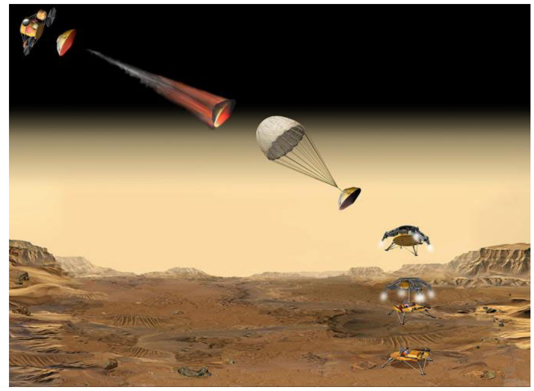
Chapter 2

Vectors



Courtesy NASA/JPL-Caltech

Summary (see examples in Hw 1, 2, 3)

Circa 1900 A.D., J. Williard Gibbs invented a useful combination of magnitude and direction called ***vectors*** and their higher-dimensional counterparts ***dyadics***, ***triadics***, and ***polyadics***. Vectors are an important **geometrical tool** e.g., for surveying, motion analysis, lasers, optics, computer graphics, animation, CAD/CAE (computer aided drawing/engineering), and FEA.

Symbol	Description	Details
$\vec{\mathbf{0}}$, $\widehat{\mathbf{u}}$	Zero vector and unit vector.	Sections 2.3, 2.4
$+ \quad - \quad *$	Vector addition, negation, subtraction, and multiplication/division with a scalar.	Sections 2.6 - 2.8
$\cdot \quad \times$	Vector dot product and cross product.	Sections 2.10, 2.11
${}^{F}\frac{d}{dt}$	Vector differentiation.	Chapters 7, 8

2.1 Examples of scalars, vectors, and dyadics

- A ***scalar*** is a non-directional quantity (e.g., a real number). Examples include:

time	density	volume	mass	moment of inertia	temperature
distance	speed	angle	weight	potential energy	kinetic energy

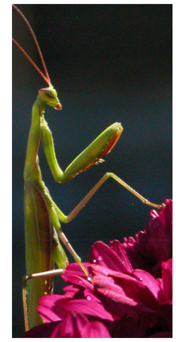
- A ***vector*** is a quantity that has magnitude and ***one*** associated direction. For example, a ***velocity vector*** has speed (how fast something is moving) and direction (which way it is going). A ***force vector*** has magnitude (how hard something is pushed) and direction (which way it is shoved). Examples include:

position vector	velocity	acceleration	linear momentum	force
impulse	angular velocity	angular acceleration	angular momentum	torque

- A ***dyad*** is a quantity with magnitude and ***two*** associated directions. For example, ***stress*** associates with area and force (both regarded as vectors). A ***dyadic*** is the **sum of dyads**. For example, an ***inertia dyadic*** (Chapter 16) is the sum of dyads associated with moments and products of inertia.

- A ***triad*** is a quantity that has magnitude and ***three*** directions. A ***triadic*** is the sum of triads.

2.2 Definition of a vector



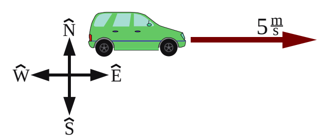
Courtesy Bro. Claude Rheaume LaSalette.

A ***vector*** is defined as a quantity having ***magnitude*** and ***direction***.[a]
Vectors are represented pictorially with straight or curved arrows (examples below).
Vectors are typeset with an arrow and bold-faced font, e.g., $\vec{\mathbf{v}}$ denotes a vector.

Certain vectors have additional special properties. For example, a ***position vector*** $\vec{\mathbf{r}}$ is associated with two points and has units of distance.

[a] A vector's ***magnitude*** is a real non-negative number. A vector's ***direction*** can be resolved into ***orientation*** and ***sense***. For example, a highway has an orientation (e.g., east-west) and a vehicle traveling east has a sense. Knowing both the orientation of a line and the sense on the line gives direction. Changing a vector's orientation or sense changes its direction.

Example of a vector: Consider the statement "the car is moving East at 5 $\frac{\text{m}}{\text{s}}$".
It is ***convenient*** to represent the car's speed and direction with the velocity vector $\vec{\mathbf{v}} = 5 * \widehat{\mathbf{East}} = 5\,\widehat{\mathbf{East}}$ (a hat designates the direction $\widehat{\mathbf{East}}$ as a ***unit vector***). The car's speed is always a real non-negative number equal to $|\vec{\mathbf{v}}|$, the ***magnitude*** of $\vec{\mathbf{v}}$. The combination of ***magnitude*** and ***direction*** is a ***vector***.



The velocity of a car moving West with speed 5 is $\vec{\mathbf{v}} = 5\,\widehat{\mathbf{West}} = -5\,\widehat{\mathbf{East}}$. The negative sign in $-5\,\widehat{\mathbf{East}}$ is associated with the vector's direction (the vector's magnitude is inherently non-negative). When a vector is written in terms of a scalar x that can be **positive** or zero or **negative**, e.g., as $x\,\widehat{\mathbf{East}}$, x is called the $\widehat{\mathbf{East}}$ ***measure*** of the vector, whereas the vector's non-negative ***magnitude*** is abs(x).

2.3 Zero vector $\vec{\mathbf{0}}$ and its properties

A ***zero vector*** $\vec{\mathbf{0}}$ is defined as a vector whose magnitude is zero.[1]

Addition of a vector $\vec{\mathbf{v}}$ with a zero vector:	$\vec{\mathbf{v}} + \vec{\mathbf{0}} = \vec{\mathbf{v}}$	
Dot product with a zero vector:	$\vec{\mathbf{v}} \cdot \vec{\mathbf{0}} \underset{(2)}{=} 0$	$\vec{\mathbf{0}}$ is ***perpendicular*** to all vectors
Cross product with a zero vector:	$\vec{\mathbf{v}} \times \vec{\mathbf{0}} \underset{(5)}{=} \vec{\mathbf{0}}$	$\vec{\mathbf{0}}$ is ***parallel*** to all vectors
Derivative of the zero vector:	$\frac{^F d\,\vec{\mathbf{0}}}{dt} = \vec{\mathbf{0}}$	F is any reference frame

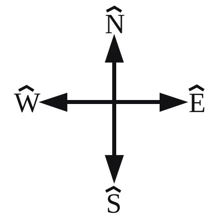
Vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ are said to be "***perpendicular***" if $\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = 0$ whereas $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ are "***parallel***" if $\vec{\mathbf{a}} \times \vec{\mathbf{b}} = \vec{\mathbf{0}}$.
Note: Some say $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ are "***parallel***" only if $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ have the same direction and anti-parallel if $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ have opposite directions.

