

Homework 1. Chapters 2.
Basis independent vectors and their properties

Show work – except for ♣ fill-in-blanks (print .pdf from www.MotionGenesis.com ⇒ [Textbooks](#) ⇒ [Resources](#)).

1.1 ♣ Solving problems – what engineers do.

Understanding dynamics results from **doing** problems. Many problems are guided to help you synthesize processes (imitation). You are encouraged to work by yourself or with colleagues/instructors and use the textbook's reference theory and other resources.



Confucius 500 B.C.

“I hear and I forget.
 I see and I remember.
 I and I understand.”

“By three methods we may learn wisdom:
 First, by reflection, which is noblest;
 Second, by imitation, which is easiest;
 Third by experience, which is the bitterest.”

1.2 ♣ What is a vector? (Section 2.2)

Two properties (attributes) of a vector are and .

1.3 ♣ What is a zero vector? (Section 2.3)

A zero vector $\vec{0}$ has a magnitude of 0/1/2/∞.

A zero vector $\vec{0}$ has no direction. True/False.

A zero vector $\vec{0}$ is *parallel* to any vector \vec{v} . True/False.

A zero vector $\vec{0}$ is *perpendicular* to any vector \vec{v} . True/False.

1.4 ♣ Unit vectors. (Section 2.4)

A unit vector has a magnitude of **0/1/2/∞.**

All unit vectors are equal. **True/False.**

1.5 ♣ Draw the following vectors: (Section 2.2)

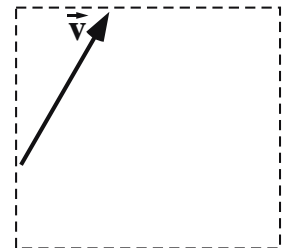
- Long, horizontally-right vector \vec{a}
- Short, vertically-upward vector \vec{b}
- Outwardly-directed unit vector \vec{c} .



1.6 ♣ Optional: Vector magnitude and direction (orientation and sense). (Section 2.2)

The figure to the right shows a vector \vec{v} . **Draw** the following vectors.

- \vec{a} : Same magnitude and same direction as \vec{v} ($\vec{a} = \vec{v}$).
- \vec{b} : Same magnitude and orientation as \vec{v} , but different sense.
- \vec{c} : Same direction as \vec{v} , but different magnitude.
- \vec{d} : Same magnitude as \vec{v} , but different direction (orientation).
- \vec{e} : Different magnitude and different direction (orientation) as \vec{v} .



1.7 ♣ Magnitude of a vector. (Section 2.2)

Consider a real number x and a horizontally-right pointing unit vector \hat{i} .

The **magnitude** of the vector $-x\hat{i}$ is (circle one): positive negative non-negative non-positive.

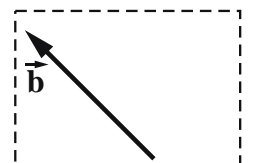
1.8 ♣ Negating a vector. (Section 2.8)

Complete the figure to the right by **drawing** the vector $-\vec{b}$.

Negating the vector \vec{b} results in a vector with different (circle all that apply):

magnitude direction orientation sense

Historical note: Negative numbers (e.g., -3) were not widely accepted until 1800 A.D.



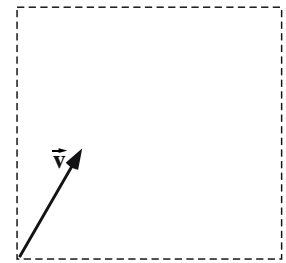
1.9 ♣ **Optional: Multiplying a vector by a scalar.** (Section 2.7)

Complete the figure to the right by drawing the vectors $2\vec{v}$ and $-2\vec{v}$.

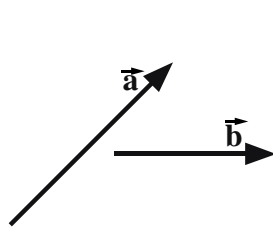
The following statements involve a vector \vec{v} and a real non-zero scalar s ($s \neq 0$).

If a statement is true, provide a numerical value for s that supports your answer

- $s\vec{v}$ can have a different *magnitude* than \vec{v} . True/False $s =$
- $s\vec{v}$ can have a different *direction* than \vec{v} . True/False $s =$
- $s\vec{v}$ can have a different *sense* than \vec{v} . True/False $s =$
- $s\vec{v}$ can have a different *orientation* than \vec{v} . True/False $s =$



1.10 ♣ **Optional: Graphical vector addition/subtraction - draw.** (Sections 2.6,2.8)



Draw $\vec{a} + \vec{b}$



Draw $\vec{a} - \vec{b}$

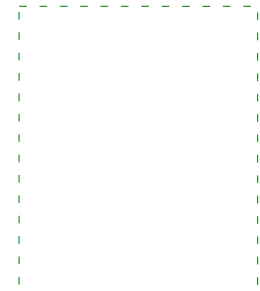


1.11 ♣ **Visual representation of a vector dot-product.** (Section 2.10)

Write the *definition* of the dot-product of a vector \vec{a} with a vector \vec{b} . Include a *sketch* with *each symbol* in the right-hand-side of your definition clearly labeled. The sketch should include \vec{a} , \vec{b} , $|\vec{a}|$, $|\vec{b}|$, ...

Result:

$$\vec{a} \cdot \vec{b} \triangleq \text{$$



1.12 ♣ **Visual representation of a vector cross-product.** (Section 2.11)

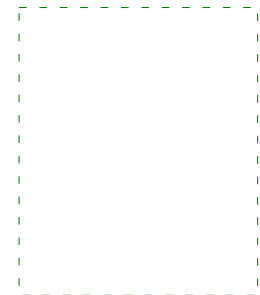
Write the *definition* of the cross-product of a vector \vec{a} with a vector \vec{b} . Include a *sketch* with *each symbol* in your definition labeled and described.

Result:

$$\vec{a} \times \vec{b} \triangleq \text{} \text{}(\theta) \hat{u}$$

where \hat{u} is

and θ is



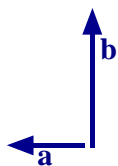
1.13 ♣ **Properties of vector dot-products and cross-products.** (Sections 2.10.1 and 2.11.1)

When \vec{a} is parallel to \vec{b} :	$\vec{a} \cdot \vec{b} = 0$	True/False	$\vec{a} \times \vec{b} = \vec{0}$	True/False
When \vec{a} is perpendicular to \vec{b} :	$\vec{a} \cdot \vec{b} = 0$	True/False	$\vec{a} \times \vec{b} = \vec{0}$	True/False
For arbitrary vectors \vec{a} and \vec{b} :	$\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$	True/False	$\vec{a} \times \vec{b} = \vec{b} \times \vec{a}$	True/False

1.14 ♣ **Calculating vector dot-products and cross-products via definitions.** (Sections 2.10 and 2.11)

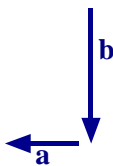
Draw a unit vector $\hat{\mathbf{k}}$ outward-normal to the plane of the paper.

Knowing vector $\vec{\mathbf{a}}$ has magnitude 2 and vector $\vec{\mathbf{b}}$ has magnitude 4, calculate the following dot-products and cross-products via their definitions (2^+ significant digits).



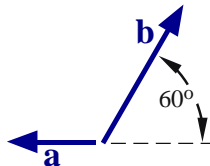
$$\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = \boxed{}$$

$$\vec{\mathbf{a}} \times \vec{\mathbf{b}} = \boxed{}$$



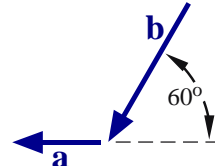
$$\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = \boxed{}$$

$$\vec{\mathbf{a}} \times \vec{\mathbf{b}} = \boxed{}$$



$$\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = \boxed{}$$

$$\vec{\mathbf{a}} \times \vec{\mathbf{b}} = \boxed{}$$



$$\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = \boxed{}$$

$$\vec{\mathbf{a}} \times \vec{\mathbf{b}} = \boxed{}$$

1.15 ♣ **Property of scalar triple product.** (Section 2.12).

For arbitrary non-zero vectors $\vec{\mathbf{a}}, \vec{\mathbf{b}}, \vec{\mathbf{c}}$: $\vec{\mathbf{a}} \cdot (\vec{\mathbf{b}} \times \vec{\mathbf{c}}) = (\vec{\mathbf{a}} \times \vec{\mathbf{b}}) \cdot \vec{\mathbf{c}}$ Never/Sometimes/Always
 A property of the *scalar triple product* is $\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} \times \vec{\mathbf{a}} = 0$. True/False.

