Chapter 1

Math review

Courtesy NASA

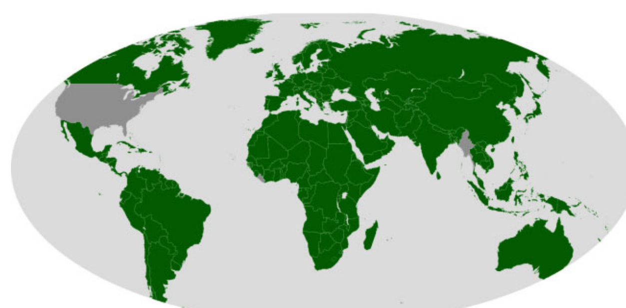
Math is a foundation for science, medicine, engineering, construction, and business. Math provides ***concepts*** (pictures, words, ideas), ***calculations*** (mathematical operations, symbols, equations, definitions), and ***context*** (situations in which the concepts and calculations are relevant and useful). More generally, math is a language and set of rules that helps us count, quantify, calculate, manipulate, relate, define, extrapolate, and abstract "stuff".[1] Advances in math depend on pictures[2] words, symbols, equations, and precise ***definitions***.[3] For example, consider the following ***definition*** of π.

Object	Example	Approximate age of human comprehension
Picture		Toddlers
Spoken word	"circle"	Pre-school
Written word	"circle", "diameter", "circumference"	Elementary school
Symbol	d for diameter, c for circumference	Middle school
Equation	$c \;=\; \pi\, d$	Middle/high school
Definition	$\pi \;\triangleq\; \frac{c}{d}$	($\triangleq$ means "***defined as***") University

1.1 Unit systems - SI and U.S.

Units quantify the measurement of "stuff". The ***SI*** system was first adopted by France on December 10, 1799 and is now used in all countries other than Liberia, Myanmar, and the United States.

The ***SI*** (metric) system uses a base-10 number system and decimals (not fractions) and has measures for length, mass, force, temperature, time, etc.



Countries using SI units (green) vs. U.S. units (grey).

NIST (National Institute of Standards & Technology) defines physical constants and conversion factors (e.g., conversion from U.S. to SI units).

Length	$1 \text{ inch} \;\triangleq\; 2.54 \text{ cm}$		
Mass	$1 \text{ lbm} \;\approx\; 0.45359237 \text{ kg}$	$1 \text{ slug} \;\approx\; g_{\text{US}} \text{ lbm}$	$g_{\text{US}} \;\approx\; 32.17404855643044$
Force	$1 \text{ Newton} \;\triangleq\; 1 \,\frac{\text{kg m}}{\text{s}^2}$	$1 \text{ lbf} \;\triangleq\; 1 \,\frac{\text{slug ft}}{\text{s}^2}$	$1 \text{ lbf} \;\triangleq\; g_{\text{US}} \,\frac{\text{lbm ft}}{\text{s}^2}$

Inaccurate unit conversions have caused ***many*** failures. In 1999, NASA lost a $125,000,000 Mars orbiter because one engineering team used SI units while another used U.S. units. In 1983, an Air Canada

[1]For example, the "idea" of ***value*** (answering **"how much something is worth"**) is quantified through money.

[2]***Art*** is ***not*** reserved for the sophisticated and educated with knowledge and historical context for art. Appreciation for shapes, colors, and emotional expression in art is available to humans on a basic (primitive/subconscious) level.

[3]Kurt Godel (1906-1978) demonstrated that any reasonably powerful mathematical system contains seemingly true statements that cannot be proven. Certain concepts are difficult to precisely define. For example, physicists call mass a "***fundamental quantity***" because it eludes mathematical definition (and humans have an inherent sense of mass).

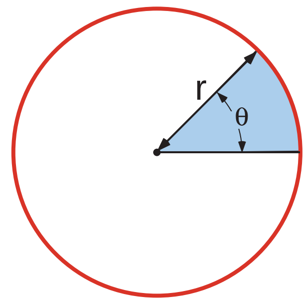
Boeing 767 ran out of fuel mid-flight because of a kg to lbm unit conversion.[4]

1.2 Geometry: Ancient Euclid and modern vectors

Geometry is the study of figures (e.g., lines, curves, surfaces, solids) and their properties (e.g., distance, area, and volume). Geometry plays a central role in construction, farming, engineering, medicine, science, etc.

