Basis independent vectors and their properties

Show work – except for ♣ fill-in-blanks-problems (print .pdf from www.MotionGenesis.com ⇒ Textbooks ⇒ Resources).

1.1 ♣ Solving problems – what engineers do.

Understanding dynamics results from **doing** problems. Many problems are guided to help you synthesize processes (imitation). You are encouraged to work by yourself or with colleagues/instructors and use the textbook's reference theory and other resources.

Confucius 500 B.C.
"I hear and I forget.
I see and I remember.
I _____ and I understand."

"By three methods we may learn wisdom:
First, by reflection, which is noblest;
Second, by imitation, which is easiest;
Third by experience, which is the bitterest."

1.2 ♣ What is a vector? (Section 2.2)

Two properties (attributes) of a vector are __________ and __________.

1.3 ♣ What is a zero vector? (Section 2.3)

A zero vector $\vec{\mathbf{0}}$ has a magnitude of **0/1/2/∞**.
A zero vector $\vec{\mathbf{0}}$ has no direction. **True/False**.
A zero vector $\vec{\mathbf{0}}$ is ***parallel*** to any vector $\vec{\mathbf{v}}$. **True/False**.
A zero vector $\vec{\mathbf{0}}$ is ***perpendicular*** to any vector $\vec{\mathbf{v}}$. **True/False**.

1.4 ♣ Unit vectors. (Section 2.4)

A unit vector has a magnitude of **0/1/2/∞**.
All unit vectors are equal. **True/False**.

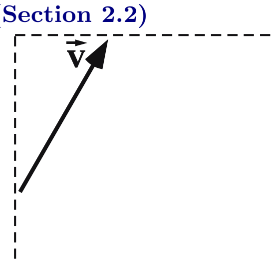
1.5 ♣ Draw the following vectors: (Section 2.2)

- Long, horizontally-right vector $\vec{\mathbf{a}}$
- Short, vertically-upward vector $\vec{\mathbf{b}}$
- Outwardly-directed **unit** vector $\widehat{\mathbf{c}}$.

1.6 ♣ Optional: Vector magnitude and direction (orientation and sense). (Section 2.2)

The figure to the right shows a vector $\vec{\mathbf{v}}$. **Draw** the following vectors.

- $\vec{\mathbf{a}}$: Same magnitude and same direction as $\vec{\mathbf{v}}$ ($\vec{\mathbf{a}} = \vec{\mathbf{v}}$).
- $\vec{\mathbf{b}}$: Same magnitude and orientation as $\vec{\mathbf{v}}$, but different sense.
- $\vec{\mathbf{c}}$: Same direction as $\vec{\mathbf{v}}$, but different magnitude.
- $\vec{\mathbf{d}}$: Same magnitude as $\vec{\mathbf{v}}$, but different direction (orientation).
- $\vec{\mathbf{e}}$: Different magnitude and different direction (orientation) as $\vec{\mathbf{v}}$.



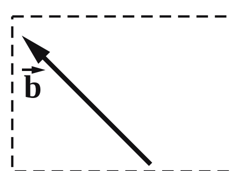
1.7 ♣ Magnitude of a vector. (Section 2.2)

Consider a real number x and a horizontally-right pointing unit vector $\widehat{\mathbf{i}}$.
The ***magnitude*** of the vector $-x\,\widehat{\mathbf{i}}$ is (circle **one**): positive negative non-negative non-positive.

1.8 ♣ Negating a vector. (Section 2.8)

Complete the figure to the right by **drawing** the vector $-\vec{\mathbf{b}}$.
Negating the vector $\vec{\mathbf{b}}$ results in a vector with different (circle **all** that apply):

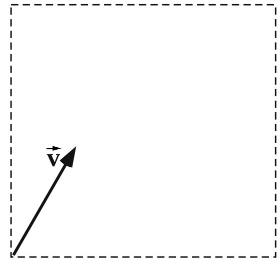
magnitude direction orientation sense



Historical note: Negative numbers (e.g., −3) were not widely accepted until 1800 A.D.

1.9 ♣ Optional: Multiplying a vector by a scalar. (Section 2.7)

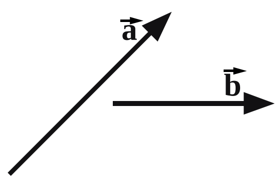
Complete the figure to the right by drawing the vectors $2\,\vec{\mathbf{v}}$ and $-2\,\vec{\mathbf{v}}$.



The following statements involve a vector $\vec{\mathbf{v}}$ and a real non-zero scalar s ($s \neq 0$).
If a statement is true, provide a numerical value for s that supports your answer

$s\,\vec{\mathbf{v}}$ can have a different ***magnitude*** than $\vec{\mathbf{v}}$.	**True/False**	$s =$
$s\,\vec{\mathbf{v}}$ can have a different ***direction*** than $\vec{\mathbf{v}}$.	**True/False**	$s =$
$s\,\vec{\mathbf{v}}$ can have a different ***sense*** than $\vec{\mathbf{v}}$.	**True/False**	$s =$
$s\,\vec{\mathbf{v}}$ can have a different ***orientation*** than $\vec{\mathbf{v}}$.	**True/False**	$s =$

1.10 ♣ Optional: Graphical vector addition/subtraction - draw. (Sections 2.6,2.8)



Draw $\vec{\mathbf{a}} + \vec{\mathbf{b}}$

Draw $\vec{\mathbf{a}} - \vec{\mathbf{b}}$

1.11 ♣ Visual representation of a vector dot-product. (Section 2.9)

Write the ***definition*** of the dot-product of a vector $\vec{\mathbf{a}}$ with a vector $\vec{\mathbf{b}}$.
Include a ***sketch*** with ***each symbol*** in the right-hand-side of your definition clearly labeled. The sketch should include $\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$, $|\vec{\mathbf{a}}|$, $|\vec{\mathbf{b}}|$, ...

Result:

$$\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} \triangleq$$

1.12 ♣ Visual representation of a vector cross-product. (Section 2.10)

Write the ***definition*** of the cross-product of a vector $\vec{\mathbf{a}}$ with a vector $\vec{\mathbf{b}}$.
Include a ***sketch*** with ***each symbol*** in your definition labeled and described.

Result:

$$\vec{\mathbf{a}} \times \vec{\mathbf{b}} \triangleq \quad (\theta)\,\hat{\mathbf{u}}$$

where $\hat{\mathbf{u}}$ is

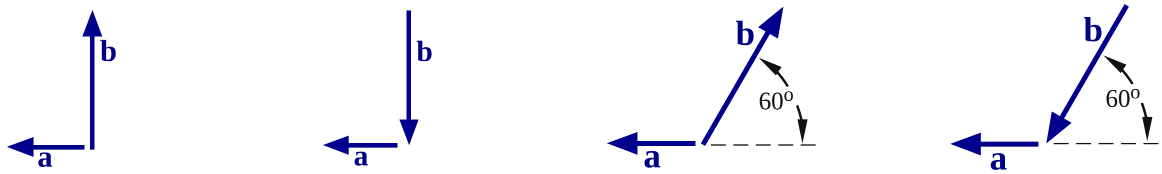
and θ is

1.13 ♣ Properties of vector dot-products and cross-products. (Sections 2.9.1 and 2.10.1)

When $\vec{\mathbf{a}}$ is parallel to $\vec{\mathbf{b}}$:	$\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = 0$	**True/False**	$\vec{\mathbf{a}} \times \vec{\mathbf{b}} = \vec{\mathbf{0}}$	**True/False**
When $\vec{\mathbf{a}}$ is perpendicular to $\vec{\mathbf{b}}$:	$\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = 0$	**True/False**	$\vec{\mathbf{a}} \times \vec{\mathbf{b}} = \vec{\mathbf{0}}$	**True/False**
For arbitrary vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$:	$\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = \vec{\mathbf{b}} \cdot \vec{\mathbf{a}}$	**True/False**	$\vec{\mathbf{a}} \times \vec{\mathbf{b}} = \vec{\mathbf{b}} \times \vec{\mathbf{a}}$	**True/False**

1.14 ♣ Calculating vector dot-products and cross-products via definitions. **(Sections 2.9 and 2.10)**

Draw a unit vector $\widehat{\mathbf{k}}$ outward-normal to the plane of the paper.
Knowing vector $\vec{\mathbf{a}}$ has magnitude 2 and vector $\vec{\mathbf{b}}$ has magnitude 4, calculate the following dot-products and cross-products via their **definitions** (2^+ significant digits).



$\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = \boxed{}$ $\quad$ $\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = \boxed{}$ $\quad$ $\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = \boxed{}$ $\quad$ $\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = \boxed{}$

$\vec{\mathbf{a}} \times \vec{\mathbf{b}} = \boxed{}$ $\quad$ $\vec{\mathbf{a}} \times \vec{\mathbf{b}} = \boxed{}$ $\quad$ $\vec{\mathbf{a}} \times \vec{\mathbf{b}} = \boxed{}$ $\quad$ $\vec{\mathbf{a}} \times \vec{\mathbf{b}} = \boxed{}$

1.15 ♣ Property of scalar triple product. **(Section 2.11).**

For arbitrary non-zero vectors $\vec{\mathbf{a}}, \vec{\mathbf{b}}, \vec{\mathbf{c}}$: $\quad \vec{\mathbf{a}} \cdot (\vec{\mathbf{b}} \times \vec{\mathbf{c}}) = (\vec{\mathbf{a}} \times \vec{\mathbf{b}}) \cdot \vec{\mathbf{c}}$ **Never/Sometimes/Always**

A property of the ***scalar triple product*** is $\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} \times \vec{\mathbf{a}} = 0$. **True/False**.

1.16 Property of vector triple cross-product. **(Sections 2.10.1 and 2.11)**

Complete the following equation: $\quad \vec{\mathbf{a}} \times \left(\vec{\mathbf{b}} \times \vec{\mathbf{c}}\right) = \vec{\mathbf{b}}\left(\boxed{}\right) - \vec{\mathbf{c}}\left(\boxed{}\right)$

Circle true or false (show supporting work): $\quad \vec{\mathbf{a}} \times (\vec{\mathbf{b}} \times \vec{\mathbf{c}}) = (\vec{\mathbf{a}} \times \vec{\mathbf{b}}) \times \vec{\mathbf{c}} + \vec{\mathbf{b}} \times (\vec{\mathbf{a}} \times \vec{\mathbf{c}})$ **True/False**

1.17 ♣ Optional: Proof of magnitude of vector cross product property. **(Sections 2.9 and 2.10)**

Letting $\widehat{\boldsymbol{\lambda}}$ be a ***unit vector*** and $\vec{\mathbf{v}}$ be ***any vector***, prove[1] $\left|\vec{\mathbf{v}} \times \widehat{\boldsymbol{\lambda}}\right|^2 = \vec{\mathbf{v}} \cdot \vec{\mathbf{v}} - (\vec{\mathbf{v}} \cdot \widehat{\boldsymbol{\lambda}})^2$.

