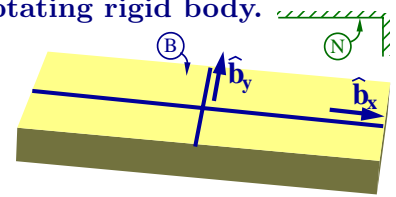


15.6 Calculate: Angular momentum and kinetic energy of a 3D rotating rigid body.

Consider a rigid body B rotating in a Newtonian reference frame N . Right-handed orthogonal unit vectors $\hat{\mathbf{b}}_x, \hat{\mathbf{b}}_y, \hat{\mathbf{b}}_z$ are fixed in B and parallel to B 's principal inertia axes about B_{cm} (B 's center of mass).



Quantity	Symbol	Type
B 's moment of inertia about B_{cm} for $\hat{\mathbf{b}}_x$	I_{xx}	Constant
B 's moment of inertia about B_{cm} for $\hat{\mathbf{b}}_y$	I_{yy}	Constant
B 's moment of inertia about B_{cm} for $\hat{\mathbf{b}}_z$	I_{zz}	Constant
$\hat{\mathbf{b}}_x$ measure of ${}^N\vec{\omega}^B$	ω_x	Variable
$\hat{\mathbf{b}}_y$ measure of ${}^N\vec{\omega}^B$	ω_y	Variable
$\hat{\mathbf{b}}_z$ measure of ${}^N\vec{\omega}^B$	ω_z	Variable

$${}^N\vec{\omega}^B = \omega_x \hat{\mathbf{b}}_x + \omega_y \hat{\mathbf{b}}_y + \omega_z \hat{\mathbf{b}}_z$$

$$\bar{\mathbf{I}}^{B/B_{cm}} = I_{xx} \hat{\mathbf{b}}_x \hat{\mathbf{b}}_x + I_{yy} \hat{\mathbf{b}}_y \hat{\mathbf{b}}_y + I_{zz} \hat{\mathbf{b}}_z \hat{\mathbf{b}}_z$$

$$= \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix}_{\hat{\mathbf{b}}_{xyz}}$$

Calculate B 's angular momentum about B_{cm} in N and B 's rotational kinetic energy in N .

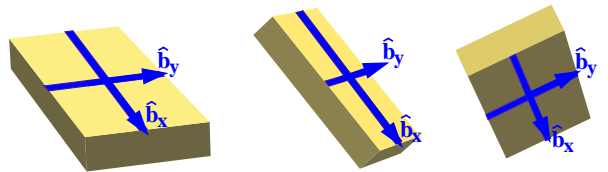
Result: ${}^N\vec{H}^{B/B_{cm}} \stackrel{(15.1)}{=} I_{xx} \omega_x \hat{\mathbf{b}}_x + I_{yy} \omega_y \hat{\mathbf{b}}_y + \text{[yellow box]} \hat{\mathbf{b}}_z$ ${}^N K_{\text{rotation}}^B \stackrel{(15.5)}{=} \frac{1}{2} (I_{xx} \omega_x^2 + I_{yy} \omega_y^2 + \text{[yellow box]})$

15.7 3D spinning rigid body: Guess and check conservation of energy/angular momentum.

This problem refers to Homeworks 6.18 and 15.6.

$$\vec{M}^{\vec{O}} \stackrel{(20.4)}{=} \frac{N d \vec{H}}{dt}$$

$$\Rightarrow \begin{aligned} \dot{\omega}_x &= [(I_{yy} - I_{zz}) \omega_z \omega_y] / I_{xx} \\ \dot{\omega}_y &= [(I_{zz} - I_{xx}) \omega_x \omega_z] / I_{yy} \\ \dot{\omega}_z &= [(I_{xx} - I_{yy}) \omega_y \omega_x] / I_{zz} \end{aligned}$$



- Using your intuition and/or **conservation of angular momentum/mechanical energy** (Sections 20.9 and 23.2), circle the quantities below that you **guess** remain constant (are “conserved”).
- Next, solve the ODEs in Hw 6.18 for $0 \leq t \leq 4$ with initial values $\omega_x = 7, \omega_y = 0.2, \omega_z = 0.2$ (use MotionGenesis, MATLAB®, or ...). Output $t, \omega_x, \omega_y, \omega_z, H_x, H_y, H_z, H_{mag} \triangleq |\vec{H}|$, and K .
- Circle the quantities that remain constant (are “**conserved**”) while solving the ODEs.