Many students spend 2^+ years learning ancient ($\approx$300 BC) 2D Euclidean geometry and trigonometry (trigonometry translates to "triangle measurement"). The invention of ***vectors*** (Gibbs $\approx$ 1900 AD) and its easy-to-use vector addition, dot-products, and cross-products have **greatly simplified** 2D and 3D geometry. Unfortunately, few instructors teach geometry or trigonometry with vectors.

1.3 Circles and their properties



The ratio of ***any*** circle's ***circumference*** to its ***diameter*** is the number[a]

$$\pi \;=\; 3.14159265358979323846264338327950288419716939937510582\ldots$$

$\boldsymbol{\pi}$ is called an "***irrational number***" because it is not a whole number or fraction, nor does it terminate or repeat. It is chaotic, disorderly, and has no discernible pattern (π has been memorized to $67,890^+$ digits).

The ***arc-length*** of a portion of the circle's periphery and the ***area*** of a wedge of the circle can be calculated in terms of the circle's ***radius*** r and the ***angle*** θ as shown right.[6]

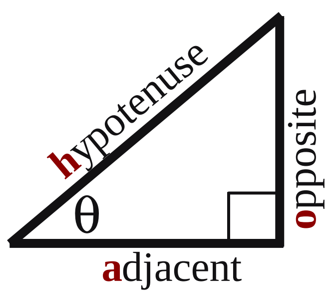
Arc-length $= \theta\, r$	Area of wedge $= \frac{\theta}{2}\, r^2$
Circumference $= 2\,\pi\, r$	Area of circle $= \pi\, r^2$

[a] The symbol $\boldsymbol{\pi}$ was popularized by Euler circa 1750, but the value $\pi \approx 3.14$ was known in Egypt circa 3000 BC.[5]

1.4 Triangles and ratios of their sides (sine, cosine, tangent)

A triangle ("three angles") is a 3-sided planar geometric shape widely used in construction, engineering, and science.

<u>S</u>oh<u>C</u>ah<u>T</u>oa is a ***mnemonic*** for memorizing the definitions of ***<u>S</u>ine***, ***<u>C</u>osine***, and ***<u>T</u>angent*** (ratios of various sides of a right triangle).



$$\boxed{\begin{aligned} \sin(\theta) &\triangleq \frac{\text{opposite}}{\text{hypotenuse}} \\ \cos(\theta) &\triangleq \frac{\text{adjacent}}{\text{hypotenuse}} \\ \tan(\theta) &\triangleq \frac{\textit{opposite}}{\textit{adjacent}} = \frac{\sin(\theta)}{\cos(\theta)} \end{aligned}} \tag{1}$$

The ***Pythagorean theorem*** in equation (2) relates lengths of sides of a right triangle. Combining the definitions of $\sin(\theta)$ and $\cos(\theta)$ with the Pythagorean theorem gives the second relationship to the right.

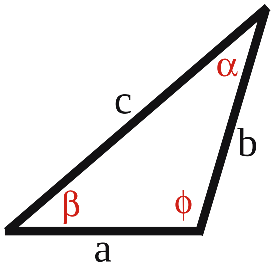
$$\boxed{\begin{aligned} &\text{hypotenuse}^2 = \text{adjacent}^2 + \text{opposite}^2 \\ &\sin^2(\theta) + \cos^2(\theta) \underset{(1)}{=} 1 \end{aligned}} \tag{2}$$

Note: Numbers under = refer to equation numbers, e.g., (1) under = means "refers to equation (1)".

[4] Ironically, Thomas Jefferson helped United States become the first country (in 1792) to use a monetary system with decimals and a base-10 number system. The historical origin of U.S. units trace to 2575 B.C. and through ancient Egypt, Greece, and Rome. The ***inch*** approximates the width of a man's thumb. The ***foot*** approximates a foot <u>with</u> shoe and was somewhat standardized in England to King Henry I. The ***mile*** "mille passus" is 1000 paces (2 steps) of a Roman soldier. An Australian study found that switching from British units to metric units freed $\frac{1}{2}$-year in science education. U.S. lawmakers have consistently failed to legislate changes in federal systems, e.g., road signs, NASA, DOD, and NSF.

[5] An ***angle*** involves two lines (or vectors) and is measured in radians or degrees. A radian is the ratio of the arc-length of a circle to its radius. A degree is an archaic unit of angle measurement based on the ancient Babylonian year which had 360 days (12 months $*$ 30 days). Each degree represents one day of Earth's travel about the sun and the degree symbol's circular appearance $^\circ$ is a reminder that 360° measures the Earth's quasi-circular travel around the sun.