[1] The direction of a zero vector $\vec{\mathbf{0}}$ is arbitrary and may be regarded as having **any** direction so that $\vec{\mathbf{0}}$ is ***parallel*** to all vectors, $\vec{\mathbf{0}}$ is ***perpendicular*** to all vectors, all zero vectors are equal, and one may use the definite pronoun "the" instead of the indefinite "a" e.g., "the zero vector". It is improper to say the ***zero vector*** has no direction as a vector is **defined** to have both magnitude and direction. It is also improper to say a ***zero vector*** has all directions as a vector is defined to have a magnitude and **a** direction (as contrasted with a dyad which has 2 directions or triad which has 3 directions).

2.4 Unit $\widehat{\text{v}}$ectors

A ***unit vector*** is defined as a vector whose magnitude is 1, and is typeset with a special hat, e.g., $\widehat{\mathbf{u}}$.

Unit vectors can be "***sign posts***", e.g., unit vectors $\widehat{\mathbf{N}}$, $\widehat{\mathbf{S}}$, $\widehat{\mathbf{W}}$, $\widehat{\mathbf{E}}$ for local Earth directions **North**, **South**, **West**, **East**. The direction of unit vectors are chosen to simplify communication and to produce efficient equations. Other useful "sign posts" are:



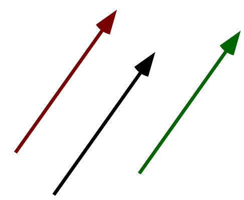
- Unit vector directed from one point to another point
- Unit vector directed locally vertical
- Unit vector parallel to the edge of an object
- Unit vector tangent to a curve or perpendicular to a surface

A unit vector can be defined so it has the same direction as an arbitrary non-zero vector $\vec{\mathbf{v}}$ by dividing $\vec{\mathbf{v}}$ by $|\vec{\mathbf{v}}|$ (the magnitude of $\vec{\mathbf{v}}$).

To avoid divide-by-zero problems during numerical computation, approximate the unit vector with a "small" positive real number ϵ in the denominator.

$$\textbf{unit}\vec{\textbf{V}}\textbf{ector} = \frac{\vec{\mathbf{v}}}{|\vec{\mathbf{v}}|} \approx \frac{\vec{\mathbf{v}}}{|\vec{\mathbf{v}}| + \epsilon} \tag{1}$$

2.5 Equal vectors (=)

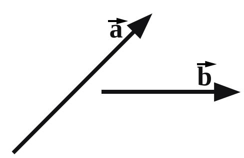


Two vectors are "equal" when they have the same magnitude and same direction. Shown to the right are three ***equal vectors***. Although each has a different location, the vectors are equal because they have the same magnitude and direction.[b]

[b]Some vectors have additional properties. For example, a position vector is associated with two points. Two position vectors are ***equal position vectors*** when, they have the same magnitude, same direction, and are associated with the same points. Two force vectors are ***equal force vectors*** when they have the same magnitude, direction, and point of application.

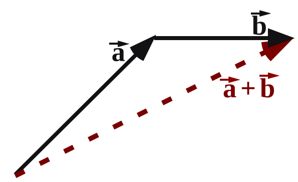
2.6 Vector addition (+)

As graphically shown to the right, adding two vectors $\vec{\mathbf{a}} + \vec{\mathbf{b}}$ produces a vector.[a] First, vector $\vec{\mathbf{b}}$ is translated[b] so its tail is at the tip of $\vec{\mathbf{a}}$. Next, the vector $\vec{\mathbf{a}} + \vec{\mathbf{b}}$ is drawn from the tail of $\vec{\mathbf{a}}$ to the tip of the translated $\vec{\mathbf{b}}$.



Properties of vector addition

Commutative law:	$\vec{\mathbf{a}} + \vec{\mathbf{b}} = \vec{\mathbf{b}} + \vec{\mathbf{a}}$
Associative law:	$(\vec{\mathbf{a}} + \vec{\mathbf{b}}) + \vec{\mathbf{c}} = \vec{\mathbf{a}} + (\vec{\mathbf{b}} + \vec{\mathbf{c}}) = \vec{\mathbf{a}} + \vec{\mathbf{b}} + \vec{\mathbf{c}}$
Addition of zero vector:	$\vec{\mathbf{a}} + \vec{\mathbf{0}} = \vec{\mathbf{a}}$



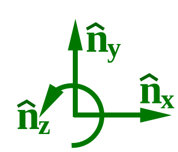
[a]It does not make sense to add vectors with different units, e.g., it is nonsensical to add a velocity vector with units of $\frac{\text{m}}{\text{s}}$ with an angular velocity vector with units of $\frac{\text{rad}}{\text{sec}}$.

[b]Translating $\vec{\mathbf{b}}$ does *not* change the magnitude or direction of $\vec{\mathbf{b}}$, and so produces an equal $\vec{\mathbf{b}}$.

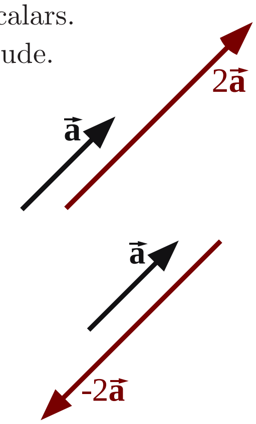
Example: Vector addition (+)

Shown to the right is an example of how to add vector $\vec{\mathbf{w}}$ to vector $\vec{\mathbf{v}}$, each which is expressed in terms of orthogonal unit vectors $\widehat{\mathbf{n}}_\text{x}$, $\widehat{\mathbf{n}}_\text{y}$, $\widehat{\mathbf{n}}_\text{z}$.

$$\begin{aligned} \vec{\mathbf{v}} &= 7\,\widehat{\mathbf{n}}_\text{x} + 5\,\widehat{\mathbf{n}}_\text{y} + 4\,\widehat{\mathbf{n}}_\text{z} \\ +\ \vec{\mathbf{w}} &= 2\,\widehat{\mathbf{n}}_\text{x} + 3\,\widehat{\mathbf{n}}_\text{y} + 2\,\widehat{\mathbf{n}}_\text{z} \\ \hline \vec{\mathbf{v}} + \vec{\mathbf{w}} &= 9\,\widehat{\mathbf{n}}_\text{x} + 8\,\widehat{\mathbf{n}}_\text{y} + 6\,\widehat{\mathbf{n}}_\text{z} \end{aligned}$$



2.7 Vector multiplied or divided by a scalar $(*$ or $/)$



To the right is a graphical representation of multiplying an arbitrary vector $\vec{\mathbf{a}}$ by real scalars.