ω_x	ω_y	ω_z	K
$H_x = \vec{H} \cdot \hat{\mathbf{b}}_x$	$H_y = \vec{H} \cdot \hat{\mathbf{b}}_y$	$H_z = \vec{H} \cdot \hat{\mathbf{b}}_z$	$ \vec{H} $

Hint: To determine if a quantity such as ω_x is constant during numerical integration, look at the numerical values of ω_x or **zoom** into the plot of ω_x vs. t and check if ω_x is constant to within a reasonable multiplier (e.g., 10) of numerical integrator accuracy. In MotionGenesis and MATLAB®, the default numerical integrator accuracy is $\approx 1 \times 10^{-6}$.

Solution at www.MotionGenesis.com \Rightarrow [Get Started](#) \Rightarrow Solving 1st-order ODEs.

Add the line: `Hx = Ixx*wx; Hy = Iyy*wy; Hz = Izz*wz; Hmag = sqrt(Hx^2 + Hy^2 + Hz^2), etc.`

Add the line: `Output t sec, Hx kg*m^2/sec, Hy kg*m^2/sec, Hz kg*m^2/sec, Hmag kg*m^2/sec, etc.`

View/plot the numerical results (numbers) and determine if they stay constant to within $\approx 1 \times 10^{-5}$.

Optional: Simulate delayed 3D spin instability for wingnut or T-handle with: $I_{xx} < I_{yy} < I_{zz}$, spin is initially mostly about $\hat{\mathbf{b}}_y$ (intermediate moment-of-inertia axis), and $(I_{yy} - I_{xx})(I_{zz} - I_{yy})$ is small, e.g., by using $\omega_x(0) = 0.2 \frac{\text{rad}}{\text{sec}}, \omega_y(0) = 7.0 \frac{\text{rad}}{\text{sec}}, \omega_z(0) = 0.2 \frac{\text{rad}}{\text{sec}}, I_{xx} = 1.9 \text{ kg m}^2, I_{yy} = 2.0 \text{ kg m}^2, I_{zz} = 3.0 \text{ kg m}^2$, Video and Dzhanibekov analysis at www.MotionGenesis.com \Rightarrow [Get Started](#) \Rightarrow **Spin stability**

To analytically determine if H_x is constant, time-differentiate H_x and check if the resulting expression is 0. Since $H_x = I_{xx} \omega_x$ and I_{xx} is constant, ω_x is constant if-and-only-if H_x is constant.

- Using this analytical test and a similar test for $\frac{d(\vec{H} \cdot \hat{\mathbf{n}})}{dt}$, circle the quantities below that remain constant (are **conserved**).

ω_x	ω_y	ω_z	
$H_x = \vec{H} \cdot \hat{\mathbf{b}}_x$	$H_y = \vec{H} \cdot \hat{\mathbf{b}}_y$	$H_z = \vec{H} \cdot \hat{\mathbf{b}}_z$	$ \vec{H} $
$H_1 = \vec{H} \cdot \hat{\mathbf{n}}_x$	$H_2 = \vec{H} \cdot \hat{\mathbf{n}}_y$	$H_3 = \vec{H} \cdot \hat{\mathbf{n}}_z$	

where $\hat{\mathbf{n}}_x, \hat{\mathbf{n}}_y, \hat{\mathbf{n}}_z$ are unit vectors fixed in the Newtonian frame N .

$$\begin{aligned} \frac{dH_x}{dt} &= \frac{d(\vec{H} \cdot \hat{\mathbf{b}}_x)}{dt} \\ &\stackrel{(6.3)}{=} \frac{N d \vec{H}}{dt} \cdot \hat{\mathbf{b}}_x + \vec{H} \cdot \frac{N d \hat{\mathbf{b}}_x}{dt} \\ &= \vec{M} \cdot \hat{\mathbf{b}}_x + \vec{H} \cdot \frac{N d \hat{\mathbf{b}}_x}{dt} \\ &\stackrel{(7.1)}{=} \vec{H} \cdot {}^N\vec{\omega}^B \times \hat{\mathbf{b}}_x \\ &= \vec{H} \cdot (\omega_z \hat{\mathbf{b}}_y - \omega_y \hat{\mathbf{b}}_z) \\ &= (I_{yy} - I_{zz}) \omega_y \omega_z \neq 0 \text{ since } I_{yy} \neq I_{zz} \text{ and } \omega_y(0) \neq 0 \text{ and } \omega_z(0) \neq 0 \\ &\text{[or more generally } \omega_y(t) \neq 0 \text{ and } \omega_z(t) \neq 0\text{].} \end{aligned}$$