1.4.1 Properties of sine and cosine and useful trigonometric formulas



Law of cosines	Euclid of Alexandria Egypt, 300 BC
Law of sines	Ptolemy of Alexandria Egypt, 100 AD
Addition formula for sine	Ptolemy of Alexandria Egypt, 100 AD

$$c^2 = a^2 + b^2 - 2\,a\,b\cos(\phi) \qquad \textbf{Law of cosines} \tag{3}$$

$$\frac{\sin(\alpha)}{a} = \frac{\sin(\beta)}{b} = \frac{\sin(\phi)}{c} \qquad \textbf{Law of sines} \tag{4}$$

$$\sin(-\alpha) = -\sin(\alpha) \tag{5}$$

$$\cos(-\alpha) = \cos(\alpha) \tag{6}$$

$$\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \sin(\beta)\cos(\alpha) \qquad \textbf{Addition formula for sine} \tag{7}$$

$$\cos(\alpha+\beta) \underset{(7)}{=} \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta) \qquad \textbf{Addition formula for cosine} \tag{8}$$

$$\sin(x) = \sin(x + 2\,\pi\,n) \qquad \text{n} = 1, 2, 3, \ldots \qquad \text{Sine is periodic} \tag{9}$$

$$-\sin(x) \underset{(7)}{=} \sin(x \pm \pi\,n) \qquad \text{n} = 1, 3, 5, \ldots \tag{10}$$

$$\cos(x) = \cos(x + 2\,\pi\,n) \qquad \text{n} = 1, 2, 3, \ldots \qquad \text{Cosine is periodic} \tag{11}$$

$$-\cos(x) \underset{(8)}{=} \cos(x \pm \pi\,n) \qquad \text{n} = 1, 3, 5, \ldots \tag{12}$$

$$\sin(x) \underset{(8)}{=} \cos(x - \frac{\pi}{2}) = \cos(-x + \frac{\pi}{2}) \tag{13}$$

$$\cos(x) \underset{(7)}{=} \sin(x + \frac{\pi}{2}) = \sin(-x + \frac{\pi}{2}) \tag{14}$$

$$\sin(x) \underset{(7)}{=} 2\,\sin(\frac{x}{2})\,\cos(\frac{x}{2}) \qquad \textbf{or} \qquad \sin(2\,x) = 2\,\sin(x)\,\cos(x) \tag{15}$$

$$\sin^2(x) \underset{(8)}{=} \frac{1 - \cos(2\,x)}{2} \qquad\qquad \cos^2(x) \underset{(8)}{=} \frac{1 + \cos(2\,x)}{2} \tag{16}$$

$$\sin^2(\frac{x}{2}) \underset{(16)}{=} \frac{1 - \cos(x)}{2} \qquad \textbf{or} \qquad \cos(x) = 1 - 2\sin^2(\frac{x}{2}) \tag{17}$$

$$\cos^2(\frac{x}{2}) \underset{(16)}{=} \frac{1 + \cos(x)}{2} \qquad \textbf{or} \qquad \cos(x) = 2\cos^2(\frac{x}{2}) - 1 \tag{18}$$

$$\cos(b) - \cos(a) \underset{(7)}{=} 2\,\sin\left(\frac{a+b}{2}\right)\,\sin\left(\frac{a-b}{2}\right) \qquad \text{Useful for beat phenomenon analysis} \tag{19}$$

$$\cos(\omega_2\,t + \phi_2) - \cos(\omega_1\,t + \phi_1) \underset{(19)}{=} 2\,\sin\left[(\frac{\omega_1+\omega_2}{2})\,t + \frac{\phi_1+\phi_2}{2}\right]\,\sin\left[(\frac{\omega_1-\omega_2}{2})\,t + \frac{\phi_1-\phi_2}{2}\right] \tag{20}$$