1.18 ♣ Vector exponentiation: $\vec{\mathbf{v}}^2$ and $\vec{\mathbf{v}}^3$. Complete the 3-step proofs. **(Section 2.9)**

Step 1: Complete the **definition** of $\vec{\mathbf{v}}^2$ in terms of $|\vec{\mathbf{v}}|$.
Step 2: Use the **definition** of the dot-product to show how $\vec{\mathbf{v}} \cdot \vec{\mathbf{v}}$ can be expressed in terms of $|\vec{\mathbf{v}}|$.
Step 3: Combine these two definitions to provide an alternate way to calculate $\vec{\mathbf{v}}^2$ with a vector dot-product.

Result: $\quad \vec{\mathbf{v}}^2 \triangleq |\vec{\mathbf{v}}|^{\boxed{}} \qquad \vec{\mathbf{v}} \cdot \vec{\mathbf{v}} \underset{(2.2)}{=} \boxed{} \qquad \vec{\mathbf{v}}^2 = \boxed{} \cdot \boxed{}$

Complete the 3-step proof that relates $\vec{\mathbf{v}}^3$ to $\vec{\mathbf{v}} \cdot \vec{\mathbf{v}}$ raised to a real number.

Result: $\quad \vec{\mathbf{v}}^3 \triangleq |\vec{\mathbf{v}}|^{\boxed{}} \underset{(2.4)}{=} \left(\sqrt{\boxed{}}\right)^{\boxed{}} = (\vec{\mathbf{v}} \cdot \vec{\mathbf{v}})^{\boxed{}}$

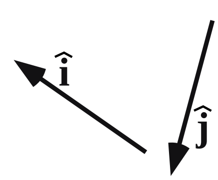
1.19 ♣ $|c\,\widehat{\mathbf{a}}_{\mathrm{x}}|$ Calculate vector magnitude with dot products. **(Section 2.9 and Hw 1.18)**

Show how the vector dot-product can be used to show that the magnitude of the vector $c\,\widehat{\mathbf{a}}_{\mathrm{x}}$ (c is a positive or **negative** number and $\widehat{\mathbf{a}}_{\mathrm{x}}$ is a unit vector) can be written solely in terms of c (without $\widehat{\mathbf{a}}_{\mathrm{x}}$).

Result: $\quad |c\,\widehat{\mathbf{a}}_{\mathrm{x}}| = \sqrt{\boxed{} \cdot \boxed{}} = \sqrt{c^2 * \boxed{} \cdot \boxed{}} = \sqrt{\boxed{}} = \mathrm{abs}(c)$

1.20 † Magnitude of the vector $\vec{\mathbf{v}}$. **Show work.** **(Section 2.9)**

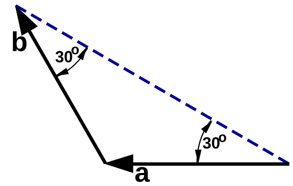
Knowing the angle between a unit vector $\widehat{\mathbf{i}}$ and unit vector $\widehat{\mathbf{j}}$ is 110°, calculate a numerical value for the magnitude of $\vec{\mathbf{v}} = 3\,\widehat{\mathbf{i}} + 4\,\widehat{\mathbf{j}}$.



Result: $\quad |\vec{\mathbf{v}}| \approx \boxed{4.098}$ $\qquad$ Note: The answer is ***not*** 5.

[1] One way to prove this is to write $(\vec{\mathbf{v}} \times \widehat{\boldsymbol{\lambda}})^2 = (\vec{\mathbf{v}} \times \widehat{\boldsymbol{\lambda}}) \cdot (\vec{\mathbf{v}} \times \widehat{\boldsymbol{\lambda}}) \underset{(2.10)}{=} \vec{\mathbf{v}} \cdot [\widehat{\boldsymbol{\lambda}} \times (\vec{\mathbf{v}} \times \widehat{\boldsymbol{\lambda}})]$ and then use the vector triple cross-product property $\vec{\mathbf{a}} \times (\vec{\mathbf{b}} \times \vec{\mathbf{c}}) = \vec{\mathbf{b}}(\vec{\mathbf{a}} \cdot \vec{\mathbf{c}}) - \vec{\mathbf{c}}(\vec{\mathbf{a}} \cdot \vec{\mathbf{b}})$ from Section 2.10. Alternately, it is helpful to write $\vec{\mathbf{v}} = v_\perp \widehat{\boldsymbol{\lambda}}_\perp + v_{||} \widehat{\boldsymbol{\lambda}}$ where $v_\perp \widehat{\boldsymbol{\lambda}}_\perp$ is the component of $\vec{\mathbf{v}}$ that is perpendicular to $\widehat{\boldsymbol{\lambda}}$ and $v_{||} \widehat{\boldsymbol{\lambda}}$ is the component of $\vec{\mathbf{v}}$ that is parallel to $\widehat{\boldsymbol{\lambda}}$.

1.21 ♣ **Angle between vectors.** **(Section 2.9)**



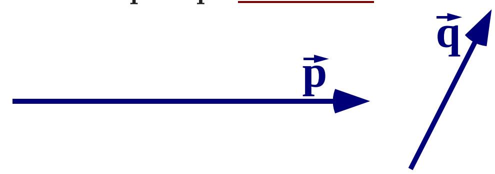
Referring to the figure to the right, find the numerical value for the angle between vector $\vec{\mathbf{a}}$ and vector $\vec{\mathbf{b}}$.

$$\angle(\vec{\mathbf{a}}, \vec{\mathbf{b}}) \;=\; \boxed{\quad}^\circ$$

1.22 Visual estimation of vector dot/cross-products. **Show work.** **(Sections 2.9 and 2.10)**

Estimate (e.g., using your pinky) the magnitude of the vector $\vec{\mathbf{p}}$ shown below. Note: 1 inch $\triangleq$ 2.54 cm.
Estimate the angle between $\vec{\mathbf{p}}$ and $\vec{\mathbf{q}}$, $\vec{\mathbf{p}} \cdot \vec{\mathbf{q}}$, and the magnitude of $\vec{\mathbf{p}} \times \vec{\mathbf{q}}$. **Show work**.
Result: (Provide numerical results with 1 or more significant digits).

$\lvert\vec{\mathbf{p}}\rvert \;\approx\; \boxed{\quad}$ cm	$\angle(\vec{\mathbf{p}}, \vec{\mathbf{q}}) \;\approx\; \boxed{\quad}^\circ$
$\vec{\mathbf{p}} \cdot \vec{\mathbf{q}} \;\approx\; \boxed{\quad}$ cm^2	$\lvert\vec{\mathbf{p}} \times \vec{\mathbf{q}}\rvert \;\approx\; \boxed{\quad}$ cm^2



1.23 ♣ **Form the *unit* vector $\hat{\mathbf{u}}$ having the same direction as $c\,\hat{\mathbf{a}}_x$.** **(Section 2.4)**

Result: $\hat{\mathbf{u}} \;=\; \boxed{\quad}\;\hat{\mathbf{a}}_x$ Note: $\hat{\mathbf{a}}_x$ is a unit vector and c is a non-zero real number, e.g., 3 or −3

1.24 ♣ **Coefficient of $\hat{\mathbf{u}}$ in cross products – definitions and trig functions.** **(Section 2.10)**

The ***cross product*** of vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ can be written in terms of a real scalar s as $\vec{\mathbf{a}} \times \vec{\mathbf{b}} \;=\; s\,\hat{\mathbf{u}}$ where $\hat{\mathbf{u}}$ is a unit vector perpendicular to both $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ in a direction defined by the **right-hand rule**. The coefficient s of the unit vector $\hat{\mathbf{u}}$ is inherently non-negative. **True/False**.

1.25 ♣ **Orthogonal vectors: Insights via drawing.** **(Section 2.10)**

Consider three unit vectors $\hat{\mathbf{a}}$, $\hat{\mathbf{b}}$, and $\hat{\mathbf{c}}$.
Vector $\hat{\mathbf{a}}$ is perpendicular to vector $\hat{\mathbf{b}}$.
Vector $\hat{\mathbf{b}}$ is perpendicular to vector $\hat{\mathbf{c}}$.
Vector $\hat{\mathbf{a}}$ is not parallel to vector $\hat{\mathbf{c}}$.

In **all** cases, $\hat{\mathbf{a}}$ is perpendicular to $\hat{\mathbf{c}}$. **True/False**.
Explain your answer by **drawing** $\hat{\mathbf{a}}$, $\hat{\mathbf{b}}$, $\hat{\mathbf{c}}$ and relevant angles.

1.26 Calculating distance between a point and a line via cross-products. **(Section 2.10.2)**

Draw a horizontally-right unit vector $\hat{\mathbf{a}}_x$ and vertically-upward unit vector $\hat{\mathbf{a}}_y$.
Draw a point Q whose position vector from a point P is $\vec{\mathbf{r}}^{\,Q/P} = 5\,\hat{\mathbf{a}}_x$.
Draw a line L that passes through point P and is parallel to $\hat{\mathbf{u}} \;=\; \frac{3}{5}\,\hat{\mathbf{a}}_x \;+\; \frac{4}{5}\,\hat{\mathbf{a}}_y$.
Calculate the ***distance*** d between Q and L using both formulas in equation (2.9).
Result: $\mathrm{d} \underset{(2.9)}{=} \boxed{\qquad\qquad} = \boxed{4}$ $\qquad \mathrm{d} \underset{(2.9)}{=} \boxed{\qquad\qquad} = \boxed{4}$

1.27 ♣ **Ranges of angles from dot-product and cross-product calculations.** **(Sections 2.9 and 2.10)**

Given: Unit vectors $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ and the numerical values of $c \triangleq \hat{\mathbf{a}} \cdot \hat{\mathbf{b}}$ and $s \triangleq \lvert\hat{\mathbf{a}} \times \hat{\mathbf{b}}\rvert$.
Complete the following table with an appropriate range of numerical values.

