Chapter 28

MIPSI: Classic particle pendulum

The pendulum to the right consists of a small heavy metallic ball tied to a light cable which is connected to the ceiling. This chapter investigates this system with the MIPSI process (**M**odeling, **I**dentifiers, **P**hysics, **S**implify/solve, **I**nterpret/design) and shows how to use various methods to form its equations of motion.

Newton's law $\vec{\mathbf{F}} = m\,\vec{\mathbf{a}}$	Angular momentum principle
Euler's rigid body equation	Power/energy-rate principle
Conservation of mechanical energy	Lagrange's method
MG road-maps (see Section 22.6.8)	Kane's method

Reproduction of Foucault pendulum. Pantheon, Paris France (Andrew Schmidt).

28.1 Modeling the classic particle pendulum

A ***model*** is a simplified representation of a complex system. Creating an ***accurate*** model requires good ***judgement*** to differentiate between what can be simplified and what cannot. Capturing a system's essence and extracting its meaningful parts is essential to engineering analysis and design – yet is difficult to teach. Some assumptions an engineer may make when modeling a pendulum are:

1. **The support is rigid (inflexible) and the cable is firmly (not loosely) attached to it. The cable is rigid (inflexible) and does not break.**
   Analyzing a flexible cable that elongates/vibrates (see Homework 8.14) or a cable that supports tension but not compression (see Homework 10.23) is significantly more difficult than analyzing a straight inextensible cable.[1]

2. **The cable is massless (significantly lighter than the objects it supports).**
   One indicator of this assumption's validity is whether or not the cable's kinetic energy is substantially smaller than the attached particle's kinetic energy. A small mass or moment of inertia ***cannot*** be ignored if it is associated with large kinetic energy.[2] Kinetic energy can help for modeling a massive spring by determining how to lump some of the spring's mass with each body attached to the spring.

3. **The massive object at the distal end is a particle or rigid body.**
   This assumption allows replacement of forces (e.g., gravity) on the object by a simpler equivalent set.

[1] One can test the inextensible-cable-assumption by numerical simulation of a pendulum with an extensible cable and observing that, as the cable's stiffness is increased, the solution approaches the rigid-cable solution (see Homework 8.15). Note: Computer time to numerically simulate motion increases with cable stiffness. When the period of the cable's extensional oscillations are much shorter than the pendulum's period, the system is said to have ***stiff differential equations***.

[2] For example, it may be ***unreasonable*** to ignore the small moment of inertia associated with a rapidly spinning small motor that is geared to a slowly spinning large object. Rotational kinetic energy scales with $I\,\omega^2$, so a rapidly spinning small motor may have more kinetic energy than a slowly spinning massive object.

4. **The massive body can be modeled as a particle (body is small and dense).**
This assumption seems reasonable if the the body's orientation is not of interest and the body's rotational kinetic energy is substantially smaller than its translational kinetic energy.

5. **The cable has a simple rotation relative to the room (simple angular velocity).**
This implies the system's motion can be described with **one** dependent variable (e.g., θ), and is valid if motion occurs over a short period of time (less than a few hours). Over longer periods of time, the Earth's rotation causes the particle to move out of plane (Foucault pendulum).

6. **The Earth is a Newtonian reference frame.**
Newton's law of motion $\vec{\mathbf{F}} = m\,\vec{\mathbf{a}}$ requires a non-accelerating and non-rotating reference frame. Although Earth is rotating (daily around its axis and yearly about the sun), the acceleration associated with these motions is ***assumed*** to be insignificantly small. Foucault showed the Earth's rotation has a substantial effect on daily motions of a simple pendulum.

7. **Earth's gravitational attraction can be approximated as a uniform field.**
In reality, gravitational forces vary as an inverse-square law. Hence, Earth's gravitational forces on the pendulum decrease as the pendulum swings up (away from Earth) and increase as it swings down (closer to Earth). Note: Uniform gravity is not used for analyzing satellite orientation (in motion for years).

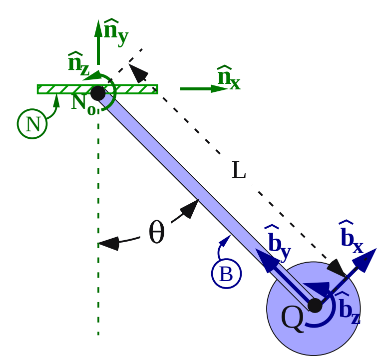
8. **Other than Earth, gravitational forces are negligible.**
Other massive objects (tables, people, the moon, sun, black holes, etc.) have negligible gravitational attraction. An example helps validate this assumption. The magnitude of gravitational force between two lead spheres of radius 1 ft (30 cm) and mass 3035 lbm (1377 kg) whose centers are only 3 ft (91 cm) apart (closest point is 1 foot apart) is 0.000034 lbf (0.00015 N), $\approx$ 90,000,000 times smaller than Earth's gravitational force exerted on the sphere.