- Multiplying a vector by a **positive** number (other than 1) changes the vector's magnitude.
- Multiplying a vector by a **negative** number changes the vector's magnitude **and** reverses the ***sense*** of the vector.
- Dividing a vector $\vec{\mathbf{a}}$ by a scalar s is defined as $\frac{\vec{\mathbf{a}}}{s} \triangleq \frac{1}{s} * \vec{\mathbf{a}}$.

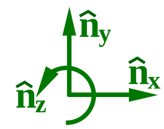
Properties of multiplication of a vector by a scalar s_1 or s_2

Commutative law:	$s_1 \vec{\mathbf{a}} = \vec{\mathbf{a}}\, s_1$
Associative law:	$s_1 (s_2 \vec{\mathbf{a}}) = (s_1 s_2) \vec{\mathbf{a}} = s_2 (s_1 \vec{\mathbf{a}}) = s_1 s_2 \vec{\mathbf{a}}$
Distributive law:	$(s_1 + s_2) \vec{\mathbf{a}} = s_1 \vec{\mathbf{a}} + s_2 \vec{\mathbf{a}}$
Distributive law:	$s_1 (\vec{\mathbf{a}} + \vec{\mathbf{b}}) = s_1 \vec{\mathbf{a}} + s_1 \vec{\mathbf{b}}$
Multiplication by zero:	$0 * \vec{\mathbf{a}} = \vec{\mathbf{0}}$

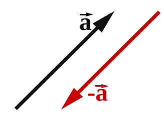
Example: Vector scalar multiplication and division $(*$ and $/)$

Given: $\vec{\mathbf{v}} = 7\,\hat{\mathbf{n}}_x + 5\,\hat{\mathbf{n}}_y + 4\,\hat{\mathbf{n}}_z$

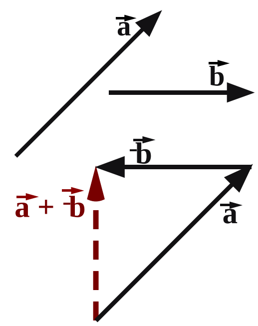
then: $5\,\vec{\mathbf{v}} = 35\,\hat{\mathbf{n}}_x + 25\,\hat{\mathbf{n}}_y + 20\,\hat{\mathbf{n}}_z$ and $\frac{\vec{\mathbf{v}}}{-2} = -3.5\,\hat{\mathbf{n}}_x - 2.5\,\hat{\mathbf{n}}_y - 2\,\hat{\mathbf{n}}_z$



2.8 Vector negation and subtraction $(-)$



Negation: As shown right, negating a vector (multiplying by -1) reverses the vector's ***sense*** (it points in the opposite direction). Negation does not change the vector's magnitude or orientation.



Subtraction: As the drawing to the right shows, subtracting a vector $\vec{\mathbf{b}}$ from a vector $\vec{\mathbf{a}}$ is simply addition and negation.[a]

$$\boxed{\vec{\mathbf{a}} - \vec{\mathbf{b}} \triangleq \vec{\mathbf{a}} + -\vec{\mathbf{b}}}$$

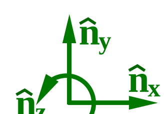
After negating vector $\vec{\mathbf{b}}$, it is translated so the tail of $-\vec{\mathbf{b}}$ is at the tip of $\vec{\mathbf{a}}$. Next, vector $\vec{\mathbf{a}} + -\vec{\mathbf{b}}$ is drawn from the tail of $\vec{\mathbf{a}}$ to the tip of the translated $-\vec{\mathbf{b}}$.

[a] In most/all mathematics, subtraction is defined as negation and addition.

Example: Vector subtraction $(-)$

Shown right is an example of how to subtract vector $\vec{\mathbf{w}}$ from vector $\vec{\mathbf{v}}$, when each is expressed in terms of orthogonal unit vectors $\hat{\mathbf{n}}_x$, $\hat{\mathbf{n}}_y$, $\hat{\mathbf{n}}_z$.

$$\begin{array}{rcl} \vec{\mathbf{v}} & = & 7\,\hat{\mathbf{n}}_x + 5\,\hat{\mathbf{n}}_y + 4\,\hat{\mathbf{n}}_z \\ -\ \vec{\mathbf{w}} & = & 2\,\hat{\mathbf{n}}_x + 3\,\hat{\mathbf{n}}_y + 2\,\hat{\mathbf{n}}_z \\ \hline \vec{\mathbf{v}} - \vec{\mathbf{w}} & = & 5\,\hat{\mathbf{n}}_x + 2\,\hat{\mathbf{n}}_y + 2\,\hat{\mathbf{n}}_z \end{array}$$

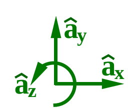


2.9 Words: Vectors vs. column matrices in the context of $\vec{\mathbf{F}} = m\,\vec{\mathbf{a}}$

Although mathematics uses the word[2] "vector" to describe a column matrix, a column matrix does ***not*** have direction. To associate direction, attach a basis e.g., as shown below.

$$\hat{\mathbf{a}}_x + 2\,\hat{\mathbf{a}}_y + 3\,\hat{\mathbf{a}}_z = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} \hat{\mathbf{a}}_x \\ \hat{\mathbf{a}}_y \\ \hat{\mathbf{a}}_z \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{a}}_x & \hat{\mathbf{a}}_y & \hat{\mathbf{a}}_z \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}_{\hat{\mathbf{a}}_{xyz}}$$

where $\hat{\mathbf{a}}_x$, $\hat{\mathbf{a}}_y$, $\hat{\mathbf{a}}_z$ are orthogonal unit vectors



Note: Although it can be helpful to represent vectors with a column matrix and orthogonal unit vectors, it is not always desirable or efficient. Postponing resolution of vectors into orthogonal components allows maximum use of simplifying vector properties and avoids simplifications such as $\sin^2(\theta) + \cos^2(\theta) = 1$ (see Homework 2.9).

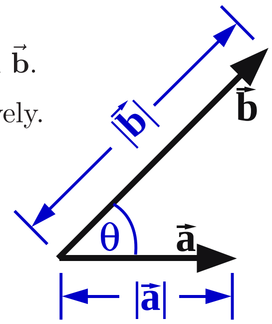
[2] Words have context. Some words are contranyms (opposite meanings) such as "fast" and "bolt" (move quickly or fasten).

2.10 Vector dot product $(\cdot)$

Equation (2) defines the ***dot product*** of vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$.

- $|\vec{\mathbf{a}}|$ and $|\vec{\mathbf{b}}|$ are the magnitudes of $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$, respectively.
- θ is the smallest angle between $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ $(0 \le \theta \le \pi)$.