Quantity	Range of values
c	$\boxed{\quad} \le c \le \boxed{\quad}$
s	$\boxed{\quad} \le s \le \boxed{\quad}$
Angle θ_c between $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ that can be uniquely determined **solely** from c	$\boxed{\quad}^\circ \le \theta_c \le \boxed{\quad}^\circ$
Angle θ_s between $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ that can be uniquely determined **solely** from s	$\boxed{\quad}^\circ \le \theta_s \le \boxed{\quad}^\circ$
Angle θ between $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$	$\boxed{\quad}^\circ \le \theta \le \boxed{\quad}^\circ$

Note: The range of θ_s differs from the (correct) range for θ. Hence, s and θ_s are insufficient to calculate θ.

1.28 ♣ Vector operations and units. **(Chapter 2)**

Circle the vector operations below (scalar multiplication, addition, dot-product, etc.) that are **defined** for a position vector $\vec{\mathbf{a}}$ (with **units** of m) and a velocity vector $\vec{\mathbf{b}}$ (with **units** of $\frac{\text{m}}{\text{s}}$).

$$-\vec{\mathbf{a}} \qquad 5\,\vec{\mathbf{a}} \qquad \vec{\mathbf{a}}\,/\,5 \qquad \vec{\mathbf{a}} + \vec{\mathbf{b}} \qquad \vec{\mathbf{a}} \cdot \vec{\mathbf{b}} \qquad \vec{\mathbf{a}} \times \vec{\mathbf{b}}$$

1.29 ♣ Using vector identities to simplify expressions **(refer to Homework 1.13).**

One reason to treat vectors as ***basis-independent*** quantities is to simplify vector expressions ***without*** resolving the vectors into orthogonal "$\vec{\mathbf{x}}, \vec{\mathbf{y}}, \vec{\mathbf{z}}$" or "$\vec{\mathbf{i}}, \vec{\mathbf{j}}, \vec{\mathbf{k}}$" components. Simplify the following vector expressions using various properties of dot-products and cross-products.

Express your results in terms of dot-products $\cdot$ and cross-products $\times$ of the arbitrary vectors $\vec{\mathbf{u}}$, $\vec{\mathbf{v}}$, $\vec{\mathbf{w}}$ (i.e., $\vec{\mathbf{u}}$, $\vec{\mathbf{v}}$, $\vec{\mathbf{w}}$ are not orthogonal).

Vector expression	Simplified vector expression
$(3\,\vec{\mathbf{u}} - 2\,\vec{\mathbf{v}}) \times (\vec{\mathbf{u}} + \vec{\mathbf{v}})$	$\boxed{\quad}\,\vec{\mathbf{u}} \times \vec{\mathbf{v}}$
$(3\,\vec{\mathbf{u}} - 2\,\vec{\mathbf{v}}) \cdot (\vec{\mathbf{u}} + \vec{\mathbf{v}})$	$\boxed{\quad}\,\vec{\mathbf{u}}^2 - \boxed{\quad}\,\vec{\mathbf{v}}^2 + \boxed{\quad}\,\vec{\mathbf{u}} \cdot \vec{\mathbf{v}}$
$(\vec{\mathbf{u}} - \vec{\mathbf{v}}) \cdot (\vec{\mathbf{u}} + \vec{\mathbf{v}})$	$\boxed{\quad} - \boxed{\quad}$
$(3\,\vec{\mathbf{u}} - 2\,\vec{\mathbf{v}}) \times (\vec{\mathbf{u}} + \vec{\mathbf{v}}) \cdot (2\,\vec{\mathbf{u}} - 7\,\vec{\mathbf{v}})$	$\boxed{\quad}$
$(\vec{\mathbf{u}} + \vec{\mathbf{v}}) \times (\vec{\mathbf{v}} + 2\,\vec{\mathbf{w}}) \cdot (\vec{\mathbf{w}} + 2\,\vec{\mathbf{u}})$	$\boxed{\quad}\,\vec{\mathbf{u}} \times \vec{\mathbf{v}} \cdot \vec{\mathbf{w}}$

1.30 ♣ Vector concepts: Solving a vector equation. **(Section 2.9.5)**

Consider the vector equation to the right and the process that follows that solves for $\dot{\theta}$ ($\widehat{\mathbf{a}}_{\mathrm{x}}$ is a unit vector and v_x, $\dot{\theta}$, R are scalars).

$$v_x\,\widehat{\mathbf{a}}_{\mathrm{x}} = \dot{\theta}\,R\,\widehat{\mathbf{a}}_{\mathrm{x}}$$

$$\dot{\theta} = \frac{v_x\,\widehat{\mathbf{a}}_{\mathrm{x}}}{R\,\widehat{\mathbf{a}}_{\mathrm{x}}} = \frac{v_x}{R}$$

This process is a valid way to solve for $\dot{\theta}$. **True/False**.

Explain:

1.31 Changing a vector equation to scalar equations. **Show work.** **(Section 2.9.5)**

Draw three mutually orthogonal unit vectors $\widehat{\mathbf{p}}$, $\widehat{\mathbf{q}}$, $\widehat{\mathbf{r}}$.

(a) Use a vector operation (e.g., $+$, $-$, $*$, $\cdot$, $\times$) to transform the following **vector** equation into **one scalar** equation and subsequently solve the scalar equation.

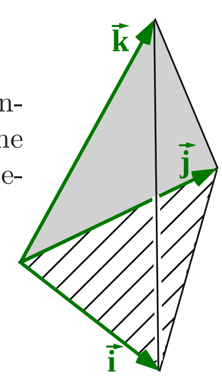
$$(2x - 4)\,\widehat{\mathbf{p}} = \vec{\mathbf{0}} \quad \overset{??}{\Rightarrow} \quad x = 2$$

(b) Show ***every*** vector operation (e.g., $+$, $-$, $*$, $\cdot$, or $\times$) that transforms the following **vector** equation into **three scalar** equations and subsequently solve the scalar equations for x, y, z.

$$(2\,x - 4)\,\widehat{\mathbf{p}} + (3\,y - 9)\,\widehat{\mathbf{q}} + (4\,z - 16)\,\widehat{\mathbf{r}} = \vec{\mathbf{0}}$$

Result: $x = 2 \qquad y = 3 \qquad z = \boxed{\quad}$

(c) † The figure to the right shows three **non-zero**, **non-orthogonal**, non-coplanar vectors $\vec{\mathbf{i}}$, $\vec{\mathbf{j}}$, $\vec{\mathbf{k}}$. Show **every** vector operation that transforms the following **vector** equation into **three** uncoupled **scalar** equations and subsequently solve those scalar equations for x, y, z (think $\times$ $\cdot$, not matrix algebra).



$$(2\,x - 4)\,\vec{\mathbf{i}} + (3\,y - 9)\,\vec{\mathbf{j}} + (4\,z - 16)\,\vec{\mathbf{k}} = \vec{\mathbf{0}}$$

Result: $x = 2 \qquad y = 3 \qquad z = \boxed{\quad}$

1.32 ♣ Number of independent scalar equations from one vector equation. **(Section 2.9.5)**

$$\boxed{\vec{\mathbf{F}}^{S} = \vec{\mathbf{0}}}$$

Consider the **vector** equation shown to the right that can be useful for static analyses of any system S.

Complete the blanks in the table to the right with **all** integers that could be equal to the number of ***independent scalar*** equations produced by the previous vector equation for any system S.

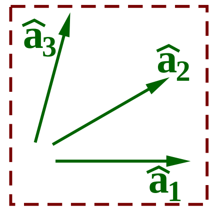
Hint: See Homework 1.31 for ideas.

System type	Integer(s)
1D (line)	0, ____
2D (planar)	0, ____
3D (spatial)	0, ____

Note: Regard 1D/linear as meaning $\vec{\mathbf{F}}^{S}$ can be expressed in terms of a single unit vector $\hat{\mathbf{i}}$ whereas 2D/planar means $\vec{\mathbf{F}}^{S}$ can be expressed in terms of two non-parallel unit vectors $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$, and 3D/spatial means $\vec{\mathbf{F}}^{S}$ can be expressed in terms of three non-coplanar unit vectors $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, $\hat{\mathbf{k}}$.

1.33 ♣ Vector concepts: Solving a vector equation **(just circle true or false and fill-in the blank).**

Consider the following vector equation written in terms of the scalars x, y, z and three unique non-orthogonal ***coplanar*** unit vectors $\hat{\mathbf{a}}_1$, $\hat{\mathbf{a}}_2$, $\hat{\mathbf{a}}_3$.



$$(2\,x - 4)\;\hat{\mathbf{a}}_1 \;+\; (3\,y - 9)\;\hat{\mathbf{a}}_2 \;+\; (4\,z - 16)\;\hat{\mathbf{a}}_3 \;=\; \vec{\mathbf{0}}$$

The **unique** solution to this vector equation is $x = 2,\; y = 3,\; z = 4$. **True/False**.

Explain: $\hat{\mathbf{a}}_2$ can be expressed in terms of $\hat{\mathbf{a}}_1$ and $\hat{\mathbf{a}}_3$ (i.e., $\hat{\mathbf{a}}_2$ is a linear combination of $\hat{\mathbf{a}}_1$ and $\hat{\mathbf{a}}_3$).
Hence the vector equation produces ____ linearly independent scalar equations.

1.34 ♣ A vector revolution in geometry. **(Chapter 2)**

The relatively new invention of vectors (Gibbs ≈ 1900 AD) has revolutionized Euclidean geometry (Euclid ≈ 300 BC). For each geometrical quantity below, circle the vector operation(s) (either the dot-product, cross-product, or both) that is ***most*** useful for their calculation.

Length: $\cdot$ $\times$	Angle: $\cdot$ $\times$
Area: $\cdot$ $\times$	Volume: $\cdot$ $\times$

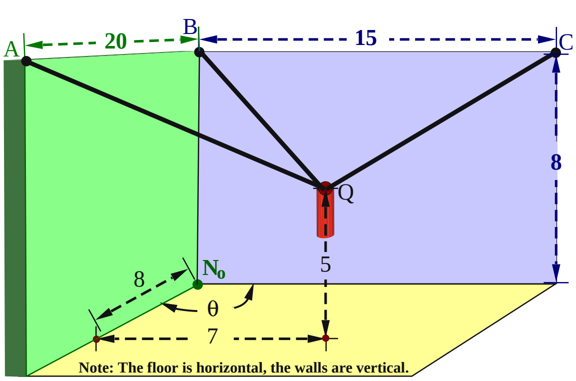
1.35 † Microphone cable lengths (non-orthogonal walls) "It's just geometry". **Show work.**

A microphone Q is attached to three pegs A, B, C by three cables. Knowing the peg locations, microphone location, and the angle θ between the vertical walls, express L_A, L_B, L_C solely in terms of numbers and θ. Next, complete the table by calculating L_B when $\theta = 120°$.