1.16 **Property of vector triple cross-product.** (Sections 2.11.1 and 2.12)

Complete the following equation: $\vec{\mathbf{a}} \times (\vec{\mathbf{b}} \times \vec{\mathbf{c}}) = \vec{\mathbf{b}} (\boxed{}) - \vec{\mathbf{c}} (\boxed{})$

Circle true or false (show supporting work): $\vec{\mathbf{a}} \times (\vec{\mathbf{b}} \times \vec{\mathbf{c}}) = (\vec{\mathbf{a}} \times \vec{\mathbf{b}}) \times \vec{\mathbf{c}} + \vec{\mathbf{b}} \times (\vec{\mathbf{a}} \times \vec{\mathbf{c}})$ True/False

1.17 ♣ **Optional: Proof of magnitude of vector cross product property.** (Sections 2.10 and 2.11)

Letting $\hat{\boldsymbol{\lambda}}$ be a *unit vector* and $\vec{\mathbf{v}}$ be *any vector*, prove¹ $|\vec{\mathbf{v}} \times \hat{\boldsymbol{\lambda}}|^2 = \vec{\mathbf{v}} \cdot \vec{\mathbf{v}} - (\vec{\mathbf{v}} \cdot \hat{\boldsymbol{\lambda}})^2$.

1.18 ♣ **Vector exponentiation: $\vec{\mathbf{v}}^2$ and $\vec{\mathbf{v}}^3$.** Complete the 3-step proofs. (Section 2.10)

Step 1: Complete the **definition** of $\vec{\mathbf{v}}^2$ in terms of $|\vec{\mathbf{v}}|$.

Step 2: Use the **definition** of the dot-product to show how $\vec{\mathbf{v}} \cdot \vec{\mathbf{v}}$ can be expressed in terms of $|\vec{\mathbf{v}}|$.

Step 3: Combine these two definitions to provide an alternate way to calculate $\vec{\mathbf{v}}^2$ with a vector dot-product.

Result:

$$\vec{\mathbf{v}}^2 \triangleq |\vec{\mathbf{v}}|^{\boxed{}}$$

$$\vec{\mathbf{v}} \cdot \vec{\mathbf{v}} \stackrel{(2.2)}{=} \boxed{}$$

$$\vec{\mathbf{v}}^2 = \boxed{} \cdot \boxed{}$$

Complete the 3-step proof that relates $\vec{\mathbf{v}}^3$ to $\vec{\mathbf{v}} \cdot \vec{\mathbf{v}}$ raised to a real number.

Result:

$$\vec{\mathbf{v}}^3 \triangleq |\vec{\mathbf{v}}|^{\boxed{}} \stackrel{(2.4)}{=} (\sqrt{\boxed{}})^{\boxed{}} = (\vec{\mathbf{v}} \cdot \vec{\mathbf{v}})^{\boxed{}}$$

1.19 ♣ **$|c\hat{\mathbf{a}}_x|$ Calculate vector magnitude with dot products.** (Section 2.10 and Hw 1.18)

Show how the vector dot-product can be used to show that the magnitude of the vector $c\hat{\mathbf{a}}_x$ (c is a positive or **negative** number and $\hat{\mathbf{a}}_x$ is a unit vector) can be written solely in terms of c (without $\hat{\mathbf{a}}_x$).

Result:

$$|c\hat{\mathbf{a}}_x| = \sqrt{\boxed{} \cdot \boxed{}} = \sqrt{c^2 * \boxed{} \cdot \boxed{}} = \sqrt{\boxed{}} = \text{abs}(c)$$

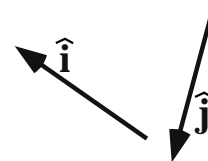
1.20 † **Magnitude of the vector $\vec{\mathbf{v}}$.** Show work. (Section 2.10)

Knowing the angle between a unit vector $\hat{\mathbf{i}}$ and unit vector $\hat{\mathbf{j}}$ is 110° , calculate a numerical value for the magnitude of $\vec{\mathbf{v}} = 3\hat{\mathbf{i}} + 4\hat{\mathbf{j}}$.

Result:

$$|\vec{\mathbf{v}}| \approx \boxed{4.098}$$

Note: The answer is not 5.

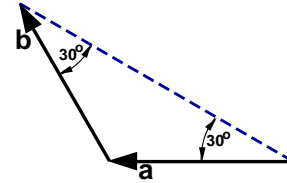


¹One way to prove this is to write $(\vec{\mathbf{v}} \times \hat{\boldsymbol{\lambda}})^2 = (\vec{\mathbf{v}} \times \hat{\boldsymbol{\lambda}}) \cdot (\vec{\mathbf{v}} \times \hat{\boldsymbol{\lambda}}) \stackrel{(2.13)}{=} \vec{\mathbf{v}} \cdot [\hat{\boldsymbol{\lambda}} \times (\vec{\mathbf{v}} \times \hat{\boldsymbol{\lambda}})]$ and then use the vector triple cross-product property $\vec{\mathbf{a}} \times (\vec{\mathbf{b}} \times \vec{\mathbf{c}}) = \vec{\mathbf{b}}(\vec{\mathbf{a}} \cdot \vec{\mathbf{c}}) - \vec{\mathbf{c}}(\vec{\mathbf{a}} \cdot \vec{\mathbf{b}})$ from Section 2.11. Alternately, it is helpful to write $\vec{\mathbf{v}} = \vec{\mathbf{v}}_\perp \hat{\boldsymbol{\lambda}}_\perp + \vec{\mathbf{v}}_\parallel \hat{\boldsymbol{\lambda}}$ where $\vec{\mathbf{v}}_\perp \hat{\boldsymbol{\lambda}}_\perp$ is the component of $\vec{\mathbf{v}}$ that is perpendicular to $\hat{\boldsymbol{\lambda}}$ and $\vec{\mathbf{v}}_\parallel \hat{\boldsymbol{\lambda}}$ is the component of $\vec{\mathbf{v}}$ that is parallel to $\hat{\boldsymbol{\lambda}}$.

1.21 ♣ **Angle between vectors.** (Section 2.10)

Referring to the figure to the right, find the numerical value for the angle between vector \vec{a} and vector \vec{b} .

$$\angle(\vec{a}, \vec{b}) = \text{[]}^\circ$$



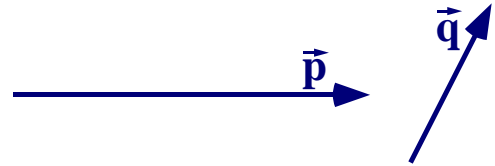
1.22 **Visual estimation of vector dot/cross-products.** Show work. (Sections 2.10 and 2.11)

Estimate (e.g., using your pinky) the magnitude of the vector \vec{p} shown below. Note: 1 inch \triangleq 2.54 cm.

Estimate the angle between \vec{p} and \vec{q} , $\vec{p} \cdot \vec{q}$, and the magnitude of $\vec{p} \times \vec{q}$. Show work.

Result: (Provide numerical results with 1 or more significant digits).

$ \vec{p} \approx \text{[]} \text{ cm}$	$\angle(\vec{p}, \vec{q}) \approx \text{[]}^\circ$
$\vec{p} \cdot \vec{q} \approx \text{[]} \text{ cm}^2$	$ \vec{p} \times \vec{q} \approx \text{[]} \text{ cm}^2$



1.23 ♣ **Form the unit vector \hat{u} having the same direction as $c\hat{a}_x$.** (Section 2.4)

Result: $\hat{u} = \text{[]} \hat{a}_x$ Note: \hat{a}_x is a unit vector and c is a non-zero real number, e.g., 3 or -3

1.24 ♣ **Coefficient of \hat{u} in cross products – definitions and trig functions.** (Section 2.11)

The **cross product** of vectors \vec{a} and \vec{b} can be written in terms of a real scalar s as $\vec{a} \times \vec{b} = s\hat{u}$ where \hat{u} is a unit vector perpendicular to both \vec{a} and \vec{b} in a direction defined by the **right-hand rule**. The coefficient s of the unit vector \hat{u} is inherently non-negative. **True/False.**

1.25 ♣ **Orthogonal vectors: Insights via drawing.** (Section 2.11)

Consider three unit vectors \hat{a} , \hat{b} , and \hat{c} .
 Vector \hat{a} is perpendicular to vector \hat{b} .
 Vector \hat{b} is perpendicular to vector \hat{c} .
 Vector \hat{a} is not parallel to vector \hat{c} .

In **all** cases, \hat{a} is perpendicular to \hat{c} . **True/False.**

Explain your answer by drawing \hat{a} , \hat{b} , \hat{c} and relevant angles.