Equation (3) is a rearrangement of equation (2) that is useful for calculating the angle θ between two vectors.
Note: $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ are "***perpendicular***" when $\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = 0$.
Note: Dot-products encapsulate the ***law of cosines***.



$$\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} \triangleq |\vec{\mathbf{a}}|\,|\vec{\mathbf{b}}|\,\cos(\theta) \qquad (2)$$

$$\cos(\theta) \underset{(2)}{=} \frac{\vec{\mathbf{a}} \cdot \vec{\mathbf{b}}}{|\vec{\mathbf{a}}|\;|\vec{\mathbf{b}}|} \qquad (3)$$

Use **acos** to calculate θ.

Equation (2) shows $\vec{\mathbf{v}} \cdot \vec{\mathbf{v}} = |\vec{\mathbf{v}}|^2$. Hence, the dot product can calculate a vector's ***magnitude*** as shown for $|\vec{\mathbf{v}}|$ in equation (4).

Equation (4) also defines ***vector exponentiation*** $\vec{\mathbf{v}}^n$ (vector $\vec{\mathbf{v}}$ raised to scalar power n) as a non-negative scalar.

Example: Kinetic energy $K = \frac{1}{2}\,m\,\vec{\mathbf{v}}^2 \underset{(4)}{=} \frac{1}{2}\,m\,\vec{\mathbf{v}} \cdot \vec{\mathbf{v}}$

$$\begin{aligned} \vec{\mathbf{v}}^2 \triangleq |\vec{\mathbf{v}}|^2 &= \vec{\mathbf{v}} \cdot \vec{\mathbf{v}} \\ |\vec{\mathbf{v}}| &= {}^{+}\!\sqrt{\vec{\mathbf{v}} \cdot \vec{\mathbf{v}}} \\ \vec{\mathbf{v}}^n \triangleq |\vec{\mathbf{v}}|^n &= {}^{+}(\vec{\mathbf{v}} \cdot \vec{\mathbf{v}})^{\frac{n}{2}} \end{aligned} \qquad (4)$$

2.10.1 Properties of the dot-product $(\cdot)$

Dot product with a zero vector	$\vec{\mathbf{a}} \cdot \vec{\mathbf{0}} = 0$	
Dot product of ***perpendicular*** vectors	$\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = 0$	if $\vec{\mathbf{a}} \perp \vec{\mathbf{b}}$
Dot product of parallel vectors	$\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = \pm\,|\vec{\mathbf{a}}|\,|\vec{\mathbf{b}}|$	if $\vec{\mathbf{a}} \parallel \vec{\mathbf{b}}$
Dot product with vectors scaled by s_1 and s_2	$s_1\,\vec{\mathbf{a}} \cdot s_2\,\vec{\mathbf{b}} = s_1\,s_2\,(\vec{\mathbf{a}} \cdot \vec{\mathbf{b}})$	
Commutative law	$\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = \vec{\mathbf{b}} \cdot \vec{\mathbf{a}}$	
Distributive law	$\vec{\mathbf{a}} \cdot (\vec{\mathbf{b}} + \vec{\mathbf{c}}) = \vec{\mathbf{a}} \cdot \vec{\mathbf{b}} + \vec{\mathbf{a}} \cdot \vec{\mathbf{c}}$	
Distributive law	$(\vec{\mathbf{a}} + \vec{\mathbf{b}}) \cdot (\vec{\mathbf{c}} + \vec{\mathbf{d}}) = \vec{\mathbf{a}} \cdot \vec{\mathbf{c}} + \vec{\mathbf{a}} \cdot \vec{\mathbf{d}} + \vec{\mathbf{b}} \cdot \vec{\mathbf{c}} + \vec{\mathbf{b}} \cdot \vec{\mathbf{d}}$	

Note: The distributive law for dot-products and cross-products is proved in [36, pgs. 23-24, 32-34].

2.10.2 Uses for the dot-product $(\cdot)$

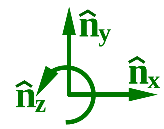
- Calculating an ***angle*** between two vectors [see equation (3) and example in Section 3.3] or determining when two vectors are ***perpendicular***, e.g., $\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = 0$.
- Calculating a vector's ***magnitude*** [see equation (4) and ***distance*** examples in Sections 3.2 and 3.3].
- Changing a ***vector equation*** into a ***scalar equation*** (see Homework 2.31).
- Calculating a ***unit vector*** in the direction of a vector $\vec{\mathbf{v}}$ [see equation (1)] $\quad \mathbf{unit}\vec{\mathbf{V}}\mathbf{ector} \underset{(1)}{=} \dfrac{\vec{\mathbf{v}}}{|\vec{\mathbf{v}}|}$
- ***Projection*** of a vector $\vec{\mathbf{v}}$ in the direction of $\vec{\mathbf{b}}$ is defined: $\vec{\mathbf{v}} \cdot \dfrac{\vec{\mathbf{b}}}{|\vec{\mathbf{b}}|}$

  See Section 4.6 for ***projections***, ***measures***, ***coefficients***, ***components***.

  Projection of $\vec{\mathbf{v}}$ onto the plane N perpendicular to $\widehat{\mathbf{n}}$: $\quad \vec{\mathbf{v}}_N = \vec{\mathbf{v}} - (\vec{\mathbf{v}} \cdot \widehat{\mathbf{n}})\,\widehat{\mathbf{n}} = \widehat{\mathbf{n}} \times (\vec{\mathbf{v}} \times \widehat{\mathbf{n}})$.
  Context: $\vec{\mathbf{v}}$ is a vector "bound" to a point v_o whose position vector $\vec{\mathbf{r}}$ from a point N_o fixed in N has $\vec{\mathbf{r}} \cdot \widehat{\mathbf{n}} > 0$.
  Example: Projection of a position vector $\vec{\mathbf{r}}$ (from N_o to a point R, where $\vec{\mathbf{r}} \cdot \widehat{\mathbf{n}} > 0$) onto N: $\vec{\mathbf{r}} - (\vec{\mathbf{r}} \cdot \widehat{\mathbf{n}})\,\widehat{\mathbf{n}}$.

  Projection of a parallelogram with edges characterized by $\vec{\mathbf{p}}$ and $\vec{\mathbf{q}}$ onto plane N: $\quad \left|\vec{\mathbf{p}}_N \times \vec{\mathbf{q}}_N\right|\widehat{\mathbf{n}}$.
  Magnitude of $\vec{\mathbf{p}}$'s projection on N crossed-with $\vec{\mathbf{q}}$'s projection on N times $\widehat{\mathbf{n}}$: $\quad \left|\vec{\mathbf{p}} \times \vec{\mathbf{q}} + [(\vec{\mathbf{p}} \cdot \widehat{\mathbf{n}})\,\vec{\mathbf{q}} - (\vec{\mathbf{q}} \cdot \widehat{\mathbf{n}})\,\vec{\mathbf{p}}] \times \widehat{\mathbf{n}}\right|\widehat{\mathbf{n}}$.