Hint: To do this **efficiently**, do **not** introduce an **orthogonal** set of unit vectors.

Hint: Use the distributive property of the vector dot-product as shown in Section 2.9.1 and Homework 2.4.

Note: Synthesis problems are difficult. Engineers solve problems. Think, talk, draw, sleep, walk, get help, ...



Distance between A and B	20 m
Distance between B and C	15 m
Distance between N_o and B	8 m
Distance along back wall (see picture)	7 m
Q's height above N_o	5 m
Distance along side wall (see picture)	8 m
L_A: Length of cable joining A and Q	16.9 m
L_B: Length of cable joining B and Q	8.1 m
L_C: Length of cable joining C and Q	14.2 m

Result:

$L_A = \sqrt{202 - 168\,\cos(\theta)}$ $\qquad L_B =$ ____ $\qquad L_C = \sqrt{137 - \_\_\_\_\,\cos(\theta)}$

Show work – except for ♣ fill-in-blanks-problems (print .pdf from www.MotionGenesis.com ⇒ Textbooks ⇒ Resources).

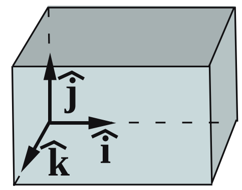
2.1 ♣ Right-handed, orthogonal, unitary, vector basis. (Section 4.1)

Draw a right-handed orthogonal (mutually perpendicular) vector basis consisting of the unit vectors $\hat{\mathbf{a}}_x$, $\hat{\mathbf{a}}_y$, $\hat{\mathbf{a}}_z$.

2.2 ♣ Adding and subtracting vectors with bases. (Sections 2.6 and 2.8)

Shown right are right-handed orthogonal unit vectors $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, $\hat{\mathbf{k}}$ and vectors $\vec{\mathbf{u}}$, $\vec{\mathbf{v}}$, and $\vec{\mathbf{w}}$. Form the vector sums and differences below.
Result:

$$\begin{aligned}\vec{\mathbf{u}} &= 2\,\hat{\mathbf{i}} + 3\,\hat{\mathbf{j}} + 4\,\hat{\mathbf{k}}\\ \vec{\mathbf{v}} &= x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}\\ \vec{\mathbf{w}} &= 5\,\hat{\mathbf{i}} - 6\,\hat{\mathbf{j}} + 7\,\hat{\mathbf{k}}\end{aligned}$$

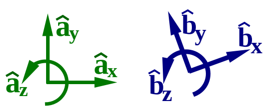


$$\vec{\mathbf{u}} + \vec{\mathbf{v}} = (2+x)\,\hat{\mathbf{i}} + \underline{\quad\quad}\,\hat{\mathbf{j}} + \underline{\quad\quad}\,\hat{\mathbf{k}} \qquad \vec{\mathbf{u}} - \vec{\mathbf{v}} = (2-x)\,\hat{\mathbf{i}} + \underline{\quad\quad}\,\hat{\mathbf{j}} + \underline{\quad\quad}\,\hat{\mathbf{k}}$$

2.3 ♣ Column matrices and vectors (Hint: What is a vector – Hw 1.2). (Section 2.1)

The vector $\hat{\mathbf{a}}_x + 2\,\hat{\mathbf{a}}_y + 3\,\hat{\mathbf{a}}_z$ is equal to the column matrix $\begin{bmatrix}1\\2\\3\end{bmatrix}$. **True/False**.
Note: $\hat{\mathbf{a}}_x$, $\hat{\mathbf{a}}_y$, $\hat{\mathbf{a}}_z$ are orthogonal unit vectors as drawn in Hw 2.1.

Adding the following vectors and column matrices produce equivalent results. **True/False**.
Note: $\hat{\mathbf{a}}_x$, $\hat{\mathbf{a}}_y$, $\hat{\mathbf{a}}_z$ and $\hat{\mathbf{b}}_x$, $\hat{\mathbf{b}}_y$, $\hat{\mathbf{b}}_z$ are sets of orthogonal unit vectors.



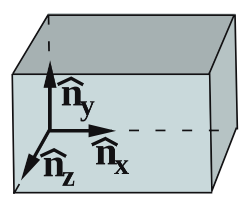
$$\hat{\mathbf{a}}_x + 2\,\hat{\mathbf{a}}_y + 3\,\hat{\mathbf{a}}_z \;+\; 4\,\hat{\mathbf{b}}_x + 5\,\hat{\mathbf{b}}_y + 6\,\hat{\mathbf{b}}_z \;=\; \begin{bmatrix}1\\2\\3\end{bmatrix} + \begin{bmatrix}4\\5\\6\end{bmatrix} = \begin{bmatrix}5\\7\\9\end{bmatrix}$$

Explain: ____________________

2.4 ♣ Calculating vector dot products with bases. (Sections 2.9 and 2.9.3)

Given: Right-handed orthogonal unit vectors $\hat{\mathbf{n}}_x$, $\hat{\mathbf{n}}_y$, $\hat{\mathbf{n}}_z$ and:

$$\begin{aligned}\vec{\mathbf{u}} &= 2\,\hat{\mathbf{n}}_x + 3\,\hat{\mathbf{n}}_y + 4\,\hat{\mathbf{n}}_z\\ \vec{\mathbf{v}} &= x\,\hat{\mathbf{n}}_x + y\,\hat{\mathbf{n}}_y + z\,\hat{\mathbf{n}}_z\\ \vec{\mathbf{w}} &= 5\,\hat{\mathbf{n}}_x - 6\,\hat{\mathbf{n}}_y + 7\,\hat{\mathbf{n}}_z\end{aligned}$$



(a) Use the distributive law for dot products to write $\vec{\mathbf{u}} \cdot \vec{\mathbf{v}}$ in terms of $\hat{\mathbf{n}}_x \cdot \hat{\mathbf{n}}_x$, $\hat{\mathbf{n}}_x \cdot \hat{\mathbf{n}}_y$, etc.
Result:

$$\begin{aligned}\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = {} & 2\,x\,\hat{\mathbf{n}}_x \cdot \hat{\mathbf{n}}_x + 2\,y\,\hat{\mathbf{n}}_x \cdot \hat{\mathbf{n}}_y + 2\,z\,\hat{\mathbf{n}}_x \cdot \hat{\mathbf{n}}_z\\ & + 3\,x\,\hat{\mathbf{n}}_y \cdot \hat{\mathbf{n}}_x + 3\,y\,\underline{\quad} \cdot \underline{\quad} + \underline{\quad\quad}\\ & + \underline{\quad\quad} + \underline{\quad\quad} + \underline{\quad\quad}\end{aligned}$$

(b) Use the definition of the dot product in equation (2.2) to calculate $\hat{\mathbf{n}}_x \cdot \hat{\mathbf{n}}_x$, $\hat{\mathbf{n}}_x \cdot \hat{\mathbf{n}}_y$, etc.
Result:

$$\begin{aligned}\hat{\mathbf{n}}_x \cdot \hat{\mathbf{n}}_x &= \underline{\quad} & \hat{\mathbf{n}}_x \cdot \hat{\mathbf{n}}_y &= \underline{\quad} & \hat{\mathbf{n}}_x \cdot \hat{\mathbf{n}}_z &= \underline{\quad}\\ \hat{\mathbf{n}}_y \cdot \hat{\mathbf{n}}_x &= \underline{\quad} & \hat{\mathbf{n}}_y \cdot \hat{\mathbf{n}}_y &= \underline{\quad} & \hat{\mathbf{n}}_y \cdot \hat{\mathbf{n}}_z &= 0\\ \hat{\mathbf{n}}_z \cdot \hat{\mathbf{n}}_x &= \underline{\quad} & \hat{\mathbf{n}}_z \cdot \hat{\mathbf{n}}_y &= \underline{\quad} & \hat{\mathbf{n}}_z \cdot \hat{\mathbf{n}}_z &= 1\end{aligned}$$

(c) In view of your previous two results, calculate $\vec{\mathbf{u}} \cdot \vec{\mathbf{v}}$.
Result:

$$\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = \underline{\quad\quad\quad}$$

(d) As shown below for the arbitrary vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ (also shown in Section 2.9.3), the dot product $\vec{\mathbf{a}} \cdot \vec{\mathbf{b}}$ is relatively easy to calculate when $\widehat{\mathbf{n}}_{\mathrm{x}}$, $\widehat{\mathbf{n}}_{\mathrm{y}}$, $\widehat{\mathbf{n}}_{\mathrm{z}}$ are ***orthogonal unit*** vectors.

$$\left.\begin{aligned} \vec{\mathbf{a}} &= a_x\,\widehat{\mathbf{n}}_{\mathrm{x}} + a_y\,\widehat{\mathbf{n}}_{\mathrm{y}} + a_z\,\widehat{\mathbf{n}}_{\mathrm{z}} \\ \vec{\mathbf{b}} &= b_x\,\widehat{\mathbf{n}}_{\mathrm{x}} + b_y\,\widehat{\mathbf{n}}_{\mathrm{y}} + b_z\,\widehat{\mathbf{n}}_{\mathrm{z}} \end{aligned}\right\} \quad \Rightarrow \quad \vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = a_x\,b_x + a_y\,b_y + a_z\,b_z$$

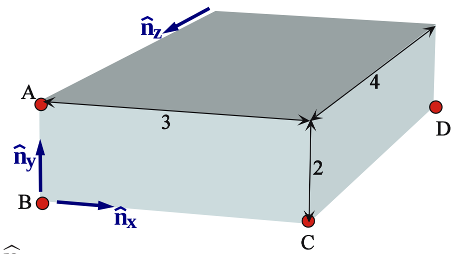
Use this **orthogonal-short-cut** to calculate:

$\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = 2\,x + 3\,y + \boxed{\quad}\,z$ $\qquad$ $\vec{\mathbf{u}} \cdot \vec{\mathbf{w}} = \boxed{\quad}$ $\qquad$ $\vec{\mathbf{v}} \cdot \vec{\mathbf{w}} = \boxed{\qquad\qquad}$

2.5 ♣ Perpendicular vectors. (Note: $\widehat{\mathbf{i}}$, $\widehat{\mathbf{j}}$, $\widehat{\mathbf{k}}$ are orthogonal unit vectors). (Section 2.9)

When vector $\vec{\mathbf{v}} = x\,\widehat{\mathbf{i}} + 2\,\widehat{\mathbf{j}} + 3\,\widehat{\mathbf{k}}$ is perpendicular to vector $\vec{\mathbf{w}} = 4\,\widehat{\mathbf{i}} + 5\,\widehat{\mathbf{j}} + 6\,\widehat{\mathbf{k}}$, $x = \boxed{\quad}$.