1.26 **Calculating distance between a point and a line via cross-products.** (Section 2.11.2)

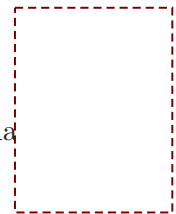
Draw a horizontally-right unit vector \hat{a}_x and vertically-upward unit vector \hat{a}_y .

Draw a point Q whose position vector from a point P is $\vec{r}^{Q/P} = 5\hat{a}_x$.

Draw a line L that passes through point P and is parallel to $\hat{u} = \frac{3}{5}\hat{a}_x + \frac{4}{5}\hat{a}_y$.

Calculate the **distance/length** d between Q and L using both formulas in equa

Result: $d = \underset{(2.9)}{\text{[]}} = \text{[4]}$ $d = \underset{(2.9)}{\text{[]}} = \text{[4]}$



1.27 ♣ **Ranges of angles from dot-product and cross-product calculations.** (Sections 2.10 and 2.11)

Given: Unit vectors \hat{a} and \hat{b} and the numerical values of $c \triangleq \hat{a} \cdot \hat{b}$ and $s \triangleq |\hat{a} \times \hat{b}|$.

Complete the following table with an appropriate range of numerical values.

Quantity	Range of values
c	$\text{[]} \leq c \leq \text{[]}$
s	$\text{[]} \leq s \leq \text{[]}$
Angle θ_c between \hat{a} and \hat{b} that can be uniquely determined solely from c	$\text{[]}^\circ \leq \theta_c \leq \text{[]}^\circ$
Angle θ_s between \hat{a} and \hat{b} that can be uniquely determined solely from s	$\text{[]}^\circ \leq \theta_s \leq \text{[]}^\circ$
Angle θ between \hat{a} and \hat{b}	$\text{[]}^\circ \leq \theta \leq \text{[]}^\circ$

Note: The range of θ_s differs from the (correct) range for θ . Hence, s and θ_s are insufficient to calculate θ .

1.28 ♣ **Vector operations and units.** (Chapter 2)

Circle the vector operations below (scalar multiplication, addition, dot-product, etc.) that are **defined** for a position vector \vec{a} (with **units** of m) and a velocity vector \vec{b} (with **units** of $\frac{m}{s}$).

$-\vec{a}$ $5\vec{a}$ $\vec{a}/5$ $\vec{a} + \vec{b}$ $\vec{a} \cdot \vec{b}$ $\vec{a} \times \vec{b}$

1.29 ♣ **Using vector identities to simplify expressions** (refer to Homework 1.13).

One reason to treat vectors as **basis-independent** quantities is to simplify vector expressions **without** resolving the vectors into orthogonal “ $\vec{x}, \vec{y}, \vec{z}$ ” or “ $\vec{i}, \vec{j}, \vec{k}$ ” components. Simplify the following vector expressions using various properties of dot-products and cross-products.

Express your results in terms of dot-products \cdot and cross-products \times of the arbitrary vectors $\vec{u}, \vec{v}, \vec{w}$ (i.e., $\vec{u}, \vec{v}, \vec{w}$ are not orthogonal).

Vector expression	Simplified vector expression
$(3\vec{u} - 2\vec{v}) \times (\vec{u} + \vec{v})$	$\square \vec{u} \times \vec{v}$
$(3\vec{u} - 2\vec{v}) \cdot (\vec{u} + \vec{v})$	$\square \vec{u}^2 - \square \vec{v}^2 + \square \vec{u} \cdot \vec{v}$
$(\vec{u} - \vec{v}) \cdot (\vec{u} + \vec{v})$	$\square - \square$
$(3\vec{u} - 2\vec{v}) \times (\vec{u} + \vec{v}) \cdot (2\vec{u} - 7\vec{v})$	\square
$(\vec{u} + \vec{v}) \times (\vec{v} + 2\vec{w}) \cdot (\vec{w} + 2\vec{u})$	$\square \vec{u} \times \vec{v} \cdot \vec{w}$

1.30 ♣ **Vector concepts: Solving a vector equation.** (Section 2.10.5)

Consider the vector equation to the right and the process that follows that solves for $\dot{\theta}$ (\hat{a}_x is a unit vector and $v_x, \dot{\theta}, R$ are scalars).

$$v_x \hat{a}_x = \dot{\theta} R \hat{a}_x$$

$$\dot{\theta} = \frac{v_x \hat{a}_x}{R \hat{a}_x} = \frac{v_x}{R}$$

This process is a valid way to solve for $\dot{\theta}$. **True/False.**

Explain:

1.31 **Changing a vector equation to scalar equations.** Show work. (Section 2.10.5)

Draw three mutually orthogonal unit vectors $\hat{p}, \hat{q}, \hat{r}$.

- (a) Use a vector operation (e.g., +, −, *, ·, ×) to transform the following **vector** equation into **one scalar** equation and subsequently solve the scalar equation.

$$(2x - 4) \hat{p} = \vec{0} \quad \xRightarrow{??} \quad x = 2$$

- (b) Show **every** vector operation (e.g., +, −, *, ·, or ×) that transforms the following **vector** equation into **three scalar** equations and subsequently solve the scalar equations for x, y, z .

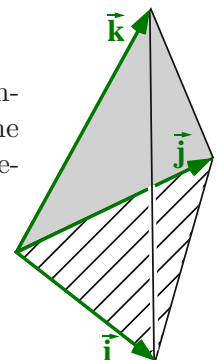
$$(2x - 4) \hat{p} + (3y - 9) \hat{q} + (4z - 16) \hat{r} = \vec{0}$$

Result: $x = 2$ $y = 3$ $z = \square$

- (c) † The figure to the right shows three **non-zero, non-orthogonal**, non-coplanar vectors $\vec{i}, \vec{j}, \vec{k}$. Show **every** vector operation that transforms the following **vector** equation into **three uncoupled scalar** equations and subsequently solve those scalar equations for x, y, z (think $\times \cdot$, not matrix algebra).

$$(2x - 4) \vec{i} + (3y - 9) \vec{j} + (4z - 16) \vec{k} = \vec{0}$$

Result: $x = 2$ $y = 3$ $z = \square$



1.32 ♣ **Number of independent scalar equations from one vector equation.** (Section 2.10.5)

Consider the **vector** equation shown to the right that can be useful for static analyses of any system S .

$$\vec{F}^S = \vec{0}$$

Complete the blanks in the table to the right with **all** integers that could be equal to the number of **independent scalar** equations produced by the previous vector equation for any system S .

System type	Integer(s)
1D (line)	0, <input type="text"/>
2D (planar)	0, <input type="text"/>
3D (spatial)	0, <input type="text"/>

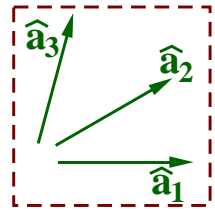
Hint: See Homework 1.31 for ideas.

Note: Regard 1D/linear as meaning \vec{F}^S can be expressed in terms of a single unit vector \hat{i} whereas 2D/planar means \vec{F}^S can be expressed in terms of two non-parallel unit vectors \hat{i} and \hat{j} , and 3D/spatial means \vec{F}^S can be expressed in terms of three non-coplanar unit vectors \hat{i} , \hat{j} , \hat{k} .

1.33 ♣ **Vector concepts: Solving a vector equation (just circle true or false and fill-in the blank).**

Consider the following vector equation written in terms of the scalars x, y, z and three unique non-orthogonal **coplanar** unit vectors $\hat{a}_1, \hat{a}_2, \hat{a}_3$.

$$(2x - 4) \hat{a}_1 + (3y - 9) \hat{a}_2 + (4z - 16) \hat{a}_3 = \vec{0}$$



The **unique** solution to this vector equation is $x = 2, y = 3, z = 4$. **True/False.**

Explain: \hat{a}_2 can be expressed in terms of \hat{a}_1 and \hat{a}_3 (i.e., \hat{a}_2 is a linear combination of \hat{a}_1 and \hat{a}_3). Hence the vector equation produces linearly independent scalar equations.

1.34 ♣ **A vector revolution in geometry.** (Chapter 2)

The relatively new invention of vectors (Gibbs \approx 1900 AD) has revolutionized Euclidean geometry (Euclid \approx 300 BC). For each geometrical quantity below, circle the vector operation(s) (either the dot-product, cross-product, or both) that is **most** useful for their calculation.