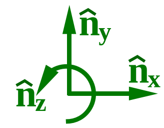
2.10.3 Special case: Dot-products with orthogonal unit vectors



When $\widehat{\mathbf{n}}_x$, $\widehat{\mathbf{n}}_y$, $\widehat{\mathbf{n}}_z$ are ***orthogonal unit*** vectors, it can be shown (see Homework 2.4)

$$(a_x\,\widehat{\mathbf{n}}_x + a_y\,\widehat{\mathbf{n}}_y + a_z\,\widehat{\mathbf{n}}_z) \cdot (b_x\,\widehat{\mathbf{n}}_x + b_y\,\widehat{\mathbf{n}}_y + b_z\,\widehat{\mathbf{n}}_z) \;=\; a_x\,b_x + a_y\,b_y + a_z\,b_z$$

2.10.4 Examples: Vector dot-products (·)



The following shows how to use dot-products with the vectors $\vec{\mathbf{v}}$ and $\vec{\mathbf{w}}$, each which is expressed in terms of the orthogonal unit vectors $\widehat{\mathbf{n}}_x$, $\widehat{\mathbf{n}}_y$, $\widehat{\mathbf{n}}_z$ shown to the right.

$$\vec{\mathbf{v}} \;=\; 7\,\widehat{\mathbf{n}}_x + 5\,\widehat{\mathbf{n}}_y + 4\,\widehat{\mathbf{n}}_z \qquad\qquad \vec{\mathbf{w}} \;=\; 2\,\widehat{\mathbf{n}}_x + 3\,\widehat{\mathbf{n}}_y + 2\,\widehat{\mathbf{n}}_z$$

$\widehat{\mathbf{n}}_x$ measure of $\vec{\mathbf{v}}$ $\qquad$ $\vec{\mathbf{v}} \cdot \widehat{\mathbf{n}}_x = 7$ (**measures** how much of $\vec{\mathbf{v}}$ is in the $\widehat{\mathbf{n}}_x$ direction).

$\vec{\mathbf{v}} \cdot \vec{\mathbf{v}} = 7^2 + 5^2 + 4^2 = 90$ $\qquad$ $|\vec{\mathbf{v}}| = \sqrt{90} \approx 9.4868$

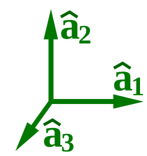
$\vec{\mathbf{w}} \cdot \vec{\mathbf{w}} = 2^2 + 3^2 + 2^2 = 17$ $\qquad$ $|\vec{\mathbf{w}}| = \sqrt{17} \approx 4.1231$

Unit vector in the direction of $\vec{\mathbf{v}}$: $\quad \dfrac{\vec{\mathbf{v}}}{|\vec{\mathbf{v}}|} = \dfrac{7\,\widehat{\mathbf{n}}_x + 5\,\widehat{\mathbf{n}}_y + 4\,\widehat{\mathbf{n}}_z}{\sqrt{90}} \approx 0.738\,\widehat{\mathbf{n}}_x + 0.527\,\widehat{\mathbf{n}}_y + 0.422\,\widehat{\mathbf{n}}_z$

Unit vector in the direction of $\vec{\mathbf{w}}$: $\quad \dfrac{\vec{\mathbf{w}}}{|\vec{\mathbf{w}}|} = \dfrac{2\,\widehat{\mathbf{n}}_x + 3\,\widehat{\mathbf{n}}_y + 2\,\widehat{\mathbf{n}}_z}{\sqrt{17}} \approx 0.485\,\widehat{\mathbf{n}}_x + 0.728\,\widehat{\mathbf{n}}_y + 0.485\,\widehat{\mathbf{n}}_z$

$\vec{\mathbf{v}} \cdot \vec{\mathbf{w}} = 7*2 + 5*3 + 4*2 = 37 \qquad \angle(\vec{\mathbf{v}}, \vec{\mathbf{w}}) = \text{acos}\big(\dfrac{37}{\sqrt{90}\;\sqrt{17}}\big) \approx 0.33 \text{ rad} \approx 18.93^\circ$

2.10.5 Dot-products to change vector equations to scalar equations (see Hw 1.31)



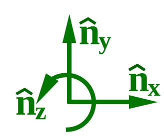
One way to form up to three linearly independent scalar equations from the vector equation $\vec{\mathbf{v}} = \vec{\mathbf{0}}$ is by dot-multiplying $\vec{\mathbf{v}} = \vec{\mathbf{0}}$ with three orthogonal unit vectors $\widehat{\mathbf{a}}_1$, $\widehat{\mathbf{a}}_2$, $\widehat{\mathbf{a}}_3$, i.e.,

Method 1: if $\vec{\mathbf{v}} = \vec{\mathbf{0}} \;\Rightarrow\; \boxed{\vec{\mathbf{v}} \cdot \widehat{\mathbf{a}}_1 = 0 \qquad \vec{\mathbf{v}} \cdot \widehat{\mathbf{a}}_2 = 0 \qquad \vec{\mathbf{v}} \cdot \widehat{\mathbf{a}}_3 = 0}$

Section 2.12.2 describes another way to form three ***different*** scalar equations from $\vec{\mathbf{v}} = \vec{\mathbf{0}}$.

2.10.6 Optional: Special case of $\vec{\mathbf{a}} \cdot \vec{\mathbf{b}}$ as matrix multiplication $a^T\,b$ (orthogonal vectors)

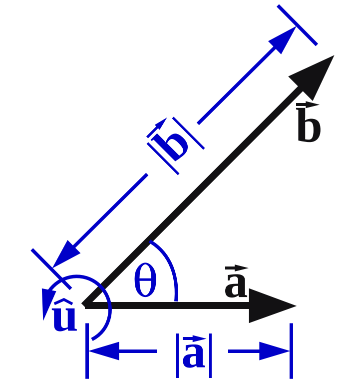
When $\vec{\mathbf{a}} = a_x\,\widehat{\mathbf{n}}_x + a_y\,\widehat{\mathbf{n}}_y + a_z\,\widehat{\mathbf{n}}_z$ and $\vec{\mathbf{b}} = b_x\,\widehat{\mathbf{n}}_x + b_y\,\widehat{\mathbf{n}}_y + b_z\,\widehat{\mathbf{n}}_z$, where $\widehat{\mathbf{n}}_x$, $\widehat{\mathbf{n}}_y$, $\widehat{\mathbf{n}}_z$ are **orthogonal unit** vectors, the dot-product $\vec{\mathbf{a}} \cdot \vec{\mathbf{b}}$ is related to the multiplication of the $\widehat{\mathbf{n}}_x$, $\widehat{\mathbf{n}}_y$, $\widehat{\mathbf{n}}_z$ row matrix representation of $\vec{\mathbf{a}}$ with the $\widehat{\mathbf{n}}_x$, $\widehat{\mathbf{n}}_y$, $\widehat{\mathbf{n}}_z$ column matrix representation of $\vec{\mathbf{b}}$ as



$$\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} \;=\; \begin{bmatrix} a_x & a_y & a_z \end{bmatrix}_{\widehat{\mathbf{n}}_{xyz}} * \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix}_{\widehat{\mathbf{n}}_{xyz}} \;=\; a_x\,b_x + a_y\,b_y + a_z\,b_z$$

Courtesy Accuray Inc.. Dot-products are heavily used in radiation and other medical equipment.