9. **Many forces (aerodynamic, friction, magnetic, electrostatic) are negligible.**
Note: Neglecting other forces (e.g., air resistance) is questionable as they affect long-term behavior.

28.2 Identifiers for the classic particle pendulum

The schematic to the right shows a light (massless) inextensible cable B with a particle Q attached to its distal end. The cable is connected to the ceiling N (a Newtonian reference frame) at point N_o.

Right-handed orthogonal unit vectors $\widehat{\mathbf{n}}_x$, $\widehat{\mathbf{n}}_y$, $\widehat{\mathbf{n}}_z$ are fixed in N with $\widehat{\mathbf{n}}_x$ horizontally-right and $\widehat{\mathbf{n}}_y$ vertically-upward. Right-handed orthogonal unit vectors $\widehat{\mathbf{b}}_x$, $\widehat{\mathbf{b}}_y$, $\widehat{\mathbf{b}}_z$ are fixed in B with $\widehat{\mathbf{b}}_y$ directed from Q to N_o and $\widehat{\mathbf{b}}_z = \widehat{\mathbf{n}}_z$ parallel to B's axis of rotation in N.

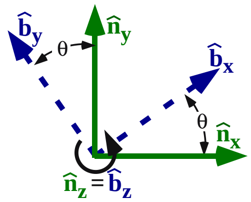


Quantity	Identifier	Type	Value
Earth's gravitational constant	g	Constant	9.8 m/s²
Mass of Q	m	Constant	2 kg
Length of B	L	Constant	1.0 m
Angle from $\widehat{\mathbf{n}}_y$ to $\widehat{\mathbf{b}}_y$ with $^+\widehat{\mathbf{n}}_z$ sense	θ	Variable	60° (initial)
Tension in cable	T	Variable	

28.3 Physics: Equations of motion of the classic particle pendulum

There are many methods for formulating equations of motion, each requiring kinematic and kinetic analysis, e.g., orientation, angular velocity, angular acceleration, position, velocity, acceleration, and force.
Note: Section 22.6.8 provides an efficient ***MG road-map*** for this problem.

- **Rotation matrix:** ${}^{b}R^{n}$ relates $\widehat{\mathbf{b}}_x$, $\widehat{\mathbf{b}}_y$, $\widehat{\mathbf{b}}_z$ to $\widehat{\mathbf{n}}_x$, $\widehat{\mathbf{n}}_y$, $\widehat{\mathbf{n}}_z$ and is found by first drawing the unit vectors in a suggestive way as shown below and using the definitions of $\sin(\theta)$ and $\cos(\theta)$.



${}^bR^n$	$\widehat{\mathbf{n}}_x$	$\widehat{\mathbf{n}}_y$	$\widehat{\mathbf{n}}_z$
$\widehat{\mathbf{b}}_x$	$\cos(\theta)$	$\sin(\theta)$	0
$\widehat{\mathbf{b}}_y$	$-\sin(\theta)$	$\cos(\theta)$	0
$\widehat{\mathbf{b}}_z$	0	0	1

(1)

- **Angular velocity and angular acceleration:**
  Since $\widehat{\mathbf{b}}_z$ is fixed in both B and N, B has a simple angular velocity in N. ${}^N\vec{\boldsymbol{\omega}}^B$ is calculated using the right-hand rule and viewing how θ increases.
  By definition ${}^N\vec{\boldsymbol{\alpha}}^B \triangleq \frac{{}^N d\, {}^N\vec{\boldsymbol{\omega}}^B}{dt}$, hence B's angular acceleration in N is:

$$ {}^N\vec{\boldsymbol{\omega}}^B = \dot{\theta}\, \widehat{\mathbf{b}}_z \quad (2)$$

$$ {}^N\vec{\boldsymbol{\alpha}}^B \underset{(2)}{=} \ddot{\theta}\, \widehat{\mathbf{b}}_z \quad (3)$$

- **Position, velocity and acceleration:**
  By inspection, Q's position vector from N_o is

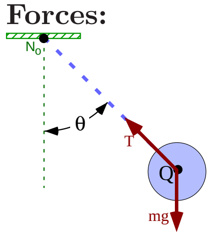
$$ \vec{\mathbf{r}}^{\,Q/N_o} = -L\, \widehat{\mathbf{b}}_y \quad (4)$$

