\mathbf{b}}_z$	I_{zz} <input type="text"/>	<input type="text"/>



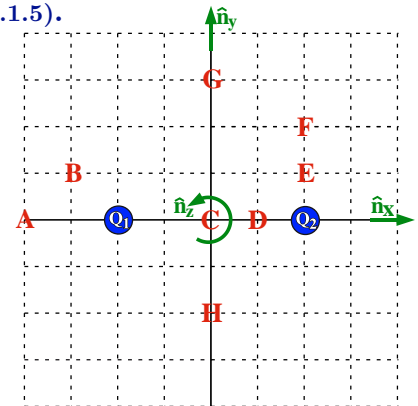
10.5 ♣ **Parallel axis theorem and moments of inertia** (Section 12.1.5).

The system S shown to the right consists of particles Q_1 and Q_2 , each of mass m , in a plane perpendicular to the unit vector $\hat{\mathbf{n}}_z$.

The *shift theorem* (also called the *parallel axis theorem*) shifts S 's moment of inertia about S_{cm} (the *mass center* of S) for the unit vector $\hat{\mathbf{n}}_z$ to an arbitrary point P in the plane using

$$I_{zz}^{S/P} = I_{zz}^{S/S_{cm}} + m^S * d^2$$

where $I_{zz}^{S/S_{cm}}$ is the system's moment of inertia about S_{cm} for $\hat{\mathbf{n}}_z$, m^S is the mass of S , and d is the distance from S_{cm} to P .



Use the shift theorem to estimate the order of S 's moment of inertia for the lines parallel to $\hat{\mathbf{n}}_z$ that pass through points A , B , C , D , E , F , G , and H , respectively. Note: Grid lines are equally spaced.

Result:

Smallest	<input type="text"/>	<input type="text"/>	<input type="text"/>	<input type="text"/>	<input type="text"/>	<input type="text"/>	<input type="text"/>	<input type="text"/>	<input type="text"/>	Largest
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Knowing each particle has mass $m = 1$ kg and the grid lines are spaced 1 m apart, calculate S 's moment of inertia about A , B , C , D , E , F , G , and H , respectively.

Result:

$I_{zz}^{S/A}$	$I_{zz}^{S/B}$	$I_{zz}^{S/C}$	$I_{zz}^{S/D}$	$I_{zz}^{S/E}$	$I_{zz}^{S/F}$	$I_{zz}^{S/G}$	$I_{zz}^{S/H}$
<input type="text"/>	<input type="text"/>	8	<input type="text"/>	<input type="text"/>	<input type="text"/>	<input type="text"/>	<input type="text"/>

(in kg m^2)

10.6 ♣ **Calculations: Product of inertia for a single particle** (Section 12.2).

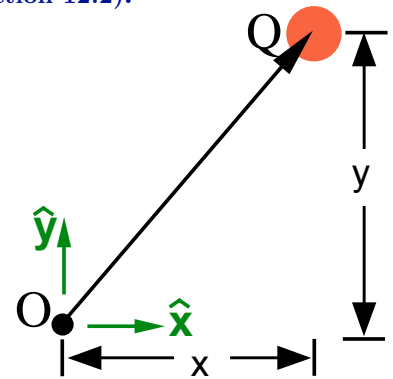
The *product of inertia* of a single particle Q about a point O for the \hat{x} and \hat{y} directions is calculated by the formula

$$I_{xy} = -mxy$$

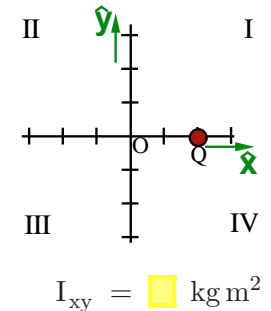
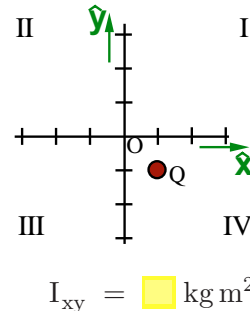
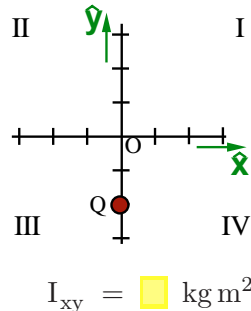
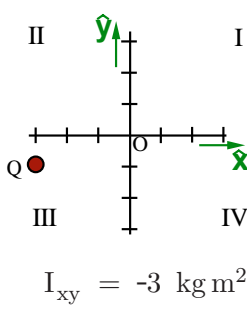
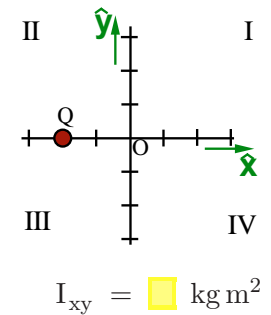
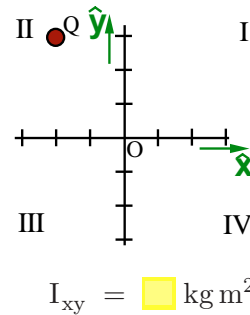
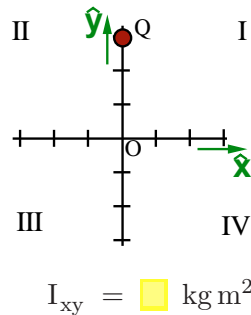
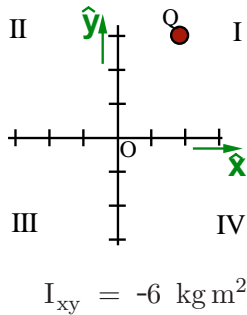
- m is the mass of Q
- x is the \hat{x} measure of Q 's position from O
- y is the \hat{y} measure of Q 's position from O

For example, if $m = 1$ kg, $x = 2$ m, and $y = 3$ m,

$$I_{xy} = -(1 \text{ kg})(2 \text{ m})(3 \text{ m}) = -6 \text{ kg m}^2$$



Knowing particle Q has a mass of 1 kg and each tick-mark represents 1 m, calculate Q 's *product of inertia* I_{xy} about point O for each figure below.



Circle the correct answer (negative, zero, or positive) for each statement about particle Q .

- When Q is in quadrant **I**, I_{xy} is **negative/zero/positive**.
- When Q is in quadrant **II**, I_{xy} is **negative/zero/positive**.
- When Q is in quadrant **III**, I_{xy} is **negative/zero/positive**.
- When Q is in quadrant **IV**, I_{xy} is **negative/zero/positive**.
- When Q is on a quadrant boundary, I_{xy} is **negative/zero/positive**.

10.7 ♣ **Sign conventions for products of inertia** (Section 12.2.4).

There are two sign conventions (\pm) for products of inertia which often lead to errors. **True/False**.