2.11 Vector cross product $(\times)$



The ***cross product*** of a vector $\vec{a}$ with a vector $\vec{b}$ is defined in equation (5).

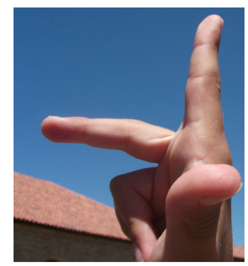
- $|\vec{a}|$ and $|\vec{b}|$ are the magnitudes of $\vec{a}$ and $\vec{b}$, respectively
- θ is the smallest angle between $\vec{a}$ and $\vec{b}$ $(0 \le \theta \le \pi)$.
- $\hat{u}$ is the unit vector **perpendicular** to both $\vec{a}$ and $\vec{b}$.
  The direction of $\hat{u}$ is determined by the ***right-hand rule***.[a]

  Note: $|\vec{a}|\,|\vec{b}|\,\sin(\theta)$ [the coefficient of $\hat{u}$ in equation (5)] is inherently ***non-negative*** because $\sin(\theta) \ge 0$ since $0 \le \theta \le \pi$. Hence, $|\vec{a} \times \vec{b}| = |\vec{a}|\,|\vec{b}|\,\sin(\theta)$.

[a] The right-hand rule is a convention, much like driving on the right-hand side of the road in North America. Until 1965, the Soviet Union used the left-hand rule.

$$\vec{a} \times \vec{b} \triangleq |\vec{a}|\,|\vec{b}|\,\sin(\theta)\,\hat{u} \qquad (5)$$

2.11.1 Properties of the cross-product $(\times)$



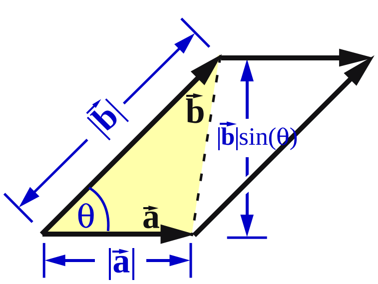
Cross product with a zero vector	$\vec{a} \times \vec{0} = \vec{0}$	
Cross product of a vector with itself	$\vec{a} \times \vec{a} = \vec{0}$	
Cross product of ***parallel*** vectors	$\vec{a} \times \vec{b} = \vec{0} \quad \text{if } \vec{a} \parallel \vec{b}$	
Cross product with vectors scaled by s_1 and s_2	$s_1\vec{a} \times s_2\vec{b} = s_1\,s_2\,(\vec{a} \times \vec{b})$	
Cross products are ***not*** commutative	$\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$	(6)
Distributive law	$\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$	
Distributive law	$(\vec{a} + \vec{b}) \times (\vec{c} + \vec{d}) = \vec{a} \times \vec{c} + \vec{a} \times \vec{d} + \vec{b} \times \vec{c} + \vec{b} \times \vec{d}$	
Cross products are ***not*** associative	$\vec{a} \times (\vec{b} \times \vec{c}) \neq (\vec{a} \times \vec{b}) \times \vec{c}$	
Vector triple cross product	$\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}\,(\vec{a} \cdot \vec{c}) - \vec{c}\,(\vec{a} \cdot \vec{b})$	(7)
When $\vec{b}$ is a unit vector	$\lvert\vec{a} \times \hat{b}\rvert^2 = \vec{a} \cdot \vec{a} - (\vec{a} \cdot \hat{b})^2$	

A mnemonic for $\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}\,(\vec{a} \cdot \vec{c}) - \vec{c}\,(\vec{a} \cdot \vec{b})$ is "**bac**k **cab**" - as in were you born in the back of a cab?
Many proofs of this formula resolve $\vec{a}$, $\vec{b}$, and $\vec{c}$ into orthogonal unit vectors (e.g., $\hat{n}_x$, $\hat{n}_y$, $\hat{n}_z$) and equate components.

2.11.2 Uses for the cross-product $(\times)$

Several uses for the cross-product in geometry, statics, and motion analysis, include calculating:

- ***Perpendicular*** vectors, e.g., $\vec{a} \times \vec{b}$ is perpendicular to both $\vec{a}$ and $\vec{b}$
- ***Moment*** of a force or linear momentum, e.g., $\vec{r} \times \vec{F}$ and $\vec{r} \times m\,\vec{v}$
- ***Velocity/acceleration*** formulas, e.g., $\vec{v} = \vec{\omega} \times \vec{r}$ and $\vec{a} = \vec{\alpha} \times \vec{r} + \vec{\omega} \times (\vec{\omega} \times \vec{r})$
- ***Area of a triangle*** whose sides have length $|\vec{a}|$ and $|\vec{b}|$

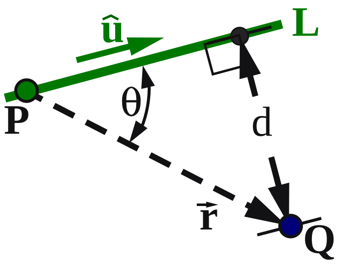


The ***area of a triangle*** Δ is half the area of a parallelogram.[a] [b]
A geometrical interpretation of $|\vec{a} \times \vec{b}|$ is the ***area of a parallelogram*** having sides of length $|\vec{a}|$ and $|\vec{b}|$, hence

$$\Delta(\vec{a}, \vec{b}) = \tfrac{1}{2}\,|\vec{a} \times \vec{b}| \qquad (8)$$

[a] Homework 2.17 shows the utility of equation (8) for ***surveying***.
[b] Section 3.3 shows the utility of a cross-product for area calculations.

- ***Distance*** d between a line L and a point Q.

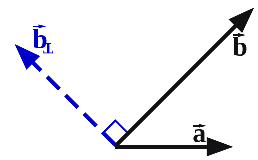


The line L (shown left) passes through point P and is parallel to the unit vector $\hat{\mathbf{u}}$. The ***distance*** d between line L and a point Q can be calculated as

$$\text{d} \;=\; \left|\vec{\mathbf{r}}^{\,Q/P} \times \hat{\mathbf{u}}\right| \;\underset{(5)}{=}\; \left|\vec{\mathbf{r}}^{\,Q/P}\right| \sin(\theta) \tag{9}$$

Note: See example in Hw 1.26. Other distance calculations are in Sections 3.2 and 3.3.