2.6 Dot product for calculating angles. (Sections 2.9 and 3.3)



The figure to the right shows a rectangular parallelepiped (block) of sides 2, 3, and 4 with points A, B, C located at corners. $\widehat{\mathbf{n}}_{\mathrm{x}}$, $\widehat{\mathbf{n}}_{\mathrm{y}}$, $\widehat{\mathbf{n}}_{\mathrm{z}}$ are right-handed orthogonal unit vectors with $\widehat{\mathbf{n}}_{\mathrm{x}}$ directed from B to C and $\widehat{\mathbf{n}}_{\mathrm{y}}$ from B to A.

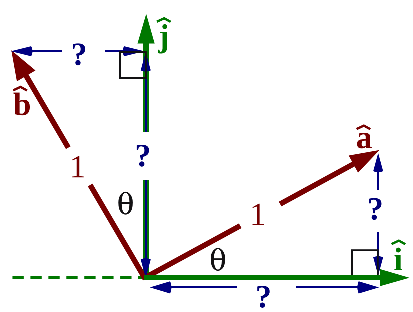
(a) Express $\vec{\mathbf{r}}^{\,C/A}$ (C's position vector from A) in terms of $\widehat{\mathbf{n}}_{\mathrm{x}}$, $\widehat{\mathbf{n}}_{\mathrm{y}}$, $\widehat{\mathbf{n}}_{\mathrm{z}}$.
Find a numerical value for $|\vec{\mathbf{r}}^{\,C/A}|^2 = \vec{\mathbf{r}}^{\,C/A} \cdot \vec{\mathbf{r}}^{\,C/A}$.
Next, use equation (2.4) to calculate the magnitude of $\vec{\mathbf{r}}^{\,C/A}$ (the distance from A to C).
Result: $\vec{\mathbf{r}}^{\,C/A} = \boxed{\quad}\,\widehat{\mathbf{n}}_{\mathrm{x}} - \boxed{\quad}\,\widehat{\mathbf{n}}_{\mathrm{y}}$ $\qquad$ $|\vec{\mathbf{r}}^{\,C/A}|^2 = \vec{\mathbf{r}}^{\,C/A} \cdot \vec{\mathbf{r}}^{\,C/A} = \boxed{\quad}$ $\qquad$ $|\vec{\mathbf{r}}^{\,C/A}| = \boxed{\quad}$

(b) Using equation (2.1), calculate the unit vector $\widehat{\mathbf{u}}$ directed from A to C in terms of $\widehat{\mathbf{n}}_{\mathrm{x}}$, $\widehat{\mathbf{n}}_{\mathrm{y}}$, $\widehat{\mathbf{n}}_{\mathrm{z}}$.
Next, find the unit vector $\widehat{\mathbf{v}}$ directed from A to D in terms of $\widehat{\mathbf{n}}_{\mathrm{x}}$, $\widehat{\mathbf{n}}_{\mathrm{y}}$, $\widehat{\mathbf{n}}_{\mathrm{z}}$.
Result:

$$\widehat{\mathbf{u}} = \frac{3\,\widehat{\mathbf{n}}_{\mathrm{x}} - 2\,\widehat{\mathbf{n}}_{\mathrm{y}}}{\sqrt{13}} \qquad \widehat{\mathbf{v}} = \frac{\boxed{\qquad\qquad}}{\boxed{\quad}}$$

(c) Calculate $\angle BAC$, the angle between line $\overline{AB}$ and line $\overline{AC}$.
Next, calculate $\angle CAD$, the angle between line $\overline{AC}$ and line $\overline{AD}$.
Result: $\angle BAC = \boxed{\quad}^\circ$ $\qquad$ $\angle CAD = \boxed{47.97^\circ}$

2.7 ♣ Vector components: Sine and cosine. (Section 1.4)



Trigonometry plays a central role in ***rotation matrices***.
Replace each **?** in the figure to the right with $\sin(\theta)$ or $\cos(\theta)$.
Express unit vectors $\widehat{\mathbf{a}}$ and $\widehat{\mathbf{b}}$ in terms of $\sin(\theta)$, $\cos(\theta)$, $\widehat{\mathbf{i}}$, $\widehat{\mathbf{j}}$.
Result:

SohCahToa

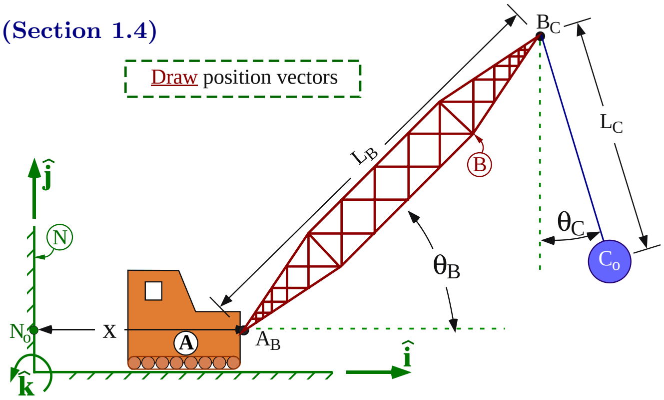
$$\widehat{\mathbf{a}} = \boxed{\qquad}\,\widehat{\mathbf{i}} + \boxed{\qquad}\,\widehat{\mathbf{j}}$$
$$\widehat{\mathbf{b}} = \boxed{\qquad}\,\widehat{\mathbf{i}} + \cos(\theta)\,\widehat{\mathbf{j}}$$

2.8 ♣ Vector components for a crane-boom. (Section 1.4)

Shown right is a crane whose cab A supports a boom B that swings a wrecking ball C_o.

$\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, $\hat{\mathbf{k}}$ are right-handed orthogonal unit vectors with $\hat{\mathbf{i}}$ horizontally-right, $\hat{\mathbf{j}}$ vertically-upward, and $\hat{\mathbf{k}}$ perpendicular to the plane containing points N_o, A_B, B_C, C_o.

Draw each position vector listed below and then use your knowledge of sine/cosine to resolve these vectors into $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ components.



A_B's position vector from N_o $\quad \vec{\mathbf{r}}^{\,A_B/N_o} = \boxed{}\,\hat{\mathbf{i}} + \boxed{}\,\hat{\mathbf{j}}$

B_C's position vector from A_B $\quad \vec{\mathbf{r}}^{\,B_C/A_B} = \boxed{}\,\hat{\mathbf{i}} + \boxed{}\,\hat{\mathbf{j}}$

C_o's position vector from B_C $\quad \vec{\mathbf{r}}^{\,C_o/B_C} = \boxed{}\,\hat{\mathbf{i}} + \boxed{}\,\hat{\mathbf{j}}$

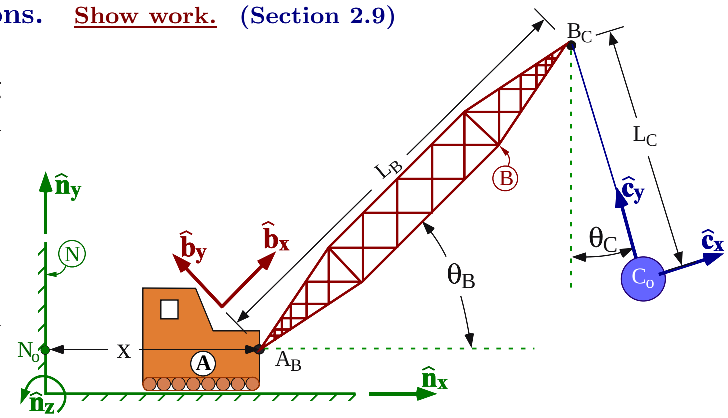
B_C's position vector from N_o $\quad \vec{\mathbf{r}}^{\,B_C/N_o} = \boxed{}\,\hat{\mathbf{i}} + \boxed{}\,\hat{\mathbf{j}}$

N_o's position vector from C_o $\quad \vec{\mathbf{r}}^{\,N_o/C_o} = \boxed{}\,\hat{\mathbf{i}} + [-L_B \sin(\theta_B) + L_C \cos(\theta_C)]\,\hat{\mathbf{j}}$

2.9 Dot products and distance calculations. Show work. (Section 2.9)

Shown right is a crane whose cab A supports a boom B that swings a wrecking ball C_o. To prevent the wrecking ball from hitting a car at point N_o, the distance between N_o and the tip of the boom (point B_C) must be controlled.

To start this problem, express B_C's position vector from N_o in terms of x, L_B, and the unit vectors $\hat{\mathbf{n}}_x$ and $\hat{\mathbf{b}}_x$.



Result: $\vec{\mathbf{r}}^{\,B_C/N_o} = \boxed{}\,\hat{\mathbf{n}}_x + \boxed{}\,\hat{\mathbf{b}}_x$

(a) ***Without*** resolving $\vec{\mathbf{r}}^{\,B_C/N_o}$ into $\hat{\mathbf{n}}_x$ and $\hat{\mathbf{n}}_y$ components (done in the next step), use equation (2.4) and the distributive property to calculate the distance between N_o and B_C in terms of x, L_B, θ_B. Then calculate its numerical value when $x = 20$ m, $L_B = 10$ m, $\theta_B = 30°$.
Result: (If necessary, complete the footnote hint below).[1]

Distance between N_o and B_C: $\quad |\vec{\mathbf{r}}^{\,B_C/N_o}| = \sqrt{\boxed{} + \boxed{} + \boxed{}} = 29.1$ m

(b) Two colleagues are confused by your use of ***mixed-bases*** vectors (i.e., $x\,\hat{\mathbf{n}}_x + L_B\,\hat{\mathbf{b}}_x$), and ask you to verify B's position vector from N_o can be expressed in the ***uniform-basis*** as shown below. Use this uniform-basis expression to verify your previous result for $|\vec{\mathbf{r}}^{\,B_C/N_o}|$.
Note: This inefficient uniform-basis approach requires the simplifying trigonometric identity $\sin^2(\theta_B) + \cos^2(\theta_B) = 1$.
Result: $\vec{\mathbf{r}}^{\,B_C/N_o} = [x + L_B \cos(\theta_B)]\,\hat{\mathbf{n}}_x + L_B \sin(\theta_B)\,\hat{\mathbf{n}}_y$ $\qquad |\vec{\mathbf{r}}^{\,B_C/N_o}|$ simplifies to previous result.