Length: • ×	Angle: • ×
Area: • ×	Volume: • ×

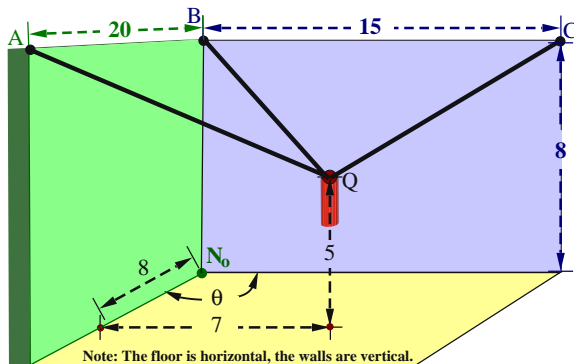
1.35 † **Microphone cable lengths (non-orthogonal walls) “It’s just geometry”.** **Show work.**

A microphone Q is attached to three pegs A, B, C by three cables. Knowing the peg locations, microphone location, and the angle θ between the vertical walls, express L_A, L_B, L_C solely in terms of numbers and θ . Next, complete the table by calculating L_B when $\theta = 120^\circ$.

Hint: To do this **efficiently**, use one set of non-orthogonal unit vectors (do **not** introduce an **orthogonal** set of unit vectors).

Hint: Use the distributive property of the vector dot-product as shown in Section 2.10.1 and Homework 2.4.

Note: Synthesis problems are difficult. Think, talk, draw, sleep, walk, get help, ... (if necessary, read Section 3.3).



Distance between A and B	20 m
Distance between B and C	15 m
Distance between N_o and B	8 m
Distance along back wall (see picture)	7 m
Q 's height above N_o	5 m
Distance along side wall (see picture)	8 m
L_A : Length of cable joining A and Q	<input type="text"/> m
L_B : Length of cable joining B and Q	<input type="text"/> m
L_C : Length of cable joining C and Q	<input type="text"/> m

Result:

$$L_A = \sqrt{202 - 168 \cos(\theta)}$$

$$L_B = \text{[Blank]}$$

$$L_C = \sqrt{137 - \text{[Blank]} \cos(\theta)}$$

Homework 2. Chapters 1, 2, 3, 4.
Vector addition, dot products, and cross products

Show work – except for ♣ fill-in-blanks (print .pdf from www.MotionGenesis.com ⇒ [Textbooks](#) ⇒ [Resources](#)).

2.1 ♣ Right-handed, orthogonal, unitary, vector basis. (Section 4.1)

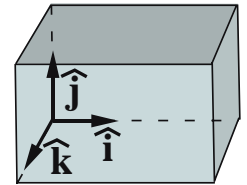
Draw a right-handed orthogonal (mutually perpendicular) vector basis consisting of the unit vectors $\hat{\mathbf{a}}_x, \hat{\mathbf{a}}_y, \hat{\mathbf{a}}_z$.



2.2 ♣ Adding and subtracting vectors with bases. (Sections 2.6 and 2.8)

Shown right are right-handed orthogonal unit vectors $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ and vectors $\vec{\mathbf{u}}, \vec{\mathbf{v}},$ and $\vec{\mathbf{w}}$.
 Form the vector sums and differences below.

$$\begin{aligned}\vec{\mathbf{u}} &= 2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 4\hat{\mathbf{k}} \\ \vec{\mathbf{v}} &= x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}} \\ \vec{\mathbf{w}} &= 5\hat{\mathbf{i}} - 6\hat{\mathbf{j}} + 7\hat{\mathbf{k}}\end{aligned}$$

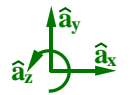


Result:

$$\vec{\mathbf{u}} + \vec{\mathbf{v}} = (2+x)\hat{\mathbf{i}} + \boxed{}\hat{\mathbf{j}} + \boxed{}\hat{\mathbf{k}} \qquad \vec{\mathbf{u}} - \vec{\mathbf{v}} = (2-x)\hat{\mathbf{i}} + \boxed{}\hat{\mathbf{j}} + \boxed{}\hat{\mathbf{k}}$$

2.3 ♣ Column matrices and vectors (Hint: What is a vector – Hw 1.2). (Section 2.9)

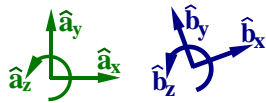
The vector $\hat{\mathbf{a}}_x + 2\hat{\mathbf{a}}_y + 3\hat{\mathbf{a}}_z$ is equal to the column matrix $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. **True/False.**
 Note: $\hat{\mathbf{a}}_x, \hat{\mathbf{a}}_y, \hat{\mathbf{a}}_z$ are orthogonal unit vectors shown right.



Adding the following vectors and column matrices produce equivalent results. **True/False.**

Note: $\hat{\mathbf{a}}_x, \hat{\mathbf{a}}_y, \hat{\mathbf{a}}_z$ and $\hat{\mathbf{b}}_x, \hat{\mathbf{b}}_y, \hat{\mathbf{b}}_z$ are the sets of orthogonal unit vectors shown below.

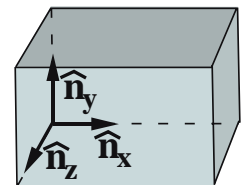
$$\hat{\mathbf{a}}_x + 2\hat{\mathbf{a}}_y + 3\hat{\mathbf{a}}_z + 4\hat{\mathbf{b}}_x + 5\hat{\mathbf{b}}_y + 6\hat{\mathbf{b}}_z = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \\ 9 \end{bmatrix}$$



2.4 ♣ Calculating vector dot products with bases. (Sections 2.10 and 2.10.3)

Given: Right-handed orthogonal unit vectors $\hat{\mathbf{n}}_x, \hat{\mathbf{n}}_y, \hat{\mathbf{n}}_z$ and:

$$\begin{aligned}\vec{\mathbf{u}} &= 2\hat{\mathbf{n}}_x + 3\hat{\mathbf{n}}_y + 4\hat{\mathbf{n}}_z \\ \vec{\mathbf{v}} &= x\hat{\mathbf{n}}_x + y\hat{\mathbf{n}}_y + z\hat{\mathbf{n}}_z \\ \vec{\mathbf{w}} &= 5\hat{\mathbf{n}}_x - 6\hat{\mathbf{n}}_y + 7\hat{\mathbf{n}}_z\end{aligned}$$



(a) Use the distributive law for dot products to write $\vec{\mathbf{u}} \cdot \vec{\mathbf{v}}$ in terms of $\hat{\mathbf{n}}_x \cdot \hat{\mathbf{n}}_x, \hat{\mathbf{n}}_x \cdot \hat{\mathbf{n}}_y,$ etc.

Result:

$$\begin{aligned}\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} &= 2x\hat{\mathbf{n}}_x \cdot \hat{\mathbf{n}}_x + 2y\hat{\mathbf{n}}_x \cdot \hat{\mathbf{n}}_y + 2z\hat{\mathbf{n}}_x \cdot \hat{\mathbf{n}}_z \\ &+ 3x\hat{\mathbf{n}}_y \cdot \hat{\mathbf{n}}_x + 3y\boxed{} \cdot \boxed{} + \boxed{} \\ &+ \boxed{} + \boxed{} + \boxed{}\end{aligned}$$

(b) Use the definition of the dot product in equation (2.2) to calculate $\hat{\mathbf{n}}_x \cdot \hat{\mathbf{n}}_x, \hat{\mathbf{n}}_x \cdot \hat{\mathbf{n}}_y,$ etc.

Result:

$$\begin{aligned}\hat{\mathbf{n}}_x \cdot \hat{\mathbf{n}}_x &= \boxed{} & \hat{\mathbf{n}}_x \cdot \hat{\mathbf{n}}_y &= \boxed{} & \hat{\mathbf{n}}_x \cdot \hat{\mathbf{n}}_z &= \boxed{} \\ \hat{\mathbf{n}}_y \cdot \hat{\mathbf{n}}_x &= \boxed{} & \hat{\mathbf{n}}_y \cdot \hat{\mathbf{n}}_y &= \boxed{} & \hat{\mathbf{n}}_y \cdot \hat{\mathbf{n}}_z &= 0 \\ \hat{\mathbf{n}}_z \cdot \hat{\mathbf{n}}_x &= \boxed{} & \hat{\mathbf{n}}_z \cdot \hat{\mathbf{n}}_y &= \boxed{} & \hat{\mathbf{n}}_z \cdot \hat{\mathbf{n}}_z &= 1\end{aligned}$$

(c) In view of your previous two results, calculate $\vec{\mathbf{u}} \cdot \vec{\mathbf{v}}$.