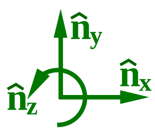
- The vector $\vec{\mathbf{b}}_\perp$ (shown right) is perpendicular to $\vec{\mathbf{b}}$ and is in the plane containing both $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$. It is calculated with the ***vector triple cross product***:



$$\vec{\mathbf{b}}_\perp \;=\; (\vec{\mathbf{a}} \times \vec{\mathbf{b}}) \times \vec{\mathbf{b}}$$

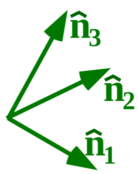
In general, $|\vec{\mathbf{b}}_\perp| \neq |\vec{\mathbf{b}}|$ and $\vec{\mathbf{b}}_\perp$ is not perpendicular to $\vec{\mathbf{a}}$.

2.11.3 Determinants and cross-products (with right-handed unit vectors)



When $\hat{\mathbf{n}}_\text{x}$, $\hat{\mathbf{n}}_\text{y}$, $\hat{\mathbf{n}}_\text{z}$ are **orthogonal unit** vectors, it can be shown (see Homework 2.13) that the cross product of $\vec{\mathbf{a}} = a_x\,\hat{\mathbf{n}}_\text{x} + a_y\,\hat{\mathbf{n}}_\text{y} + a_z\,\hat{\mathbf{n}}_\text{z}$ with $\vec{\mathbf{b}} = b_x\,\hat{\mathbf{n}}_\text{x} + b_y\,\hat{\mathbf{n}}_\text{y} + b_z\,\hat{\mathbf{n}}_\text{z}$ happens to be equal to the ***determinant*** of the following matrix.

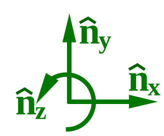
$$\vec{\mathbf{a}} \times \vec{\mathbf{b}} \;=\; \det\begin{bmatrix} \hat{\mathbf{n}}_\text{x} & \hat{\mathbf{n}}_\text{y} & \hat{\mathbf{n}}_\text{z} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{bmatrix} \;=\; \begin{array}{l} \quad\;(a_y\,b_z - a_z\,b_y)\,\hat{\mathbf{n}}_\text{x} \\ -\;(a_x\,b_z - a_z\,b_x)\,\hat{\mathbf{n}}_\text{y} \\ +\;(a_x\,b_y - a_y\,b_x)\,\hat{\mathbf{n}}_\text{z} \end{array} \tag{10}$$



Similarly for **non-orthogonal unit** vectors $\hat{\mathbf{n}}_1$, $\hat{\mathbf{n}}_2$, $\hat{\mathbf{n}}_3$ and the cross product of $\vec{\mathbf{c}} = c_1\,\hat{\mathbf{n}}_1 + c_2\,\hat{\mathbf{n}}_2 + c_3\,\hat{\mathbf{n}}_3$ with $\vec{\mathbf{d}} = d_1\,\hat{\mathbf{n}}_1 + d_2\,\hat{\mathbf{n}}_2 + d_3\,\hat{\mathbf{n}}_3$.

$$\vec{\mathbf{c}} \times \vec{\mathbf{d}} \;=\; \det\begin{bmatrix} \hat{\mathbf{n}}_2\times\hat{\mathbf{n}}_3 & \hat{\mathbf{n}}_3\times\hat{\mathbf{n}}_1 & \hat{\mathbf{n}}_1\times\hat{\mathbf{n}}_2 \\ c_1 & c_2 & c_3 \\ d_1 & d_2 & d_3 \end{bmatrix} \;=\; \begin{array}{l} \quad\;(c_2\,d_3 - c_3\,d_2)\,\hat{\mathbf{n}}_2\times\hat{\mathbf{n}}_3 \\ -\;(c_1\,d_3 - c_3\,d_1)\,\hat{\mathbf{n}}_3\times\hat{\mathbf{n}}_1 \\ +\;(c_1\,d_2 - c_2\,d_1)\,\hat{\mathbf{n}}_1\times\hat{\mathbf{n}}_2 \end{array}$$

Examples: Vector cross-products (×) with determinants.



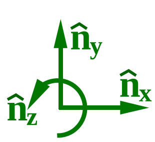
The following shows how to use cross-products with the vectors $\vec{\mathbf{v}}$ and $\vec{\mathbf{w}}$, each which is expressed in terms of the orthogonal unit vectors $\hat{\mathbf{n}}_\text{x}$, $\hat{\mathbf{n}}_\text{y}$, $\hat{\mathbf{n}}_\text{z}$ shown to the right.

$$\left.\begin{array}{l} \vec{\mathbf{v}} \;=\; 7\,\hat{\mathbf{n}}_\text{x} + 5\,\hat{\mathbf{n}}_\text{y} + 4\,\hat{\mathbf{n}}_\text{z} \\ \vec{\mathbf{w}} \;=\; 2\,\hat{\mathbf{n}}_\text{x} + 3\,\hat{\mathbf{n}}_\text{y} + 2\,\hat{\mathbf{n}}_\text{z} \end{array}\right\} \qquad \vec{\mathbf{v}} \times \vec{\mathbf{w}} \;=\; \det\begin{bmatrix} \hat{\mathbf{n}}_\text{x} & \hat{\mathbf{n}}_\text{y} & \hat{\mathbf{n}}_\text{z} \\ 7 & 5 & 4 \\ 2 & 3 & 2 \end{bmatrix} \;=\; -2\,\hat{\mathbf{n}}_\text{x} - 6\,\hat{\mathbf{n}}_\text{y} + 11\,\hat{\mathbf{n}}_\text{z}$$

Area from vectors $\vec{\mathbf{v}}$ and $\vec{\mathbf{w}}$: $\Delta(\vec{\mathbf{v}}, \vec{\mathbf{w}}) \;=\; \frac{1}{2}\,|\vec{\mathbf{v}} \times \vec{\mathbf{w}}| \;=\; \frac{1}{2}\sqrt{(-2)^2 + (-6)^2 + 11^2} \;=\; \frac{\sqrt{161}}{2} \;\approx\; 6.344$

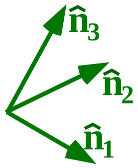
$$\vec{\mathbf{v}} \times (\vec{\mathbf{v}} \times \vec{\mathbf{w}}) \;=\; \det\begin{bmatrix} \hat{\mathbf{n}}_\text{x} & \hat{\mathbf{n}}_\text{y} & \hat{\mathbf{n}}_\text{z} \\ 7 & 5 & 4 \\ -2 & -6 & 11 \end{bmatrix} \;=\; 79\,\hat{\mathbf{n}}_\text{x} - 85\,\hat{\mathbf{n}}_\text{y} - 32\,\hat{\mathbf{n}}_\text{z}$$