  Q's velocity in N is defined as the time-derivative in N of $\vec{\mathbf{r}}^{\,Q/N_o}$.
  Q's acceleration in N is defined as the time-derivative in N of ${}^N\vec{\mathbf{v}}^Q$.

$$ {}^N\vec{\mathbf{v}}^Q \triangleq \frac{{}^N d\, \vec{\mathbf{r}}^{\,Q/N_o}}{dt} \underset{(4)}{=} \frac{{}^N d\, (-L\, \widehat{\mathbf{b}}_y)}{dt} = \frac{{}^B d\, (-L\, \widehat{\mathbf{b}}_y)}{dt} + {}^N\vec{\boldsymbol{\omega}}^B \times (-L\, \widehat{\mathbf{b}}_y) = \dot{\theta}\, L\, \widehat{\mathbf{b}}_x \quad (5)$$

$$ {}^N\vec{\mathbf{a}}^Q \triangleq \frac{{}^N d\, {}^N\vec{\mathbf{v}}^P}{dt} \underset{(5)}{=} \frac{{}^N d\, (\dot{\theta}\, L\, \widehat{\mathbf{b}}_x)}{dt} = \frac{{}^B d\, (\dot{\theta}\, L\, \widehat{\mathbf{b}}_x)}{dt} + {}^N\vec{\boldsymbol{\omega}}^B \times (\dot{\theta}\, L\, \widehat{\mathbf{b}}_x) = \ddot{\theta}\, L\, \widehat{\mathbf{b}}_x + \dot{\theta}^2\, L\, \widehat{\mathbf{b}}_y \quad (6)$$

- **Forces:**



The resultant of all forces (tension and gravity) on particle Q is[a]

$$ \vec{\mathbf{F}}^Q = T\, \widehat{\mathbf{b}}_y - m\, g\, \widehat{\mathbf{n}}_y \quad (7)$$

[a] A massless, rigid, inextensible cable can only exert a tensile force on Q in the $\widehat{\mathbf{b}}_y$ direction. The cable's inability to exert force on Q in the $\widehat{\mathbf{b}}_x$ direction is a consequence of the Newton/Euler laws and a ***free-body analysis*** of the cable.

28.3.1 $\vec{\mathbf{F}} = m\, \vec{\mathbf{a}}$ for the classic particle pendulum

$$ \boxed{\vec{\mathbf{F}}^Q = m\, {}^N\vec{\mathbf{a}}^Q} $$

Substituting equations (6) and (7) into Newton's law gives the ***vector*** equation of motion:

$$ \underset{(7)}{T\, \widehat{\mathbf{b}}_y - m\, g\, \widehat{\mathbf{n}}_y} = \underset{(6)}{m\, (L\, \ddot{\theta}\, \widehat{\mathbf{b}}_x + L\, \dot{\theta}^2\, \widehat{\mathbf{b}}_y)} $$

Scalar equations are generated from a vector equation by dot-multiplication with a vector. There are a variety of choices of vectors to use for this dot-multiplication and some are better than others. Although $\widehat{\mathbf{n}}_x$ and $\widehat{\mathbf{n}}_y$ may seem obvious, the more efficient choice is $\widehat{\mathbf{b}}_x$ and $\widehat{\mathbf{b}}_y$.

Dot-multiplication of $\vec{\mathbf{F}} = m\, \vec{\mathbf{a}}$ with $\widehat{\mathbf{n}}_x$ and $\widehat{\mathbf{n}}_y$.
Dot-multiplication of $\vec{\mathbf{F}} = m\, \vec{\mathbf{a}}$ with $\widehat{\mathbf{n}}_x$ and $\widehat{\mathbf{n}}_y$, produces a set of linear algebraic equations that are ***coupled*** in $\ddot{\theta}$ and T. Using standard linear algebra techniques, it is a straight-forward (albeit tedious) process to solve for T and $\ddot{\theta}$ (shown below-right). In general, ***free-body analyses*** result in equations that are ***coupled*** in force scalars (e.g., T) and acceleration scalars (e.g., $\ddot{\theta}$).