(c) **Optional:** Calculate the distance between N_o and C_o in terms of x, L_B, L_C, θ_B, and θ_C.
Result: $|\vec{\mathbf{r}}^{\,C_o/N_o}| = \sqrt{x^2 + L_B^2 + L_C^2 + 2\,x\,L_B \cos(\theta_B) + 2\,x\,L_C \sin(\theta_C) - 2\,L_B\,L_C \sin(\theta_B - \theta_C)}$

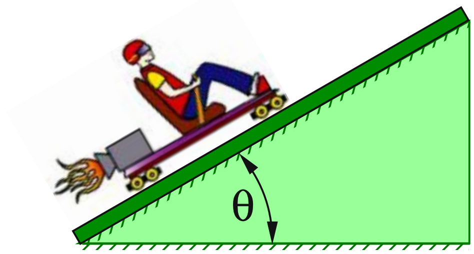
---

[1]Hint: Use the distributive property to express $\vec{\mathbf{r}}^{\,B_C/N_o} \cdot \vec{\mathbf{r}}^{\,B_C/N_o}$ in terms of x, L_B, and $\hat{\mathbf{n}}_x \cdot \hat{\mathbf{b}}_x$.
Next, use the ***dot-product definition*** in equation (2.2) to calculate $\hat{\mathbf{n}}_x \cdot \hat{\mathbf{b}}_x = \boxed{}$, and then rewrite
$\vec{\mathbf{r}}^{\,B_C/N_o} \cdot \vec{\mathbf{r}}^{\,B_C/N_o} = \boxed{}^2 + \boxed{}^2 + 2\,x\,L_B\,(\boxed{} \cdot \boxed{}) = \boxed{}^2 + \boxed{}^2 + 2\,x\,L_B \cos(\boxed{})$.
Note: The distributive property for vector dot-multiplication is $(\vec{\mathbf{a}} + \vec{\mathbf{b}}) \cdot (\vec{\mathbf{c}} + \vec{\mathbf{d}}) = \vec{\mathbf{a}} \cdot \vec{\mathbf{c}} + \vec{\mathbf{a}} \cdot \vec{\mathbf{d}} + \vec{\mathbf{b}} \cdot \vec{\mathbf{c}} + \vec{\mathbf{b}} \cdot \vec{\mathbf{d}}$.

2.10 Vector components, free-body diagram (FBD), and motion graphs for a rocket-sled.

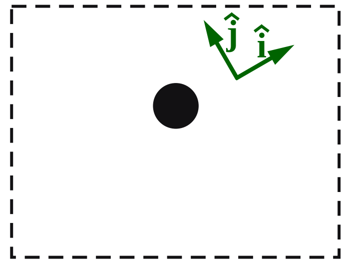
The following figure shows a rocket-sled moving along smooth (**frictionless**) inclined rails.

Description	Symbol	Type
Mass of rocket-sled and rider	m	Constant
Earth's gravitational acceleration	g	Constant
Angle between horizontal and inclined-rails	θ	Constant
$\widehat{\mathbf{i}}$ measure of thrust force on sled	F_T	Specified
$\widehat{\mathbf{j}}$ measure of normal force on sled	F_N	Variable
$\widehat{\mathbf{i}}$ measure of rocket-sled position	x	Variable



Draw a unit vector $\widehat{\mathbf{i}}$ upward-right and parallel to the rails.
Draw a unit vector $\widehat{\mathbf{j}}$ outward-normal to the rails (perpendicular to $\widehat{\mathbf{i}}$) and in the plane of the paper.

Free-body diagram



Draw a particle representing the rocket-sled.
Draw the thrust, normal, and gravity forces on the rocket-sled.
Express the net force on the rocket-sled in terms of $\widehat{\mathbf{i}}$ and $\widehat{\mathbf{j}}$.

Result: $\vec{\mathbf{F}}_{\text{Net}} = \vec{\mathbf{F}}_{\text{Thrust}} + \vec{\mathbf{F}}_{\text{Normal}} + \vec{\mathbf{F}}_{\text{gravity}}$

$$= \boxed{\qquad\qquad}\,\widehat{\mathbf{i}} + \boxed{\qquad\qquad}\,\widehat{\mathbf{j}}$$

$\vec{\mathbf{F}} = m\,\vec{\mathbf{a}}$ The rocket-sled's acceleration can be calculated as $\vec{\mathbf{a}} = \frac{d^2x}{dt^2}\,\widehat{\mathbf{i}} = \ddot{x}\,\widehat{\mathbf{i}}$.

Substitute the right-hand sides of $\vec{\mathbf{F}}_{\text{Net}}$ and $\vec{\mathbf{a}}$ into $\vec{\mathbf{F}}_{\text{Net}} = m\,\vec{\mathbf{a}}$ and solve for $\ddot{x}$ and F_N.

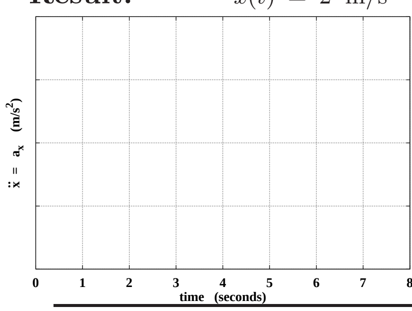
Result: $\boxed{\qquad\qquad}\,\widehat{\mathbf{i}} + \boxed{\qquad\qquad}\,\widehat{\mathbf{j}} = m\,\ddot{x}\,\widehat{\mathbf{i}}$

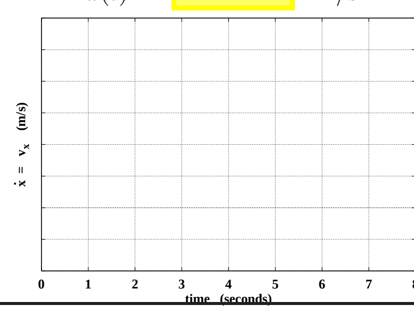
$$\widehat{\mathbf{i}}:\quad \ddot{x} = \frac{\boxed{\quad} - \boxed{\qquad}}{m} \qquad\qquad \widehat{\mathbf{j}}:\quad F_N = \boxed{\qquad}$$

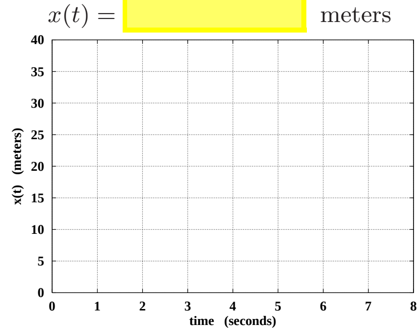
Solve for acceleration: Knowing $m = 100$ kg, $g = 10\ \frac{\text{m}}{\text{s}^2}$, $\theta = 30°$, $F_T = 700$ N, **graph** $\ddot{x}$.

Solve for velocity and position: Knowing the rocket-sled starts at $x = 8$ m, and is initially moving **downward-left** along the rail at $4\ \frac{\text{m}}{\text{s}}$, solve and **sketch** $\dot{x}(t)$ and $x(t)$ for $0 \le t \le 8$.

Result: $\ddot{x}(t) = 2\ \text{m/s}^2$ $\qquad \dot{x}(t) = \boxed{\qquad}$ m/s $\qquad x(t) = \boxed{\qquad}$ meters





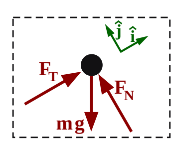


Include friction between the rails and rocket-sled, modeled via a coefficient of kinetic friction μ_k. Express $\ddot{x}$ and F_N in terms of some/all of μ_k and symbols in the table. Knowing $\mu_k \approx 0.115$ $x(0) = 8$ m, and the rocket-sled initially moves **upward-right** at $4\ \frac{\text{m}}{\text{s}}$, find $\dot{x}(t)$ and $x(t)$.

Result:

$$\ddot{x}(t) = \frac{\boxed{\qquad\qquad\qquad}}{m} \approx 1\ \frac{\text{m}}{\text{s}^2} \qquad F_N = \boxed{\qquad} \qquad \dot{x}(t) \approx \boxed{\quad} \qquad x(t) \approx \boxed{\qquad}$$

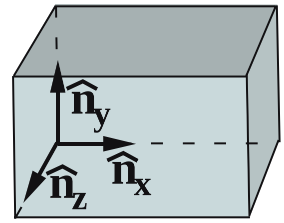
Calculate the minimum thrust (redraw FBDs) to:	Result
a. Keep the rocket-sled moving uphill at constant speed	$F_T \approx \boxed{\quad}$ N
b. Keep the rocket-sled moving downhill at constant speed	$F_T \approx \boxed{\quad}$ N



For part **b**, assume the rocket is initially moving **downward-left** at 4 m/s.
Solution at www.MotionGenesis.com ⇒ Get Started ⇒ **Rocket sled.**

2.11 ♣ **Construct a unit vector $\hat{\mathbf{u}}$ in the direction of each vector given below.** **(Section 2.9.2)**

Vector	Unit vector $\hat{\mathbf{u}}$
$3\,\hat{\mathbf{n}}_x$	$\hat{\mathbf{n}}_x$
$-3\,\hat{\mathbf{n}}_x$	
$3\,\hat{\mathbf{n}}_x - 4\,\hat{\mathbf{n}}_y$	
$3\,\hat{\mathbf{n}}_x - 4\,\hat{\mathbf{n}}_y + 12\,\hat{\mathbf{n}}_z$	
$c\,\hat{\mathbf{n}}_x$ c is a real non-zero number	or



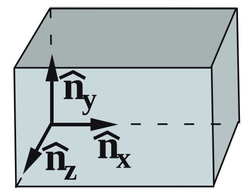
Note: $\hat{\mathbf{n}}_x$, $\hat{\mathbf{n}}_y$, $\hat{\mathbf{n}}_z$ are orthogonal unit vectors.

Note: Ensure your last answer agrees with your first two answers, e.g., if $c = 3$ or $c = -3$.