2.11.4 Optional: Cross product as skew-symmetric matrix multiplication



When $\hat{\mathbf{n}}_\text{x}$, $\hat{\mathbf{n}}_\text{y}$, $\hat{\mathbf{n}}_\text{z}$ are **orthogonal unit** vectors, the cross product of $\vec{\mathbf{a}} = a_x\,\hat{\mathbf{n}}_\text{x} + a_y\,\hat{\mathbf{n}}_\text{y} + a_z\,\hat{\mathbf{n}}_\text{z}$ with $\vec{\mathbf{b}} = b_x\,\hat{\mathbf{n}}_\text{x} + b_y\,\hat{\mathbf{n}}_\text{y} + b_z\,\hat{\mathbf{n}}_\text{z}$ can be calculated with ***skew symmetric matrix multiplication***.

$$\vec{\mathbf{a}} \times \vec{\mathbf{b}} \;\underset{(10)}{=}\; \begin{bmatrix} \hat{\mathbf{n}}_\text{x} & \hat{\mathbf{n}}_\text{y} & \hat{\mathbf{n}}_\text{z} \end{bmatrix} \begin{bmatrix} a_y\,b_z - a_z\,b_y \\ a_z\,b_x - a_x\,b_z \\ a_x\,b_y - a_y\,b_x \end{bmatrix} \;=\; \begin{bmatrix} \hat{\mathbf{n}}_\text{x} & \hat{\mathbf{n}}_\text{y} & \hat{\mathbf{n}}_\text{z} \end{bmatrix} \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix} \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix} \tag{11}$$



Similarly for **non-orthogonal unit** vectors $\hat{\mathbf{n}}_1$, $\hat{\mathbf{n}}_2$, $\hat{\mathbf{n}}_3$ and the cross product of $\vec{\mathbf{c}} = c_1\,\hat{\mathbf{n}}_1 + c_2\,\hat{\mathbf{n}}_2 + c_3\,\hat{\mathbf{n}}_3$ with $\vec{\mathbf{d}} = d_1\,\hat{\mathbf{n}}_1 + d_2\,\hat{\mathbf{n}}_2 + d_3\,\hat{\mathbf{n}}_3$.
Hw 2.14 discusses ***inefficiencies*** of calculating cross products with skew-symmetric matrix multiplication. Skew-symmetric matrices are useful for relating angular velocity to the time-derivative of a rotation matrix.

$$\vec{\mathbf{c}} \times \vec{\mathbf{d}} = \begin{bmatrix} \hat{\mathbf{n}}_2 \times \hat{\mathbf{n}}_3 & \hat{\mathbf{n}}_3 \times \hat{\mathbf{n}}_1 & \hat{\mathbf{n}}_1 \times \hat{\mathbf{n}}_2 \end{bmatrix} \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \qquad (12)$$

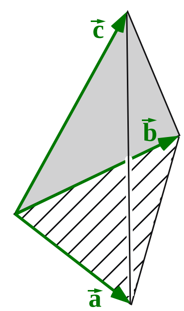
2.12 Scalar triple product ($\cdot\ \times$ or $\times\ \cdot$)

The ***scalar triple product*** of vectors $\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$, $\vec{\mathbf{c}}$ is the scalar defined in the various ways shown in equation (13). Homework 2.16 shows how ***determinants*** can calculate scalar triple products.

$$\text{ScalarTripleProduct} \triangleq \boxed{\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} \times \vec{\mathbf{c}} = \vec{\mathbf{a}} \times \vec{\mathbf{b}} \cdot \vec{\mathbf{c}}} = \vec{\mathbf{b}} \cdot \vec{\mathbf{c}} \times \vec{\mathbf{a}} = \vec{\mathbf{b}} \times \vec{\mathbf{c}} \cdot \vec{\mathbf{a}} \qquad (13)$$

Although parentheses make equation (13) clearer, i.e., ScalarTripleProduct $\triangleq \vec{\mathbf{a}} \cdot (\vec{\mathbf{b}} \times \vec{\mathbf{c}})$, the parentheses are unnecessary because the cross product $\vec{\mathbf{b}} \times \vec{\mathbf{c}}$ **must** be performed before the dot product for a sensible result to be produced.

2.12.1 Scalar triple product and the volume of a tetrahedron

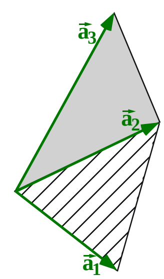


A geometrical interpretation of $\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} \times \vec{\mathbf{c}}$ is the ***volume of a parallelepiped*** having sides of length $|\vec{\mathbf{a}}|$, $|\vec{\mathbf{b}}|$, and $|\vec{\mathbf{c}}|$. The formula for the ***volume of a tetrahedron*** whose sides are described by the vectors $\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$, $\vec{\mathbf{c}}$ is

$$\text{Tetrahedron Volume} = \frac{1}{6}\ \vec{\mathbf{a}} \cdot \vec{\mathbf{b}} \times \vec{\mathbf{c}}$$

This formula is used for volume calculations (e.g., highway ***surveying*** cut and fill), 3D ***CAD***, solid geometry, and mass property calculations.

2.12.2 ($\times\ \cdot$) to change vector equations to scalar equations (see Hw 1.31)



Section 2.10.5 showed one method to form scalar equations from the vector equation $\vec{\mathbf{v}} = \vec{\mathbf{0}}$. A 2^{nd} method expresses $\vec{\mathbf{v}}$ in terms of three non-coplanar (but not necessarily orthogonal or unit) vectors $\vec{\mathbf{a}}_1$, $\vec{\mathbf{a}}_2$, $\vec{\mathbf{a}}_3$, and writes the equally valid (but generally different) set of linearly independent scalar equations shown below.

Method 2: if $\vec{\mathbf{v}} = v_1\,\vec{\mathbf{a}}_1 + v_2\,\vec{\mathbf{a}}_2 + v_3\,\vec{\mathbf{a}}_3 = \vec{\mathbf{0}} \;\Rightarrow\; \boxed{v_1 = 0 \quad v_2 = 0 \quad v_3 = 0}$

Note: The proof that $v_i = 0$ $(i = 1, 2, 3)$ follows directly by substituting $\vec{\mathbf{v}} = \vec{\mathbf{0}}$ into equation (4.2).

Vectors are used with ***surveying data*** for volume cut-and-fill dirt calculations for highway construction