1.4.2 Sine and cosine as functions (Euler, circa 1730)

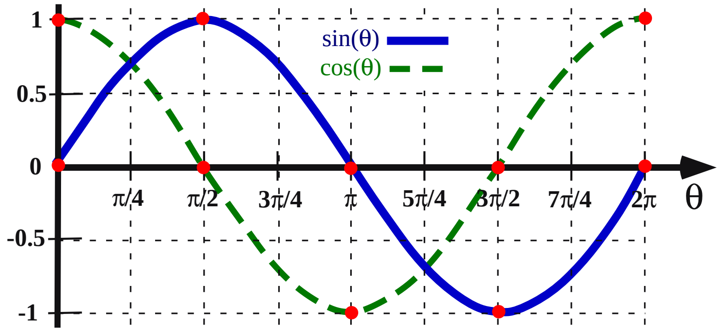
Euler's interpretation of ***cosine*** and ***sine*** as ***functions*** (not just ratios of sides of a triangle) was a major advance for trigonometry and functions.[6]

$$\boxed{\cos(\theta) \;\triangleq\; \frac{\textbf{\textit{a}}\text{djacent}}{\textbf{\textit{h}}\text{ypotenuse}}}$$

Cosine function

$$\boxed{\sin(\theta) \;\triangleq\; \frac{\textbf{\textit{o}}\text{pposite}}{\textbf{\textit{h}}\text{ypotenuse}}}$$

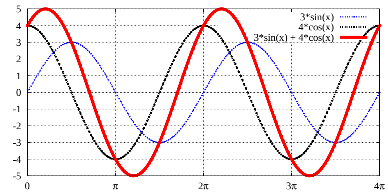
Sine function



1.4.3 The amplitude-phase formulas for sine and cosine

Two trigonometric identities that are particularly helpful in dynamic systems are the ***amplitude/phase formulas for sine and cosine.***[a]

[a] These amplitude-phase formulas are used extensively in vibration analysis. These formulas use `atan2` because A and B may be **positive**, **negative**, or **zero**.



$$\boxed{A\sin(x) + B\cos(x) \;=\; C\sin(x+\phi_s) \quad \texttt{where}\;\; C \;=\; {}^{+}\!\sqrt{A^2+B^2} \quad \texttt{and}\;\; \phi_s \;=\; \texttt{atan2}(\,B, A)} \tag{21}$$

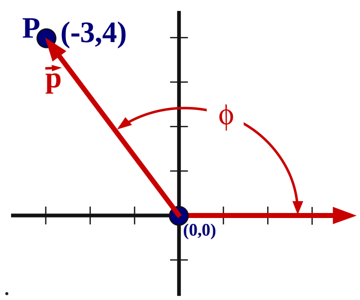
$$\boxed{A\sin(x) + B\cos(x) \;=\; C\cos(x+\phi_c) \quad \texttt{where}\;\; C \;=\; {}^{+}\!\sqrt{A^2+B^2} \quad \texttt{and}\;\; \phi_c \;=\; \texttt{atan2}(-A, B)} \tag{22}$$

1.4.4 The function atan2(y, x)

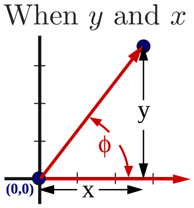
The ***atan2*** function is named because: it is similar to the atan (arc-tangent) function; it takes two arguments; it calculates an angle ϕ with range $-\pi < \phi \le \pi$ (2 times atan function's range of $\frac{-\pi}{2} < \phi \le \frac{\pi}{2}$).

To determine the angle $\phi = \texttt{atan2}(\,y,\,x\,)$

- Draw horizontal and vertical axes as shown right.
- Draw a point P located at the designated y and x values.
  Example: For $\texttt{atan2}(4, -3)$, draw point P at $y = 4$ and $x = -3$.
- Draw a vector $\vec{\mathbf{p}}$ from $(0,0)$ to point P.
- Draw angle ϕ from the ^{+}x-axis to $\vec{\mathbf{p}}$, with $^{+}$counter-clockwise sense.
- Using trigonometry, calculate the value of ϕ, e.g., $\phi \;=\; {}^{+}2.21$ rads.
- Alternately, calculate $\texttt{atan2}(\,y,\,x\,)$ with MATLAB®, **M**otion**G**enesis, Java, C, C^{++}, ...