$$ -T\, \sin(\theta) \underset{(1)}{=} m\, [L\, \ddot{\theta}\, \cos(\theta) - L\, \dot{\theta}^2\, \sin(\theta)] $$

$$ T\, \cos(\theta) - m\, g \underset{(1)}{=} m\, [L\, \ddot{\theta}\, \sin(\theta) + L\, \dot{\theta}^2\, \cos(\theta)] $$

$$ \Longrightarrow $$

$$ \ddot{\theta} = \frac{-g}{L}\, \sin(\theta) $$

$$ T = m\, g\, \cos(\theta) + m\, L\dot{\theta}^2 $$

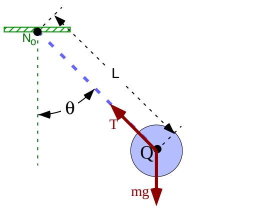
Dot-multiplication of $\vec{\mathbf{F}} = m\, \vec{\mathbf{a}}$ with $\widehat{\mathbf{b}}_x$ and $\widehat{\mathbf{b}}_y$ – a simpler set of equations.
Dot-multiplication of $\vec{\mathbf{F}} = m\, \vec{\mathbf{a}}$ with $\widehat{\mathbf{b}}_x$ and $\widehat{\mathbf{b}}_y$ produces ***simpler*** algebraic equations that are ***uncoupled***

in $\ddot{\theta}$ and T. The main point is that the choice of vectors used for dot-multiplication affects complexity and coupling of the resulting scalar equations, and this choice can be used to the analyst's benefit.

For $\widehat{\mathbf{b}}_{\mathrm{x}}$: $\quad -m\,g\,\sin\theta \underset{(1)}{=} m\,L\,\ddot{\theta}$ $\qquad\qquad$ For $\widehat{\mathbf{b}}_{\mathrm{y}}$: $\quad T \,-\, m\,g\,\cos(\theta) \underset{(1)}{=} m\,L\,\dot{\theta}^2$

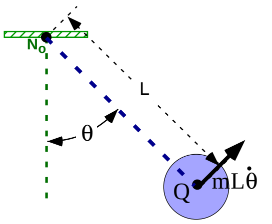
28.3.2 Angular momentum principle for the classic particle pendulum

Another way to formulate equations of motion is with the ***angular momentum principle***, which is facilitated by the analyst choosing a convenient "about-point". For the pendulum, point N_o is a convenient about-point (as designated by the ***MG road-map*** for θ in Section 22.6.8).



The moment of all forces on Q about N_o is calculated with the cross product of $\vec{\mathbf{r}}^{\,Q/N_o}$ (Q's position vector from N_o) with $\vec{\mathbf{F}}^{\,Q}$ (the resultant of all forces on Q), as

$$\vec{\mathbf{M}}^{Q/N_o} = \vec{\mathbf{r}}^{\,Q/N_o} \times \vec{\mathbf{F}}^{\,Q} \underset{(47)}{=} -L\,\widehat{\mathbf{b}}_{\mathrm{y}} \times \left(T\,\widehat{\mathbf{b}}_{\mathrm{y}} - m\,g\,\widehat{\mathbf{n}}_{\mathrm{y}}\right) \underset{(1)}{=} -m\,g\,L\,\sin(\theta)\,\widehat{\mathbf{b}}_{\mathrm{z}}$$



Q's angular momentum about N_o in N is the cross product of Q's position vector from N_o with $m\,{}^N\vec{\mathbf{v}}^{Q}$ (Q's translational momentum in N), i.e.,

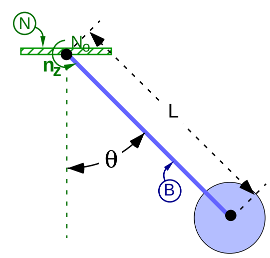
$${}^N\vec{\mathrm{H}}^{Q/N_o} \triangleq \vec{\mathbf{r}}^{\,Q/N_o} \times m\,{}^N\vec{\mathbf{v}}^{Q} \underset{(45)}{=} -L\,\widehat{\mathbf{b}}_{\mathrm{y}} \times m\,L\,\dot{\theta}\,\widehat{\mathbf{b}}_{\mathrm{x}} = m\,L^2\,\dot{\theta}\,\widehat{\mathbf{b}}_{\mathrm{z}}$$

Assembling the ingredients in the angular momentum principle, one finds

$$\boxed{\vec{\mathbf{M}}^{Q/N_o} = \frac{{}^N d\;{}^N\vec{\mathrm{H}}^{Q/N_o}}{dt}} \quad\Rightarrow\quad -m\,g\,L\,\sin(\theta)\,\widehat{\mathbf{b}}_{\mathrm{z}} = m\,L^2\,\ddot{\theta}\,\widehat{\mathbf{b}}_{\mathrm{z}}$$

Dot-multiplication of both sides of the previous vector equation with $\widehat{\mathbf{b}}_{\mathrm{z}}$, dividing both sides by mL, and subsequent rearrangement gives the classic pendulum equation $\ddot{\theta} + \frac{g}{L}\sin(\theta) = 0$.