2.12 ♣ **Calculating vector cross products with bases.** **(Section 2.10)**

Given: Right-handed orthogonal unit vectors $\hat{\mathbf{n}}_x$, $\hat{\mathbf{n}}_y$, $\hat{\mathbf{n}}_z$ and:

$$\begin{aligned}\vec{\mathbf{u}} &= 2\,\hat{\mathbf{n}}_x + 3\,\hat{\mathbf{n}}_y + 4\,\hat{\mathbf{n}}_z \\ \vec{\mathbf{v}} &= x\,\hat{\mathbf{n}}_x + y\,\hat{\mathbf{n}}_y + z\,\hat{\mathbf{n}}_z \\ \vec{\mathbf{w}} &= 5\,\hat{\mathbf{n}}_x - 6\,\hat{\mathbf{n}}_y + 7\,\hat{\mathbf{n}}_z\end{aligned}$$



(a) Use the distributive law for cross products to write $\vec{\mathbf{u}} \times \vec{\mathbf{v}}$ in terms of $\hat{\mathbf{n}}_x \times \hat{\mathbf{n}}_x$, $\hat{\mathbf{n}}_x \times \hat{\mathbf{n}}_y$, etc.
Result:

$$\begin{aligned}\vec{\mathbf{u}} \times \vec{\mathbf{v}} = \; & 2\,x\,\hat{\mathbf{n}}_x \times \hat{\mathbf{n}}_x + 2\,y\,\hat{\mathbf{n}}_x \times \hat{\mathbf{n}}_y + 2\,z\,\hat{\mathbf{n}}_x \times \hat{\mathbf{n}}_z \\ & + \boxed{\quad} + \boxed{\quad} + \boxed{\quad} \\ & + \boxed{\quad} + \boxed{\quad} + \boxed{\quad}\end{aligned}$$

(b) Use the definition of the cross product to calculate $\hat{\mathbf{n}}_x \times \hat{\mathbf{n}}_x$, $\hat{\mathbf{n}}_x \times \hat{\mathbf{n}}_y$, etc.
Result:

$$\begin{array}{lll}\hat{\mathbf{n}}_x \times \hat{\mathbf{n}}_x = \vec{\mathbf{0}} & \hat{\mathbf{n}}_x \times \hat{\mathbf{n}}_y = \hat{\mathbf{n}}_z & \hat{\mathbf{n}}_x \times \hat{\mathbf{n}}_z = -\hat{\mathbf{n}}_y \\ \hat{\mathbf{n}}_y \times \hat{\mathbf{n}}_x = \boxed{\quad} & \hat{\mathbf{n}}_y \times \hat{\mathbf{n}}_y = \boxed{\quad} & \hat{\mathbf{n}}_y \times \hat{\mathbf{n}}_z = \boxed{\quad} \\ \hat{\mathbf{n}}_z \times \hat{\mathbf{n}}_x = \boxed{\quad} & \hat{\mathbf{n}}_z \times \hat{\mathbf{n}}_y = \boxed{\quad} & \hat{\mathbf{n}}_z \times \hat{\mathbf{n}}_z = \boxed{\quad}\end{array}$$

(c) In view of your previous two results, calculate $\vec{\mathbf{u}} \times \vec{\mathbf{v}}$.
Result: $\vec{\mathbf{u}} \times \vec{\mathbf{v}} = \boxed{\quad}\,\hat{\mathbf{n}}_x + \boxed{\quad}\,\hat{\mathbf{n}}_y + \boxed{\quad}\,\hat{\mathbf{n}}_z$

2.13 **Cross products and determinants.** **(Section 2.10.3)**

Given **right-handed orthogonal unit** vectors $\hat{\mathbf{n}}_x$, $\hat{\mathbf{n}}_y$, $\hat{\mathbf{n}}_z$ and two arbitrary vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ expressed as shown to the right, show that calculating $\vec{\mathbf{a}} \times \vec{\mathbf{b}}$ with the distributive property of the cross product happens to be equal to the determinant of the matrix shown to the right.

$$\begin{aligned}\vec{\mathbf{a}} &= a_x\,\hat{\mathbf{n}}_x + a_y\,\hat{\mathbf{n}}_y + a_z\,\hat{\mathbf{n}}_z \\ \vec{\mathbf{b}} &= b_x\,\hat{\mathbf{n}}_x + b_y\,\hat{\mathbf{n}}_y + b_z\,\hat{\mathbf{n}}_z\end{aligned}$$

$$\vec{\mathbf{a}} \times \vec{\mathbf{b}} = \det \begin{bmatrix} \hat{\mathbf{n}}_x & \hat{\mathbf{n}}_y & \hat{\mathbf{n}}_z \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{bmatrix}$$

Next use this determinant method to calculate the following cross products (refer to Hw 2.12).
Result: $\vec{\mathbf{u}} \times \vec{\mathbf{v}} = (3\,z - 4\,y)\,\hat{\mathbf{n}}_x + (4\,x - 2\,z)\,\hat{\mathbf{n}}_y + (2\,y - 3\,x)\,\hat{\mathbf{n}}_z$

Optional: $\vec{\mathbf{u}} \times \vec{\mathbf{w}} = 45\,\hat{\mathbf{n}}_x + 6\,\hat{\mathbf{n}}_y - 27\,\hat{\mathbf{n}}_z$

Optional: $\vec{\mathbf{v}} \times \vec{\mathbf{w}} = (\boxed{\quad})\,\hat{\mathbf{n}}_x + (\boxed{\quad})\,\hat{\mathbf{n}}_y - (\boxed{\quad})\,\hat{\mathbf{n}}_z$

2.14 ♣ **Optional: Cross product as skew-symmetric matrix multiplication.**

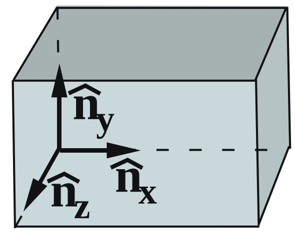
Referring to the previous problem, show that the $\hat{\mathbf{n}}_\text{x}$, $\hat{\mathbf{n}}_\text{y}$, $\hat{\mathbf{n}}_\text{z}$ coefficients of $\vec{\mathbf{a}} \times \vec{\mathbf{b}}$ happen to be equal to the elements that result from the following ***skew symmetric matrix*** multiplication.

$$\begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix} \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix}$$

After counting the number of computer operations required to multiple the 3×3 matrix by the 3×1 matrix (including multiplication by 0), and comparing the number of operations required to calculate the elements of the simplified answer, it is clear that using a matrix multiplication to calculate a cross product is **inefficient** **True/False**.

2.15 ♣ **Scalar triple product with bases.** **(Section 2.11)**

The figure shows right-handed orthogonal unit vectors $\hat{\mathbf{n}}_\text{x}$, $\hat{\mathbf{n}}_\text{y}$, $\hat{\mathbf{n}}_\text{z}$.



Given:
$$\vec{\mathbf{u}} = 2\,\hat{\mathbf{n}}_\text{x} + 3\,\hat{\mathbf{n}}_\text{y} + 4\,\hat{\mathbf{n}}_\text{z}$$
$$\vec{\mathbf{v}} = x\,\hat{\mathbf{n}}_\text{x} + y\,\hat{\mathbf{n}}_\text{y} + z\,\hat{\mathbf{n}}_\text{z}$$
$$\vec{\mathbf{w}} = 5\,\hat{\mathbf{n}}_\text{x} - 6\,\hat{\mathbf{n}}_\text{y} + 7\,\hat{\mathbf{n}}_\text{z}$$

Calculate:
$$\vec{\mathbf{u}} \times \vec{\mathbf{v}} \cdot \vec{\mathbf{u}}$$
$$\vec{\mathbf{u}} \times \vec{\mathbf{v}} \cdot \vec{\mathbf{w}}$$
$$\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} \times \vec{\mathbf{w}}$$

Result: (Although the order of operations in $\vec{\mathbf{u}} \times \vec{\mathbf{v}} \cdot \vec{\mathbf{u}}$ is unambiguous, parentheses may clarify your work.)

$$\vec{\mathbf{u}} \times \vec{\mathbf{v}} \cdot \vec{\mathbf{u}} = \boxed{}$$

$$\vec{\mathbf{u}} \times \vec{\mathbf{v}} \cdot \vec{\mathbf{w}} = \boxed{}$$

$$\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} \times \vec{\mathbf{w}} = 27\,z - 45\,x - 6\,y$$

In view of your last two results, $\vec{\mathbf{u}} \times \vec{\mathbf{v}} \cdot \vec{\mathbf{w}}$ is **equal/not equal** (circle one) to $\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} \times \vec{\mathbf{w}}$.
It is **OK/not OK** to switch the $\cdot$ and $\times$ in the scalar triple product.

2.16 ♣ **Optional: Scalar triple products and determinants.** **(Section 2.11)**

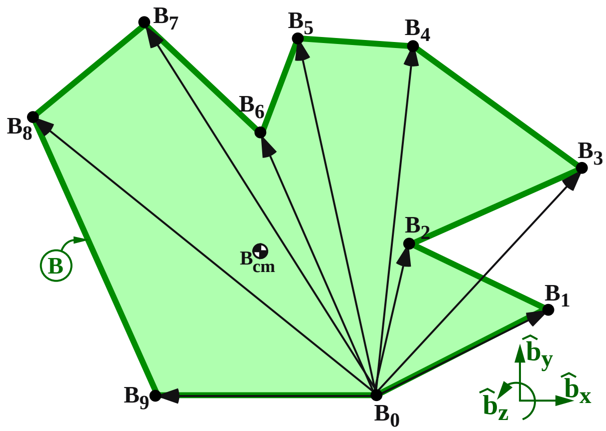
Given **right-handed orthogonal unit** vectors $\hat{\mathbf{n}}_\text{x}$, $\hat{\mathbf{n}}_\text{y}$, $\hat{\mathbf{n}}_\text{z}$ and arbitrary vectors $\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$, $\vec{\mathbf{c}}$ expressed as shown right, prove that calculating $\vec{\mathbf{a}} \cdot (\vec{\mathbf{b}} \times \vec{\mathbf{c}})$ happens to be equal to the determinant of the matrix shown to the right.

$$\vec{\mathbf{a}} = a_x\,\hat{\mathbf{n}}_\text{x} + a_y\,\hat{\mathbf{n}}_\text{y} + a_z\,\hat{\mathbf{n}}_\text{z}$$
$$\vec{\mathbf{b}} = b_x\,\hat{\mathbf{n}}_\text{x} + b_y\,\hat{\mathbf{n}}_\text{y} + b_z\,\hat{\mathbf{n}}_\text{z}$$
$$\vec{\mathbf{c}} = c_x\,\hat{\mathbf{n}}_\text{x} + c_y\,\hat{\mathbf{n}}_\text{y} + c_z\,\hat{\mathbf{n}}_\text{z}$$

$$\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} \times \vec{\mathbf{c}} = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}$$