Result:

$$\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = \boxed{}$$

(d) As shown below for the arbitrary vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ (also shown in Section 2.10.3), the dot product

$\vec{a} \cdot \vec{b}$ is relatively easy to calculate when $\hat{n}_x, \hat{n}_y, \hat{n}_z$ are *orthogonal unit* vectors.

$$\left. \begin{aligned} \vec{a} &= a_x \hat{n}_x + a_y \hat{n}_y + a_z \hat{n}_z \\ \vec{b} &= b_x \hat{n}_x + b_y \hat{n}_y + b_z \hat{n}_z \end{aligned} \right\} \Rightarrow \vec{a} \cdot \vec{b} = a_x b_x + a_y b_y + a_z b_z$$

Use this **orthogonal-short-cut** to calculate:

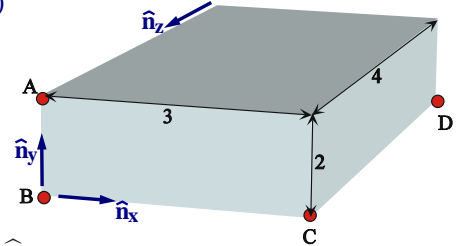
$$\vec{u} \cdot \vec{v} = 2x + 3y + \boxed{}z \quad \vec{u} \cdot \vec{w} = \boxed{} \quad \vec{v} \cdot \vec{w} = \boxed{}$$

2.5 ♣ Perpendicular vectors. (Note: $\hat{i}, \hat{j}, \hat{k}$ are orthogonal unit vectors). (Section 2.10)

When vector $\vec{v} = x\hat{i} + 2\hat{j} + 3\hat{k}$ is perpendicular to vector $\vec{w} = 4\hat{i} + 5\hat{j} + 6\hat{k}$, $x = \boxed{}$.

2.6 Dot product for calculating angles. (Sections 2.10 and 3.3)

The figure to the right shows a rectangular parallelepiped (block) of sides 2, 3, and 4 with points A, B, C located at corners. $\hat{n}_x, \hat{n}_y, \hat{n}_z$ are right-handed orthogonal unit vectors with \hat{n}_x directed from B to C and \hat{n}_y from B to A .



(a) Express $\vec{r}^{C/A}$ (C 's position vector from A) in terms of $\hat{n}_x, \hat{n}_y, \hat{n}_z$.

Find a numerical value for $|\vec{r}^{C/A}|^2 = \vec{r}^{C/A} \cdot \vec{r}^{C/A}$.

Next, use equation (2.4) to calculate the magnitude of $\vec{r}^{C/A}$ (the distance from A to C).

Result: $\vec{r}^{C/A} = \boxed{}\hat{n}_x - \boxed{}\hat{n}_y \quad |\vec{r}^{C/A}|^2 = \vec{r}^{C/A} \cdot \vec{r}^{C/A} = \boxed{} \quad |\vec{r}^{C/A}| = \boxed{}$

(b) Using equation (2.1), calculate the unit vector \hat{u} directed from A to C in terms of $\hat{n}_x, \hat{n}_y, \hat{n}_z$.

Next, find the unit vector \hat{v} directed from A to D in terms of $\hat{n}_x, \hat{n}_y, \hat{n}_z$.

Result: $\hat{u} = \frac{3\hat{n}_x - 2\hat{n}_y}{\sqrt{13}} \quad \hat{v} = \frac{\boxed{}}{\boxed{}}$

(c) Calculate $\angle BAC$, the angle between line \overline{AB} and line \overline{AC} .

Next, calculate $\angle CAD$, the angle between line \overline{AC} and line \overline{AD} .

Result: $\angle BAC = \boxed{}^\circ \quad \angle CAD = \boxed{47.97^\circ}$

2.7 ♣ Vector components: Sine and cosine. (Section 1.4)

Trigonometry plays a central role in *rotation matrices*.

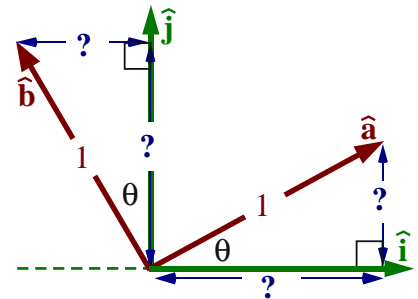
Replace each **?** in the figure to the right with $\sin(\theta)$ or $\cos(\theta)$.

Express unit vectors \hat{a} and \hat{b} in terms of $\sin(\theta), \cos(\theta), \hat{i}, \hat{j}$.

Result: $\hat{a} = \boxed{}\hat{i} + \boxed{}\hat{j}$

SohCahToa

$$\hat{b} = \boxed{}\hat{i} + \cos(\theta)\hat{j}$$

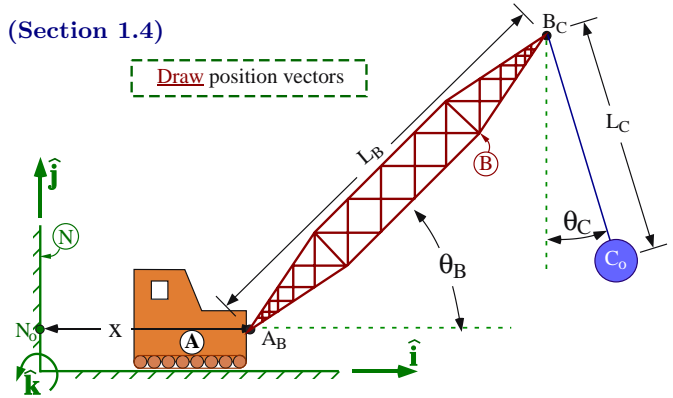


2.8 ♣ Vector components for a crane-boom. (Section 1.4)

Shown right is a crane whose cab A supports a boom B that swings a wrecking ball C_o .

\hat{i} , \hat{j} , \hat{k} are right-handed orthogonal unit vectors with \hat{i} horizontally-right, \hat{j} vertically-upward, and \hat{k} perpendicular to the plane containing points N_o , A_B , B_C , C_o .

Draw each position vector listed below and then use your knowledge of sine/cosine to resolve these vectors into \hat{i} and \hat{j} components.



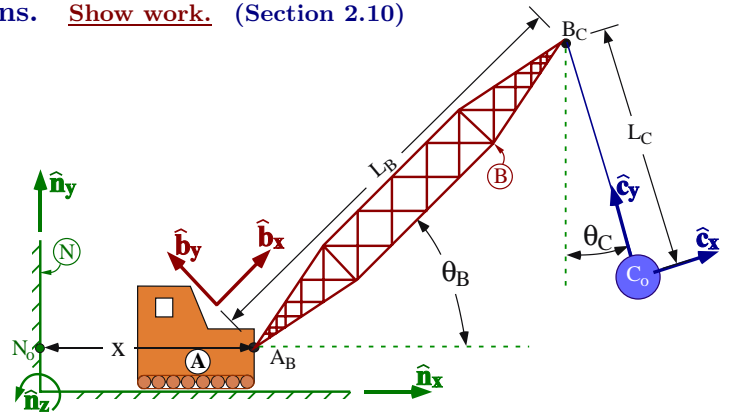
- A_B 's position vector from N_o $\vec{r}^{A_B/N_o} =$ $\hat{i} +$ \hat{j}
- B_C 's position vector from A_B $\vec{r}^{B_C/A_B} =$ $\hat{i} +$ \hat{j}
- C_o 's position vector from B_C $\vec{r}^{C_o/B_C} =$ $\hat{i} +$ \hat{j}
- B_C 's position vector from N_o $\vec{r}^{B_C/N_o} =$ $\hat{i} +$ \hat{j}
- N_o 's position vector from C_o $\vec{r}^{N_o/C_o} =$ $\hat{i} + [-L_B \sin(\theta_B) + L_C \cos(\theta_C)] \hat{j}$

2.9 Dot products and distance calculations. **Show work.** (Section 2.10)

Shown right is a crane whose cab A supports a boom B that swings a wrecking ball C_o . To prevent the wrecking ball from hitting a car at point N_o , the distance between N_o and the tip of the boom (point B_C) must be controlled.

To start this problem, express B_C 's position vector from N_o in terms of x , L_B , and the unit vectors \hat{n}_x and \hat{b}_x .

Result: $\vec{r}^{B_C/N_o} =$ $\hat{n}_x +$ \hat{b}_x



- (a) **Without** resolving \vec{r}^{B_C/N_o} into \hat{n}_x and \hat{n}_y components (done in the next step), use equation (2.4) and the distributive property to calculate the distance between N_o and B_C in terms of x , L_B , θ_B . Then calculate its numerical value when $x = 20$ m, $L_B = 10$ m, $\theta_B = 30^\circ$.