28.3.3 Euler's rigid body equation for the classic particle pendulum



Since B is a rigid body whose motion in a Newtonian reference frame N is restricted to the plane perpendicular to $\widehat{\mathbf{n}}_{\mathrm{z}}$, its equations of motion can be formed with Euler's equation for a rigid body with a ***simple angular velocity*** as

$$\boxed{\vec{\mathbf{M}}_z^{B/N_o} \underset{(2D)}{=} \mathrm{I}\;{}^N\vec{\boldsymbol{\alpha}}^{B}} \quad\Rightarrow\quad -m\,g\,L\,\sin(\theta)\,\widehat{\mathbf{b}}_{\mathrm{z}} = m\,L^2\,\ddot{\theta}\,\widehat{\mathbf{b}}_{\mathrm{z}}$$

- $\vec{\mathbf{M}}_z^{B/N_o} = -m\,g\,L\,\sin(\theta)\,\widehat{\mathbf{b}}_{\mathrm{z}}$ is the $\widehat{\mathbf{b}}_{\mathrm{z}}$ component of the moment of all forces on B about N_o, (see Section 28.3.2).
- I is B's mass moment of inertia about the line passing through N_o and parallel to $\widehat{\mathbf{b}}_{\mathrm{z}}$.
  Since B's mass is solely particle Q, $\mathrm{I} = m\,L^2$ (Q's mass multiplied by the square of Q's distance from N_o).
- ${}^N\vec{\boldsymbol{\alpha}}^{B}$ is B's angular acceleration in N, calculated in equation (3) as ${}^N\vec{\boldsymbol{\alpha}}^{B} = \ddot{\theta}\,\widehat{\mathbf{b}}_{\mathrm{z}}$.

Dot-multiplication of both sides of the vector equation with $\widehat{\mathbf{b}}_{\mathrm{z}}$, dividing by mL, and subsequent rearrangement gives the classic pendulum equation $\ddot{\theta} + \frac{g}{L}\sin(\theta) = 0$.

28.3.4 Kinetic energy for the classic particle pendulum

Kinetic energy is useful for:

- The ***power/energy-rate principle*** described in Section 23.1 (see example in Section 28.3.5)
- The ***work/energy principle*** described in Section 23.7
- Conservation of mechanical energy described in Section 24.2 (see example in Section 28.3.6).

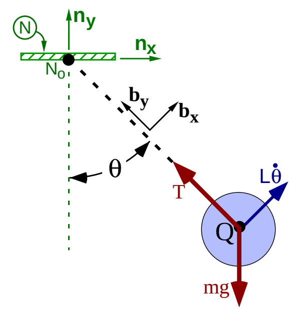
- Lagrange's equations of motion in Chapter 26 (see example in Section 28.3.8).

$$^N K^Q \triangleq \frac{1}{2}\, m\, {}^N\vec{\mathbf{v}}^Q \cdot {}^N\vec{\mathbf{v}}^Q \;=\; \frac{1}{2}\, m\, (L\,\dot\theta\,\widehat{\mathbf{b}}_{\rm x}) \cdot (L\,\dot\theta\,\widehat{\mathbf{b}}_{\rm x}) \;=\; \frac{1}{2}\, m\, L^2\, \dot\theta^2$$

28.3.5 Power/energy-rate principle for the classic particle pendulum

The following figure shows the forces and velocity for particle Q. Since this particle-pendulum has one-degree of freedom in N, the ***power/energy-rate principle*** described in Section 23.1 is useful for forming its equations of motion.[3]

The power of all forces on Q in N is defined as



$$^N\mathrm{P}^Q \triangleq \vec{\mathbf{F}}^Q \cdot {}^N\vec{\mathbf{v}}^Q \;=\; (T\,\widehat{\mathbf{b}}_{\rm y} - m\,g\,\widehat{\mathbf{n}}_{\rm y}) \cdot (L\,\dot\theta\,\widehat{\mathbf{b}}_{\rm x}) \;=\; -m\,g\,L\,\sin(\theta)\,\dot\theta$$

Since tension T does ***not*** appear in the power of Q, it is called a ***workless force*** or ***non-contributing force***. The fortuitous absence of tension (an unknown) is one of the major advantages of the power/energy-rate principle.