The function $\phi = \texttt{atan2}(\,y,\,x\,)$ returns an angle that satisfies both $\sin(\phi)$ and $\cos(\phi)$ as shown below. When y and x are continuous, ordinary or partial derivatives of `atan2` can be calculated (proof in Section 1.6.10).



$$\left.\begin{aligned} \sin(\phi) \;&=\; \frac{y}{{}^{+}\!\sqrt{x^2 \,+\, y^2}} \\ \cos(\phi) \;&=\; \frac{x}{{}^{+}\!\sqrt{x^2 \,+\, y^2}} \end{aligned}\right\} \;\Rightarrow\; \begin{aligned} &\boxed{\phi \;=\; \texttt{atan2}(\,y,\,x\,)} \quad -\pi < \phi \le \pi \\ &\boxed{\dot{\phi} \;=\; \frac{x\,\dot{y} \,-\, y\,\dot{x}}{x^2 \,+\, y^2}} \quad \boxed{\frac{\partial\theta}{\partial s} \;=\; \frac{x\,\frac{\partial y}{\partial s} \,-\, y\,\frac{\partial x}{\partial s}}{x^2 \,+\, y^2}} \end{aligned} \tag{23}$$

[6] The Babylonians used properties of right triangles for thousands of years before their proofs by Pythagoras of Samos [≈500 BC]. The definitions of ***sine***, ***cosine***, and ***tangent*** as ratios of sides of a right triangle predate 140 BC when the Greek Hipparchus made sine, cosine, and tangent tables. Euler's interpretation of sine, cosine, and tangent as ***functions*** was a breakthrough for math. Gibb's invention of vectors (≈1900 AD) significantly simplified 3D geometry and trigonometry and proofs of ***law of cosines*** (Homework 3.7), ***law of sines*** (Homework 3.8), and ***sine addition formula*** (Section 3.4), from which other trigonometric formulas are derived [***cosine addition formula*** (Homework 1.9), ***half-angle formulas***, ***double-angle formulas***, etc.]. The trigonometric identities to prove equation (19) include $\sin(a+b) \;=\; \sin(a)\cos(b) + \cos(a)\sin(b)$, $\sin^2(x) \,+\, \cos^2(x) = 1$, and $\cos^2(\frac{x}{2}) \;=\; \frac{1+\,\cos(x)}{2}$.

1.5 Types of scalars: Variable, Specified, Constant

- An ***independent variable*** is a quantity that varies independently, i.e., it does not depend on other variables. Many dynamic systems have one independent variable, namely ***time*** t.
- A ***dependent variable*** is a quantity whose value depends on the independent variable and its dependence is considered to be **unknown**, e.g., governed by an algebraic or differential equation.
- A ***specified variable*** is a quantity that varies in a **known** way, e.g., it is ***prescribed*** as a function of constants, time, and other variables, such as $x = \sin(t)$.
- A ***constant*** is a quantity whose value does not change (a constant may be **known** or **unknown**).

1.6 Differentiation

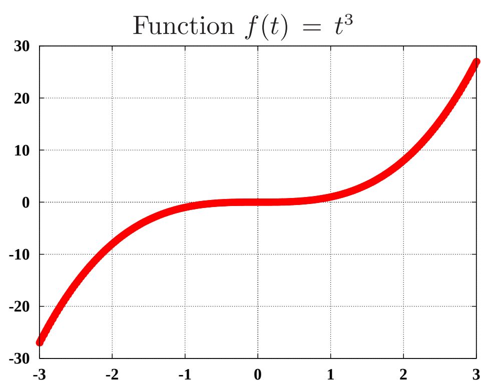
1.6.1 Definition of an ordinary derivative of a scalar function

When a function f is regarded to depend on **1** scalar variable t, it is denoted $f(t)$.

The ordinary ***1st-derivative*** of f with respect to t" is denoted in various ways as shown in equation (24).[a]

$$\boxed{f' = \dot{f} = \frac{df}{dt} = \lim_{h \to 0} \frac{f(t+h) - f(t)}{h}} \tag{24}$$

[a] The notation using a ratio (fraction) of differentials $\frac{df}{dt}$ was invented by Leibniz in 1675, the dot-notation $\dot{f}$ by Newton $\approx$ 1675, the prime notation f' by Lagrange in 1797, and the limit notation by Cauchy and Weierstrauss in 1850.



Geometrically, the 1^{st}-derivative is ***slope*** (e.g., the ***slope*** of t^3 is $3\,t^2$).

The derivative of the derivative with respect to t is called the "***2^{nd}-derivative*** of $f(t)$ with respect to t", and is denoted in various ways as shown below.

$$\boxed{f'' = \ddot{f} = \frac{d^2 f}{dt^2} \triangleq \frac{d}{dt}\left(\frac{df}{dt}\right)}$$

Geometrically, the second derivative is ***curvature***.