Result: (If necessary, complete the footnote hint below).¹

Distance between N_o and B_C : $|\vec{r}^{B_C/N_o}| = \sqrt{\text{input} + \text{input} + \text{input}} = 29.1$ m

- (b) Two colleagues are confused by your use of **mixed-bases** vectors (i.e., $x\hat{n}_x + L_B\hat{b}_x$), and ask you to verify B 's position vector from N_o can be expressed in the **uniform-basis** as shown below. Use this uniform-basis expression to verify your previous result for $|\vec{r}^{B_C/N_o}|$.

Note: This inefficient uniform-basis approach requires the simplifying trigonometric identity $\sin^2(\theta_B) + \cos^2(\theta_B) = 1$.

Result: $\vec{r}^{B_C/N_o} = [x + L_B \cos(\theta_B)] \hat{n}_x + L_B \sin(\theta_B) \hat{n}_y$ $|\vec{r}^{B_C/N_o}|$ simplifies to previous result.

- (c) **Optional:** Calculate the distance between N_o and C_o in terms of x , L_B , L_C , θ_B , and θ_C .

Result: $|\vec{r}^{C_o/N_o}| = \sqrt{x^2 + L_B^2 + L_C^2 + 2xL_B \cos(\theta_B) + 2xL_C \sin(\theta_C) - 2L_B L_C \sin(\theta_B - \theta_C)}$

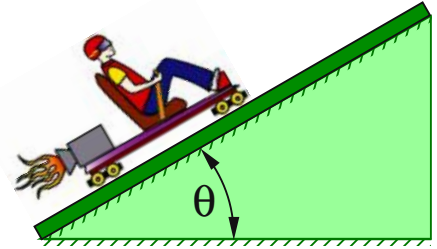
¹Hint: Use the distributive property to express $\vec{r}^{B_C/N_o} \cdot \vec{r}^{B_C/N_o}$ in terms of x , L_B , and $\hat{n}_x \cdot \hat{b}_x$. Next, use the **dot-product definition** in equation (2.2) to calculate $\hat{n}_x \cdot \hat{b}_x =$, and then rewrite $\vec{r}^{B_C/N_o} \cdot \vec{r}^{B_C/N_o} =$ ² + ² + $2xL_B$ (\cdot) = ² + ² + $2xL_B \cos$ ().

Note: The distributive property for vector dot-multiplication is $(\vec{a} + \vec{b}) \cdot (\vec{c} + \vec{d}) = \vec{a} \cdot \vec{c} + \vec{a} \cdot \vec{d} + \vec{b} \cdot \vec{c} + \vec{b} \cdot \vec{d}$.

2.10 Vector components, free-body diagram (FBD), and motion graphs for a rocket-sled.

The following figure shows a rocket-sled moving along smooth (**frictionless**) inclined rails.

Description	Symbol	Type
Mass of rocket-sled and rider	m	Constant
Earth's gravitational acceleration	g	Constant
Angle between horizontal and inclined-rails	θ	Constant
\hat{i} measure of thrust force on sled	F_T	Specified
\hat{j} measure of normal force on sled	F_N	Variable
\hat{i} measure of rocket-sled position	x	Variable



Draw a unit vector \hat{i} upward-right and parallel to the rails.

Draw a unit vector \hat{j} outward-normal to the rails (perpendicular to \hat{i}) and in the plane of the paper.

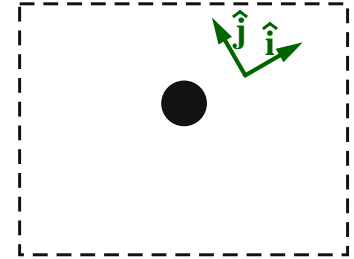
Free-body diagram

Draw a particle representing the rocket-sled.

Draw the thrust, normal, and gravity forces on the rocket-sled.

Express the net force on the rocket-sled in terms of \hat{i} and \hat{j} .

Result: $\vec{F}_{\text{Net}} = \vec{F}_{\text{Thrust}} + \vec{F}_{\text{Normal}} + \vec{F}_{\text{gravity}}$
 $= \text{[]} \hat{i} + \text{[]} \hat{j}$



$\vec{F} = m \vec{a}$ The rocket-sled's acceleration can be calculated as $\vec{a} = \frac{d^2 x}{dt^2} \hat{i} = \ddot{x} \hat{i}$.

Substitute the right-hand sides of \vec{F}_{Net} and \vec{a} into $\vec{F}_{\text{Net}} = m \vec{a}$ and solve for \ddot{x} and F_N .

Result: $\text{[]} \hat{i} + \text{[]} \hat{j} = m \ddot{x} \hat{i}$
 $\hat{i}: \ddot{x} = \frac{\text{[]} - \text{[]}}{m}$ $\hat{j}: F_N = \text{[]}$

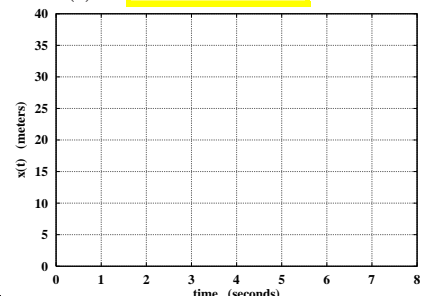
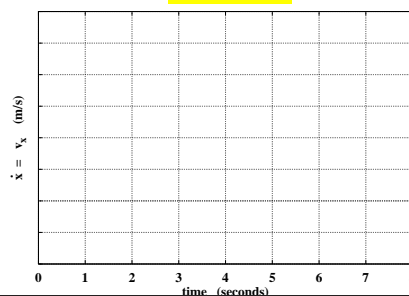
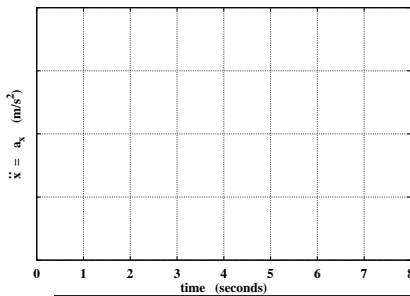
Solve for acceleration: Knowing $m = 100 \text{ kg}$, $g = 10 \frac{\text{m}}{\text{s}^2}$, $\theta = 30^\circ$, $F_T = 700 \text{ N}$, **graph** \ddot{x} .

Solve for velocity and position: Knowing the rocket-sled starts at $x = 8 \text{ m}$, and is initially moving **downward-left** along the rail at $4 \frac{\text{m}}{\text{s}}$, solve and **sketch** $\dot{x}(t)$ and $x(t)$ for $0 \leq t \leq 8$.

Result: $\ddot{x}(t) = 2 \text{ m/s}^2$

$\dot{x}(t) = \text{[]} \text{ m/s}$

$x(t) = \text{[]} \text{ meters}$

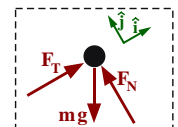


Include friction between the rails and rocket-sled, modeled via a coefficient of kinetic friction μ_k . Express \ddot{x} and F_N in terms of some/all of μ_k and symbols in the table. Knowing $\mu_k \approx 0.115$, $x(0) = 8 \text{ m}$, and the rocket-sled initially moves **upward-right** at $4 \frac{\text{m}}{\text{s}}$, find $\dot{x}(t)$ and $x(t)$.

Result: $\ddot{x}(t) = \text{[]} \approx 1 \frac{\text{m}}{\text{s}^2}$ $F_N = \text{[]}$ $\dot{x}(t) \approx \text{[]}$
 $x(t) \approx \text{[]}$

Calculate the minimum thrust (redraw FBDs) to:

	Result
a. Keep the rocket-sled moving uphill at constant speed	$F_T \approx \text{[]} \text{ N}$
b. Keep the rocket-sled moving downhill at constant speed	$F_T \approx \text{[]} \text{ N}$

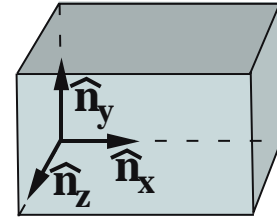


For part b, assume the rocket is initially moving **downward-left** at 4 m/s .

Solution at www.MotionGenesis.com \Rightarrow **Get Started** \Rightarrow **Rocket sled.**

2.11 ♣ Construct a unit vector \hat{u} in the direction of each vector given below. (Section 2.10.2)

Vector	Unit vector \hat{u}
$3\hat{n}_x$	\hat{n}_x
$-3\hat{n}_x$	<input type="text"/>
$3\hat{n}_x - 4\hat{n}_y$	<input type="text"/>
$3\hat{n}_x - 4\hat{n}_y + 12\hat{n}_z$	<input type="text"/>
$c\hat{n}_x$	<input type="text"/> or <input type="text"/>
c is a real non-zero number	



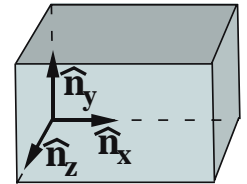
Note: $\hat{n}_x, \hat{n}_y, \hat{n}_z$ are orthogonal unit vectors.