Q's kinetic energy in N was calculated (previous section) as $^N K^Q = \frac{1}{2}\, m\, L^2\, \dot\theta^2$.

Equating power to the time-derivative of kinetic energy (i.e., using the ***power/energy-rate principle***) gives[4]

$$\boxed{^N\mathrm{P}^Q = \frac{d\, ^N K^Q}{dt}} \qquad \Rightarrow \qquad -m\,g\,L\,\sin(\theta)\,\dot\theta \;=\; m\,L^2\,\dot\theta\,\ddot\theta$$

Rearranging and factoring out $\dot\theta$, noting that in general $\dot\theta \neq 0$, and dividing by $m\,L^2$, gives

$$\left[\, m\,L^2\,\ddot\theta \;+\; m\,g\,L\,\sin(\theta) \,\right] \dot\theta = 0 \qquad \Rightarrow \qquad m\,L^2\,\ddot\theta \;+\; m\,g\,L\,\sin(\theta) = 0 \qquad \Rightarrow \qquad \ddot\theta \;+\; \frac{g}{L}\,\sin(\theta) = 0$$

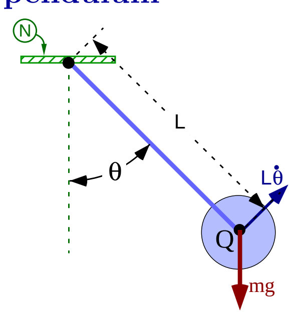
This power/energy-rate equation is useful for several reasons:

- It forms an equation of motion for systems with **one**-degree of freedom.
- It tells us that gravity g affects the Q's motion in N, but tension T does not.
- Multiplying both sides of the power/energy-rate equation by the differential dt and integrating the left-hand side with respect to $d\theta$ and the right-hand side with respect to $d\dot\theta$ produces a ***work/energy principle*** where C is an arbitrary constant of integration. Note: Since work is a function of only configuration, a potential energy exists and this integral represents ***conservation of mechanical energy***.

$$m\,g\,L\,\cos(\theta) \;+\; C \;=\; \frac{1}{2}\, m\, L^2\, \dot\theta^2 \qquad \text{or} \qquad C \;=\; \frac{1}{2}\, m\, L^2\, \dot\theta^2 \;+\; -m\,g\,L\,\cos(\theta)$$

28.3.6 Conservation of mechanical energy for the classic particle pendulum

Conservation of mechanical energy is the time-integral of the ***power/energy-rate principle*** and relates the system's kinetic and potential energies to an arbitrary constant (e.g., Constant = 0) as



$$\boxed{K \;+\; U \;=\; \text{Constant}} \qquad \Rightarrow \qquad \frac{1}{2}\, m\, L^2\, \dot\theta^2 \;+\; -m\,g\,L\,\cos(\theta) = 0$$

The time-derivative of ***conservation of mechanical energy*** produces **one** time-dependent equation of motion (namely the ***power/energy-rate principle***) as

$$m\,L^2\,\dot\theta\,\ddot\theta \;+\; m\,g\,L\,\sin(\theta)\,\dot\theta \;=\; 0$$

[3] The pendulum has **one** degree of freedom because its motion is described by **one** velocity variable, namely $\dot\theta$.

[4] Multiplying the power/energy-rate equation by the differential dt and integrating the left-hand side with respect to $d\theta$ and the right-hand side with respect to $d\dot\theta$ leads to ***conservation of mechanical energy*** $m\,g\,L\,\cos(\theta) = \frac{1}{2}\, m\, L^2\, \dot\theta^2$.

Rearranging and factoring out $\dot{\theta}$, noting that in general $\dot{\theta} \neq 0$, and dividing by $m\,L^2$, gives

$$[m\,L^2\,\ddot{\theta} \;+\; m\,g\;L\,\sin(\theta)]\,\dot{\theta} \;=\; 0 \qquad \Rightarrow \qquad m\,L^2\,\ddot{\theta} \;+\; m\,g\,L\,\sin(\theta) \;=\; 0 \qquad \Rightarrow \qquad \ddot{\theta} \;+\; \frac{g}{L}\,\sin(\theta) \;=\; 0$$

28.3.7 Kane's method for the classic particle pendulum

As described in Chapter 25, ***Kane's method*** employs ***generalized speeds*** and ***partial velocities***. With $\dot{\theta}$ as the ***generalized speeds***, ***Q's partial velocity in N for*** $\dot{\theta}$ is $\dfrac{\partial\,{}^N\vec{\mathbf{v}}^Q}{\partial\,\dot{\theta}} = \dfrac{\partial\,(L\,\dot{\theta}\,\widehat{\mathbf{b}}_{\mathrm{x}})}{\partial\,\dot{\theta}} = L\,\widehat{\mathbf{b}}_{\mathrm{x}}$.