2.17 ♣ Cross products: Commercial algorithm for area calculations (surveying). (Section 2.10.2)

One reason triangles are important is that complex **planar objects** such as the polygon B below can be decomposed into triangles. Planar measurements are important in various professions, including farming acreage, building costs, and mass and area properties of 2D objects.



$$
\begin{aligned}
\vec{\mathbf{r}}^{\,B_1/B_0} &= 2.0\,\widehat{\mathbf{b}}_\mathrm{x} + 2.0\,\widehat{\mathbf{b}}_\mathrm{y}\\
\vec{\mathbf{r}}^{\,B_2/B_0} &= 0.5\,\widehat{\mathbf{b}}_\mathrm{x} + 2.5\,\widehat{\mathbf{b}}_\mathrm{y}\\
\vec{\mathbf{r}}^{\,B_3/B_0} &= 3.0\,\widehat{\mathbf{b}}_\mathrm{x} + 4.0\,\widehat{\mathbf{b}}_\mathrm{y}\\
\vec{\mathbf{r}}^{\,B_4/B_0} &= 0.2\,\widehat{\mathbf{b}}_\mathrm{x} + 6.0\,\widehat{\mathbf{b}}_\mathrm{y}\\
\vec{\mathbf{r}}^{\,B_5/B_0} &= -0.5\,\widehat{\mathbf{b}}_\mathrm{x} + 7.0\,\widehat{\mathbf{b}}_\mathrm{y}\\
\vec{\mathbf{r}}^{\,B_6/B_0} &= -1.0\,\widehat{\mathbf{b}}_\mathrm{x} + 5.0\,\widehat{\mathbf{b}}_\mathrm{y}\\
\vec{\mathbf{r}}^{\,B_7/B_0} &= -2.0\,\widehat{\mathbf{b}}_\mathrm{x} + 7.0\,\widehat{\mathbf{b}}_\mathrm{y}\\
\vec{\mathbf{r}}^{\,B_8/B_0} &= -4.0\,\widehat{\mathbf{b}}_\mathrm{x} + 5.0\,\widehat{\mathbf{b}}_\mathrm{y}\\
\vec{\mathbf{r}}^{\,B_9/B_0} &= -2.0\,\widehat{\mathbf{b}}_\mathrm{x} + 0.0\,\widehat{\mathbf{b}}_\mathrm{y}
\end{aligned}
$$

A commercial algorithm for calculating the area of the polygon B shown above is to:

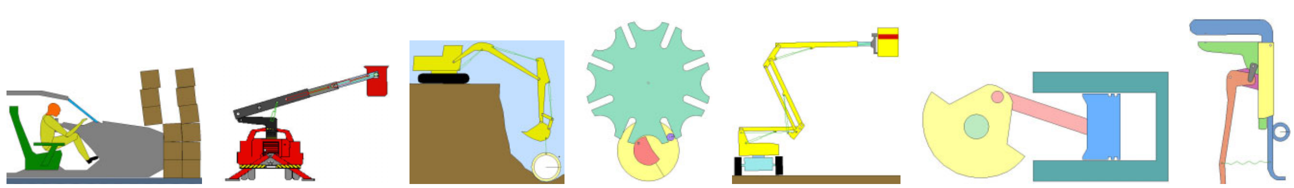
- Label a vertex B_0 and number the remaining vertices sequentially in a counter-clockwise fashion.
- Form $\vec{\mathbf{r}}^{\,B_i/B_0}$, the position vector of vertex B_i $(i = 1, 2, \ldots)$ from vertex B_0
- Calculate $\vec{\mathbf{A}}_2$ and $\vec{\mathbf{A}}_4$, the "vector-areas" of the triangles defined by vertices $B_0\,B_2\,B_3$, and $B_0\,B_4\,B_5$, respectively. Formulas for the vector-areas of each triangle are given below along with the vector sum of these areas $\vec{\mathbf{A}}$ and the polygon's area (the magnitude of $\vec{\mathbf{A}}$).

Result: (Just fill in the blanks. You only need to calculate $\vec{\mathbf{A}}_2$, $\vec{\mathbf{A}}_4$, and $\mathbf{A}$)

$\vec{\mathbf{A}}_1$	=	$\frac{1}{2} * \vec{\mathbf{r}}^{\,B_1/B_0} \times \vec{\mathbf{r}}^{\,B_2/B_0}$	=	$2\,\widehat{\mathbf{b}}_\mathrm{z}$
$\vec{\mathbf{A}}_2$	=	$\frac{1}{2} * \vec{\mathbf{r}}^{\,B_2/B_0} \times \vec{\mathbf{r}}^{\,B_3/B_0}$	=	
$\vec{\mathbf{A}}_3$	=	...	=	$8.6\,\widehat{\mathbf{b}}_\mathrm{z}$
$\vec{\mathbf{A}}_4$	=	...	=	
$\vec{\mathbf{A}}_5$	=	...	=	$2.25\,\widehat{\mathbf{b}}_\mathrm{z}$
$\vec{\mathbf{A}}_6$	=	...	=	$1.5\,\widehat{\mathbf{b}}_\mathrm{z}$
$\vec{\mathbf{A}}_7$	=	...	=	$9\,\widehat{\mathbf{b}}_\mathrm{z}$
$\vec{\mathbf{A}}_8$	=	$\frac{1}{2} * \vec{\mathbf{r}}^{\,B_8/B_0} \times \vec{\mathbf{r}}^{\,B_9/B_0}$	=	$5\,\widehat{\mathbf{b}}_\mathrm{z}$
$\mathbf{A}$	=	$\sum_{i=1}^{8} \vec{\mathbf{A}}_\mathrm{i}$	=	
Area	=	$\lvert\vec{\mathbf{A}}\rvert$	=	27.8

Accounting for overlapped areas is done with **positive** and **negative** signs on vectors.

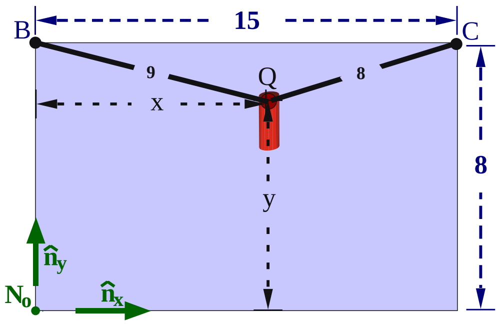
Note: Compute cross products with the distributive property $(\vec{\mathbf{a}}+\vec{\mathbf{b}}) \times (\vec{\mathbf{c}}+\vec{\mathbf{d}}) = \vec{\mathbf{a}} \times \vec{\mathbf{c}} + \vec{\mathbf{a}} \times \vec{\mathbf{d}} + \vec{\mathbf{b}} \times \vec{\mathbf{c}} + \vec{\mathbf{b}} \times \vec{\mathbf{d}}$ and the **cross-product-definition with the right-hand rule** (not determinants or special formulas in a book). Also, use the fact that $\widehat{\mathbf{b}}_\mathrm{x}$, $\widehat{\mathbf{b}}_\mathrm{y}$, $\widehat{\mathbf{b}}_\mathrm{z}$ are orthogonal unit vectors.



Planar objects, courtesy of Working Model and Design-Simulation Technologies

2.18 Locating a microphone (2D). **Show work.** **(Section 1.4)**

A microphone Q is attached to two pegs B and C by two cables. Knowing the peg locations, cable lengths, and that points B, C, Q, and N_o all lie in the same plane, determine the distance between Q and N_o. Try to do the problem first using Euclidean geometry - and then try vectors.[2]

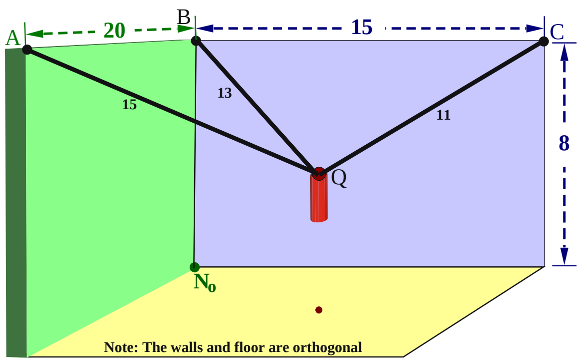


Quantity	Value
Distance from B to C	15 m
Distance from N_o to B	8 m
Length of cable joining B and Q	9 m
Length of cable joining C and Q	8 m
Distance between N_o and Q	9.01 m
If Q is above ceiling, distance $\approx$ 12 m	

Note: Although there are two "mathematical" answers to this problem, one is above the ceiling and requires the cables to be in compression.

2.19 † Locating a microphone (3D).

A microphone Q is attached to three pegs A, B, and C by three cables. Knowing the peg locations and cable lengths, determine the distance between Q and point N_o. **Show work.**[3]



Quantity	Value
Distance from A to B	20 m
Distance from B to C	15 m
Distance from N_o to B	8 m
Length of cable joining A and Q	15 m
Length of cable joining B and Q	13 m
Length of cable joining C and Q	11 m
Distance between N_o and Q	13.3 m
If Q is above ceiling, distance $\approx$ 17 m	

Note: This is part of the process of a camera targeting a football/baseball in a stadium or laser targeting cancer or ...

2.20 Optional: FBD and statics "review" problems from Homework 15.

The following problems require an understanding of dot-products, force, and ***free-body diagrams***.

Hw 15.1	Hw 15.8	Hw 15.9	Hw 15.10	Hw 15.12(a)

[2]Hint: For Euclidean geometry, use the law of cosines and SohCahToa. For vectors, see Homework 1.35.

[3]Hint: See Homework 1.35 or Section 3.3. Introduce whatever **identifiers** facilitate your work.
Although nonlinear algebraic equations are usually solved with a computer, these can also be solved "by-hand".
Solution at www.MotionGenesis.com. Alternately, www.WolframAlpha.com solves sets of nonlinear equations, e.g., type
Solve `x^2 + (-20+z)^2 + (-8+y)^2 = 225,    x^2 + z^2 + (-8+y)^2 = 169,    z^2 + (-15+x)^2 + (-8+y)^2 = 121`