Note: Ensure your last answer agrees with your first two answers, e.g., if $c = 3$ or $c = -3$.

2.12 ♣ Calculating vector cross products with bases. (Section 2.11)

Given: Right-handed orthogonal unit vectors $\hat{n}_x, \hat{n}_y, \hat{n}_z$ and:

$$\begin{aligned}\vec{u} &= 2\hat{n}_x + 3\hat{n}_y + 4\hat{n}_z \\ \vec{v} &= x\hat{n}_x + y\hat{n}_y + z\hat{n}_z \\ \vec{w} &= 5\hat{n}_x - 6\hat{n}_y + 7\hat{n}_z\end{aligned}$$



(a) Use the distributive law for cross products to write $\vec{u} \times \vec{v}$ in terms of $\hat{n}_x \times \hat{n}_x, \hat{n}_x \times \hat{n}_y,$ etc.

Result:
$$\begin{aligned}\vec{u} \times \vec{v} &= 2x\hat{n}_x \times \hat{n}_x + 2y\hat{n}_x \times \hat{n}_y + 2z\hat{n}_x \times \hat{n}_z \\ &+ \text{[]} + \text{[]} + \text{[]} \\ &+ \text{[]} + \text{[]} + \text{[]}\end{aligned}$$

(b) Use the definition of the cross product to calculate $\hat{n}_x \times \hat{n}_x, \hat{n}_x \times \hat{n}_y,$ etc.

Result:
$$\begin{aligned}\hat{n}_x \times \hat{n}_x &= \vec{0} & \hat{n}_x \times \hat{n}_y &= \hat{n}_z & \hat{n}_x \times \hat{n}_z &= -\hat{n}_y \\ \hat{n}_y \times \hat{n}_x &= \text{[]} & \hat{n}_y \times \hat{n}_y &= \text{[]} & \hat{n}_y \times \hat{n}_z &= \text{[]} \\ \hat{n}_z \times \hat{n}_x &= \text{[]} & \hat{n}_z \times \hat{n}_y &= \text{[]} & \hat{n}_z \times \hat{n}_z &= \text{[]}\end{aligned}$$

(c) In view of your previous two results, calculate $\vec{u} \times \vec{v}$.

Result:
$$\vec{u} \times \vec{v} = \text{[]}\hat{n}_x + \text{[]}\hat{n}_y + \text{[]}\hat{n}_z$$

2.13 Cross products and determinants. (Section 2.11.3)

Given **right-handed orthogonal unit** vectors $\hat{n}_x, \hat{n}_y, \hat{n}_z$ and two arbitrary vectors \vec{a} and \vec{b} expressed as shown to the right, show that calculating $\vec{a} \times \vec{b}$ with the distributive property of the cross product happens to be equal to the determinant of the matrix shown to the right.

$$\vec{a} = a_x\hat{n}_x + a_y\hat{n}_y + a_z\hat{n}_z$$

$$\vec{b} = b_x\hat{n}_x + b_y\hat{n}_y + b_z\hat{n}_z$$

$$\vec{a} \times \vec{b} = \det \begin{bmatrix} \hat{n}_x & \hat{n}_y & \hat{n}_z \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{bmatrix}$$

Next use this determinant method to calculate the following cross products (refer to Hw 2.12).

Result:
$$\vec{u} \times \vec{v} = (3z - 4y)\hat{n}_x + (4x - 2z)\hat{n}_y + (2y - 3x)\hat{n}_z$$

Optional:
$$\vec{u} \times \vec{w} = 45\hat{n}_x + 6\hat{n}_y - 27\hat{n}_z$$

Optional:
$$\vec{v} \times \vec{w} = \text{[]}\hat{n}_x + \text{[]}\hat{n}_y - \text{[]}\hat{n}_z$$

2.14 ♣ Optional: Cross product as skew-symmetric matrix multiplication.

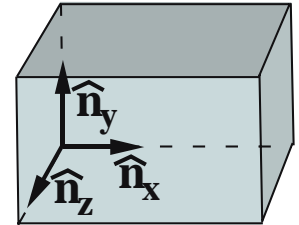
Referring to the previous problem, show that the $\hat{n}_x, \hat{n}_y, \hat{n}_z$ coefficients of $\vec{a} \times \vec{b}$ happen to be equal to the elements that result from the following *skew symmetric matrix* multiplication.

$$\begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix} \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix}$$

After counting the number of computer operations required to multiply the 3×3 matrix by the 3×1 matrix (including multiplication by 0), and comparing the number of operations required to calculate the elements of the simplified answer, it is clear that using a matrix multiplication to calculate a cross product is **inefficient** True/False.

2.15 ♣ Scalar triple product with bases. (Section 2.12)

The figure shows right-handed orthogonal unit vectors $\hat{n}_x, \hat{n}_y, \hat{n}_z$.



Given

$$\vec{u} = 2\hat{n}_x + 3\hat{n}_y + 4\hat{n}_z$$

$$\vec{v} = x\hat{n}_x + y\hat{n}_y + z\hat{n}_z$$

$$\vec{w} = 5\hat{n}_x - 6\hat{n}_y + 7\hat{n}_z$$

Calculate

$$\vec{u} \times \vec{v} \cdot \vec{u} = \square$$

$$\vec{u} \times \vec{v} \cdot \vec{w} = \square z - \square x - 6y$$

$$\vec{u} \cdot \vec{v} \times \vec{w} = \square z - 45x - \square y$$

Note: Although the order of operations in $\vec{u} \times \vec{v} \cdot \vec{u}$ is unambiguous, parentheses may clarify your work.

$\vec{u} \times \vec{v} \cdot \vec{w} = \vec{u} \cdot \vec{v} \times \vec{w}$ and it is OK to switch \cdot and \times in scalar triple products. True/False.

2.16 ♣ Optional: Scalar triple products and determinants. (Section 2.12)

Given **right-handed orthogonal unit** vectors $\hat{n}_x, \hat{n}_y, \hat{n}_z$ and arbitrary vectors $\vec{a}, \vec{b}, \vec{c}$ expressed as shown right, prove that calculating $\vec{a} \cdot (\vec{b} \times \vec{c})$ happens to be equal to the determinant of the matrix shown to the right.

$$\vec{a} = a_x \hat{n}_x + a_y \hat{n}_y + a_z \hat{n}_z$$

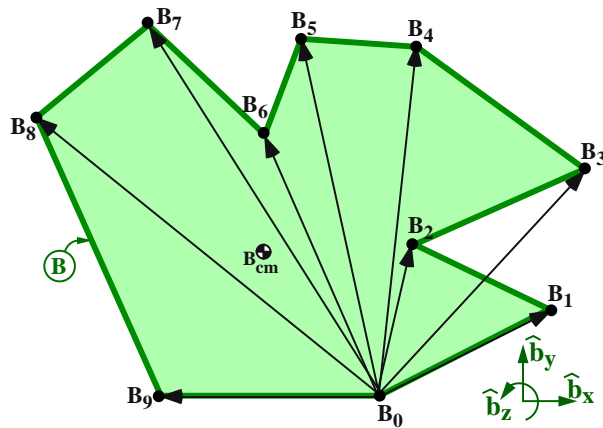
$$\vec{b} = b_x \hat{n}_x + b_y \hat{n}_y + b_z \hat{n}_z$$

$$\vec{c} = c_x \hat{n}_x + c_y \hat{n}_y + c_z \hat{n}_z$$

$$\vec{a} \cdot \vec{b} \times \vec{c} = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}$$

2.17 ♣ **Cross products: Commercial algorithm for area calculations (surveying).** (Section 2.11.2)

One reason triangles are important is that complex **planar objects** such as the polygon B below can be decomposed into triangles. Planar measurements are important in various professions, including farming acreage, building costs, and mass and area properties of 2D objects.



$$\begin{aligned} \vec{r}^{B_1/B_0} &= 2.0 \hat{b}_x + 2.0 \hat{b}_y \\ \vec{r}^{B_2/B_0} &= 0.5 \hat{b}_x + 2.5 \hat{b}_y \\ \vec{r}^{B_3/B_0} &= 3.0 \hat{b}_x + 4.0 \hat{b}_y \\ \vec{r}^{B_4/B_0} &= 0.2 \hat{b}_x + 6.0 \hat{b}_y \\ \vec{r}^{B_5/B_0} &= -0.5 \hat{b}_x + 7.0 \hat{b}_y \\ \vec{r}^{B_6/B_0} &= -1.0 \hat{b}_x + 5.0 \hat{b}_y \\ \vec{r}^{B_7/B_0} &= -2.0 \hat{b}_x + 7.0 \hat{b}_y \\ \vec{r}^{B_8/B_0} &= -4.0 \hat{b}_x + 5.0 \hat{b}_y \\ \vec{r}^{B_9/B_0} &= -2.0 \hat{b}_x + 0.0 \hat{b}_y \end{aligned}$$

A commercial algorithm for calculating the area of the polygon B shown above is to:

- Label a vertex B_0 and number the remaining vertices sequentially in a counter-clockwise fashion.
- Form \vec{r}^{B_i/B_0} , the position vector of vertex B_i ($i = 1, 2, \dots$) from vertex B_0
- Calculate \vec{A}_2 and \vec{A}_4 , the “vector-areas” of the triangles defined by vertices $B_0 B_2 B_3$, and $B_0 B_4 B_5$, respectively. Formulas for the vector-areas of each triangle are given below along with the vector sum of these areas \vec{A} and the polygon’s area (the magnitude of \vec{A}).