Kane's equations for the pendulum can be written as shown below [from equation (25.1)].
Substituting for $\vec{\mathbf{F}}^Q$ from equation (7) and for ${}^N\vec{\mathbf{a}}^Q$ from equation (6) gives

$$\left[\vec{\mathbf{F}}^Q \underset{(25.1)}{=} m^Q\,{}^N\vec{\mathbf{a}}^Q\right] \cdot \frac{\partial\,{}^N\vec{\mathbf{v}}^Q}{\partial\,\dot{\theta}} \qquad \left[(T\,\widehat{\mathbf{b}}_{\mathrm{y}} - m\,g\,\widehat{\mathbf{n}}_{\mathrm{y}}) \underset{(7)}{} \;=\; m\,(L\,\ddot{\theta}\,\widehat{\mathbf{b}}_{\mathrm{x}} \;+\; L\,\dot{\theta}^2\,\widehat{\mathbf{b}}_{\mathrm{y}}) \underset{(6)}{}\right] \cdot L\,\widehat{\mathbf{b}}_{\mathrm{x}} \qquad -m\,g\,L\,\sin(\theta) \;-\; m\,L^2\,\ddot{\theta} \;=\; 0$$

Dividing by $m\,L^2$ and subsequent rearrangement gives the classic pendulum equation $\ddot{\theta} + \frac{g}{L}\,\sin(\theta) \;=\; 0$.

28.3.8 Lagrange's method for the classic particle pendulum

As described in Chapter 26, ***Lagrange's equations of the second kind*** relate partial derivatives of potential energy U and kinetic energy K as shown below [from equation (26.1)].

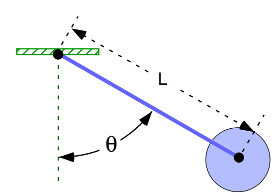
$$-\frac{\partial\,U}{\partial\,\theta} \underset{(26.1)}{=} \frac{d}{dt}\frac{\partial\,K}{\partial\,\dot{\theta}} - \frac{\partial\,K}{\partial\,\theta} \qquad \begin{array}{l} U \;=\; m\,g\,\text{height} \;=\; -m\,g\,L\,\cos(\theta) \\ K \;=\; \frac{1}{2}\,m\,{}^N\vec{\mathbf{v}}^Q \cdot {}^N\vec{\mathbf{v}}^Q \;=\; \frac{1}{2}\,m\,L^2\,\dot{\theta}^2 \end{array} \qquad m\,L^2\,\ddot{\theta} \;+\; m\,g\,L\,\sin(\theta) \;=\; 0$$

Dividing by mL^2 and rearrangement gives the classic pendulum equation $\ddot{\theta} + \frac{g}{L}\,\sin(\theta) \;=\; 0$.

28.4 Solution of the classic particle pendulum ODE

The ODE (ordinary differential equation) that governs the motion of the classic pendulum is nonlinear, homogeneous, constant-coefficient, and 2^{nd}-order. There are a variety of methods to solve this ODE for $\theta(t)$, including:

- Numerical integration with a computer program
- Analytical solution with Jacobian elliptic functions
- Analytical solution using the small angle approximation $\sin(\theta) \approx \theta$



$$\ddot{\theta} \;+\; \frac{g}{L}\,\sin(\theta) \;=\; 0$$

28.4.1 Numerical solution of pendulum ODE via MotionGenesis and/or MATLAB®

There are ***few*** mechanical systems whose motions have analytical solutions. Alternately, there are ***many*** mechanical systems whose motions can be predicted with numerical integration, (e.g., a variable-step Runga-Kutta integrator). For example, the following **M**otion**G**enesis commands numerically solve the differential equation for $\theta(t)$ for $0 \leq t \leq 6$ secs with $g = 9.8\,\frac{\text{m}}{\text{s}^2}$, $L = 1.0$ m, $\theta(t{=}0) = 60°$.

Note: Output for plotting is created every 0.02 sec by assigning a numerical integration step (`tStep` = 0.02 sec).
To make **M**otion**G**enesis create a .m file for subsequent use with MATLAB®, change the last line to `pendulum.m`.

```
Constant g = 9.8 m/s^2;    L = 1 m            % Declare g (gravity) and L (length).
Variable    theta'' = -g/L*sin(theta)         % Governing ode (ordinary differential equation).
Input       theta = 60 deg, theta' = 0 deg/sec % Initial values for theta and theta'.
Input       tFinal = 6 sec, tStep = 0.02 sec  % Solve ODE from t=0 to 6 sec.  Output every 0.02 sec.
OutputPlot  t sec, theta deg                  % Have ODE output and plot theta vs. t.
ODE() pendulum                                % Numerically integrate the ode.
```

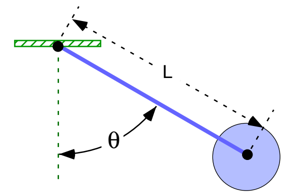

28.4.2 Analytical (closed-form) solution of the classic particle pendulum ODE

The simple particle pendulum is one of the ***few*** mechanical systems whose motion are governed by a ***nonlinear*** ODE which have a known analytical (closed-form) solution. The exact analytical solution for its ODE is a ***Jacobian elliptic function***, with an pendulum oscillation period τ_{period} that depends on K(k), the elliptic integral of the first kind with modulus $k = \sin(\frac{\theta_0}{2})$ [θ_0 is initial value of θ].[5]

$$\tau_{\text{period}} \underset{(\text{exact})}{=} 4\,K(k)\,\sqrt{\frac{L}{g}} \qquad\qquad \tau_{\text{period}} \approx \frac{2\,\pi}{\omega} \quad \text{where} \quad \omega = \sqrt{\frac{g}{L}}\,\left(1 - k^2\right)^{[0.25\,\cos(\frac{\theta_0}{2})^{0.125}]}$$

28.4.3 Simplification and analytical solution of the classic particle pendulum ODE

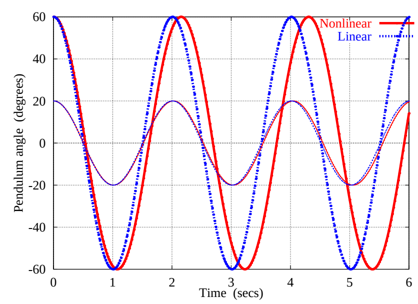
One way to approximate the pendulum's ***nonlinear*** ODE is with the ***small angle approximation*** $\sin(\theta) \approx \theta$, which makes the ODE ***linear*** and allows the solution to be written in terms of $\omega_n \triangleq \sqrt{\frac{g}{L}}$ and the initial values θ_0 and $\dot{\theta}_0$.



$$\ddot{\theta} + \frac{g}{L}\,\theta \approx 0 \qquad \Rightarrow \qquad \theta(t) \approx \frac{\dot{\theta}_0}{\omega_n}\,\sin(\omega_n\,t) + \theta_0\,\cos(\omega_n\,t)$$

The period of oscillations of the approximate analytical solution is $\tau_{\text{period}} \approx 2\,\pi\,\omega_n = 2\,\pi\,\sqrt{\frac{L}{g}}$. τ_{period} can also be experimentally determined with a real pendulum and is surprisingly well-correlated to this analytically simple model and small-angle approximation for τ_{period}.[6]

In addition to correlating the period, it is helpful to compare the motion predicted by the linear differential equation with that predicted by the exact solution of the nonlinear differential equation as is done in the figure to the right. For small initial angles, e.g., $\theta_0 = 20°$, the linear and nonlinear differential equations predict similar motions. For larger initial angles of θ, e.g., $\theta_0 = 60°$, the motion differ more, but is still surprisingly similar even though θ is large ($|\theta| > 1$ rad).



Linear and nonlinear solution

28.5 Interpretation of results for the classic particle pendulum

The investigation of the classic particle pendulum leads to several conclusions:[7]

- There are many ways to form equations of motion for a simple pendulum.
- The motion of the pendulum does not depend on the mass of the object.
- For small angles, the period of oscillation does *not* depend on the initial value of θ.
- For large initial angles, the motion predicted by the differential equation employing the small angle approximation differs from the motion predicted by the full nonlinear differential equation. The differences are surprisingly small even when θ_0 is relatively large, e.g., $\theta_0 > 1$ rad.

[5] The approximation in [29] for τ_{period} is accurate to 1% for angles up to 177° and computable with a simple calculator.

[6] The experimental determination of a pendulum's period requires a string, a tape measure, and a stop watch. It is easy to time the period of a one meter long pendulum and compare it with the analytical period of $\tau_{\text{period}} = 2\,\pi\,\sqrt{1/9.8} = 2.007$ sec.

[7] Other interesting results for this simple pendulum were reported in [73] by Vassar College Physicist Cindy Schwarz.