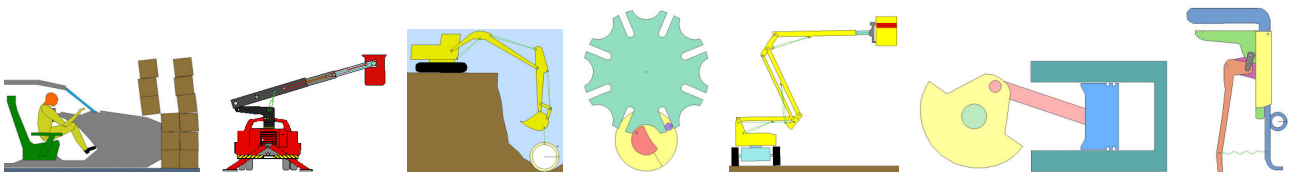
Result: (Just fill in the blanks. You only need to calculate \vec{A}_2 , \vec{A}_4 , and \mathbf{A})

$\vec{A}_1 =$	$\frac{1}{2} * \vec{r}^{B_1/B_0} \times \vec{r}^{B_2/B_0} =$	$2 \hat{b}_z$
$\vec{A}_2 =$	$\frac{1}{2} * \vec{r}^{B_2/B_0} \times \vec{r}^{B_3/B_0} =$	<input type="text"/>
$\vec{A}_3 =$...	$8.6 \hat{b}_z$
$\vec{A}_4 =$...	<input type="text"/>
$\vec{A}_5 =$...	$2.25 \hat{b}_z$
$\vec{A}_6 =$...	$1.5 \hat{b}_z$
$\vec{A}_7 =$...	$9 \hat{b}_z$
$\vec{A}_8 =$	$\frac{1}{2} * \vec{r}^{B_8/B_0} \times \vec{r}^{B_9/B_0} =$	$5 \hat{b}_z$
$\mathbf{A} =$	$\sum_{i=1}^8 \vec{A}_i =$	<input type="text"/>
Area =	$ \vec{A} $	= 27.8

Accounting for overlapped areas is done with **positive** and **negative** signs on vectors.



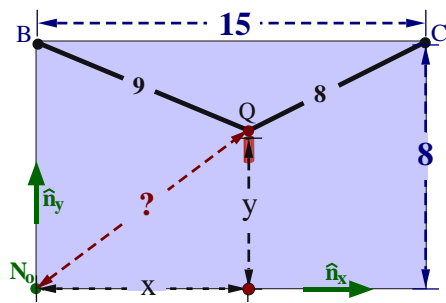
Note: Compute cross products with the distributive property $(\vec{a} + \vec{b}) \times (\vec{c} + \vec{d}) = \vec{a} \times \vec{c} + \vec{a} \times \vec{d} + \vec{b} \times \vec{c} + \vec{b} \times \vec{d}$ and the **cross-product-definition with the right-hand rule** (not determinants or special formulas in a book). Also, use the fact that \hat{b}_x , \hat{b}_y , \hat{b}_z are orthogonal unit vectors.



Planar objects, courtesy of Working Model and Design-Simulation Technologies

2.18 Locating a microphone (2D). Show work. (Section 1.4)

A microphone Q is attached to two pegs B and C by two cables. Knowing the peg locations, cable lengths, and points B , C , Q , N_o all lie in the same plane, determine the distance between Q and N_o . Try to do the problem first using Euclidean geometry - and then try vectors.²

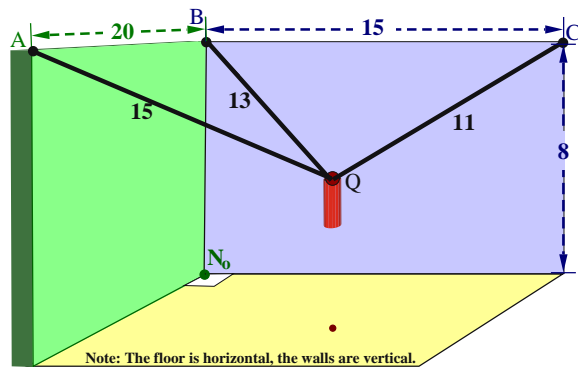


Quantity	Value
Distance from B to C	15 m
Distance from N_o to B	8 m
Length of cable joining B and Q	9 m
Length of cable joining C and Q	8 m
Distance between N_o and Q	9.01 m
If Q is above ceiling, distance ≈ 12 m	

Note: Although there are two “mathematical” answers to this problem, one is above the ceiling and requires the cables to be in compression.

2.19 † Locating a microphone (3D).

A microphone Q is attached to three pegs A , B , and C by three cables. Knowing the peg locations and cable lengths, determine the distance between Q and point N_o . Show work.³



Quantity	Value
Distance from A to B	20 m
Distance from B to C	15 m
Distance from N_o to B	8 m
Length of cable joining A and Q	15 m
Length of cable joining B and Q	13 m
Length of cable joining C and Q	11 m
Distance between N_o and Q	13.3 m
If Q is above ceiling, distance ≈ 17 m	

Note: This is part of the process of a camera targeting a football/baseball in a stadium or laser targeting cancer or ...

2.20 † Cable length to keep a window-washer’s beam stationary and horizontal.

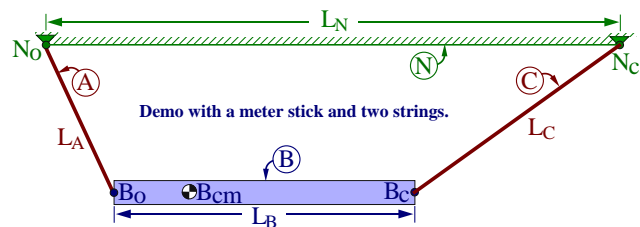
A beam B is attached to the roof of a building N by two relatively light (massless) cables A and C . Cable A attaches to the roof at point N_o of N and to the beam at point B_o of B . Cable C attaches to the roof at point N_C of N and to the beam at point B_C of B . N_o , B_o , B_{cm} , B_C , N_C are all in the same vertical plane. B_{cm} (center of mass of beam/workers) is $\frac{L_B}{4}$ from B_o .

Description	Symbol	Type	Value
Distance between N_o and N_C	L_N	Constant	16 m
Distance between B_o and B_C	L_B	Constant	7 m
Length of cable A	L_A	Constant	7 m
Length of cable C	L_C	Constant	? m

Determine L_C so the beam stays horizontal.

Result: $L_C =$ **9** m

If $L_B = L_N$, intuition/analysis predicts $L_C = L_A$ (vertical cables), independent of B_{cm} ’s location between B_o and B_C .



Stumped? www.MotionGenesis.com ⇒ [Textbooks](#) ⇒ [Resources](#)

2.21 Optional: FBD and statics “review” problems from Homework 16.

The following problems require an understanding of dot-products, force, and *free-body diagrams*.

Hw 16.1	Hw 16.12	Hw 16.13	Hw 16.14	Hw 16.16(a)
---------	----------	----------	----------	-------------

²Hint: For Euclidean geometry, use the law of cosines (at least once). For vectors, see Homework 1.35.

³Hint: See Homework 1.35 or Section 3.3. Introduce whatever **identifiers** facilitate your work.

Although nonlinear algebraic equations are usually solved with a computer, these can also be solved “by-hand”.

Solution at www.MotionGenesis.com. Alternately, www.WolframAlpha.com solves sets of nonlinear equations, e.g., type Solve $x^2 + (-20+z)^2 + (-8+y)^2 = 225$, $x^2 + z^2 + (-8+y)^2 = 169$, $z^2 + (-15+x)^2 + (-8+y)^2 = 121$