For example, the ***curvature*** of $f(t) = t^3$ is $\frac{d^2 f}{dt^2} = 6\,t$.

1.6.2 Definition of a partial derivative of a scalar function

When a function f depends on n independent scalar variables $t_1, \ldots, t_n$, it is denoted $f(t_1, \ldots, t_n)$.[7]
There are n quantities $\frac{\partial f}{\partial t_i}$ called "first ***partial derivatives*** of f with respect to t_i", defined as

$$\frac{\partial f}{\partial t_i} \triangleq \lim_{h \to 0} \frac{f(t_1, \ldots,\; t_i + h,\; \ldots, t_n) \;-\; f(t_1, \ldots,\; t_i,\; \ldots, t_n)}{h} \qquad (i = 1, \ldots, n) \tag{25}$$

The definition of the ***partial derivative*** of f with respect to t in equation (25) reduces to the ***ordinary derivative*** of f with respect to t when f is a function of ***one*** independent variable,[8] i.e., $\frac{df}{dt} = \frac{\partial f}{\partial t}$.

Since $\frac{\partial f}{\partial t_i}$ is defined as a limit and is not a ratio of differentials, one cannot cancel the ∂t_i in the denominator by multiplying through by ∂t_i. In other words ∂t_i is not an entity in its own right.

[7] Euler invented the function notation, e.g., $f(t)$, $f(x, y)$, circa 1730.
[8] Synonyms for ordinary (as in ordinary derivative) are "plain" and "boring" because f is a function of only ***one*** variable, whereas a "hot and spicy" partial derivative is a function of ***two or more variables***.

1.6.3 Definition of the differential of an independent variable and scalar function

The ***differentials*** of the independent scalar variables $t_1, \ldots, t_n$ are denoted $dt_1, \ldots, dt_n$ and defined as arbitrary **non-zero** scalar quantities having the same dimension (units) as $t_1, \ldots, t_n$. When a scalar variable f is regarded as a function of n independent scalar variables $t_1, \ldots, t_n$, one may define the:

differential of the function f $$\boxed{df \triangleq \frac{\partial f}{\partial t_1} * dt_1 + \frac{\partial f}{\partial t_2} * dt_2 + \ldots \frac{\partial f}{\partial t_n} * dt_n} \tag{26}$$

Leibniz regarded differentials as "infinitesimal" whereas Cauchy (tutored by Laplace and Lagrange, inventor of limits) did not.

When f is regarded as a function of ***one*** scalar variable t, equation (26) simplifies as shown below-left.

Since the ***differential*** dt is defined as a **non-zero** scalar quantity, divide by dt to produce the **ratio** of df to dt, i.e.,

$$df \underset{(26)}{=} \frac{\partial f}{\partial t} * dt \quad \Rightarrow \quad \frac{df}{dt} = \frac{\partial f}{\partial t} \tag{27}$$

Hence, when f is a function of ***one*** independent variable t, the symbol $\frac{df}{dt}$ can means both a **ratio** of the differential df to the differential dt and as a **limit** (or ***ordinary derivative***) in the sense of equation (24). Although "overloading" the symbol $\frac{df}{dt}$ may be confusing, it is useful - particular for integration.

1.6.4 Definition of the total derivative of a scalar function

At times, a function f can be regarded as either depending on $\mathbf{1}$ scalar quantity t, or regarded as a function of $\mathbf{n+1}$ scalar quantities $x_1, \ldots, x_n$ and t, where $x_1, \ldots, x_n$ are themselves functions of t. When f is regarded as a function of $x_1, \ldots, x_n$ and t, f is denoted $f(x_1(t), \ldots, x_n(t), t)$, and the ordinary derivative of f with respect to t is called the ***total derivative*** of f with respect to t and can be calculated as

$$\boxed{\begin{aligned} \frac{df}{dt} &= \frac{\partial f}{\partial x_1} * \frac{dx_1}{dt} + \frac{\partial f}{\partial x_2} * \frac{dx_2}{dt} + \ldots \frac{\partial f}{\partial x_n} * \frac{dx_n}{dt} + \frac{\partial f}{\partial t} \\ &= \frac{\partial f}{\partial x_1} \dot{x}_1 + \frac{\partial f}{\partial x_2} \dot{x}_2 + \ldots \frac{\partial f}{\partial x_n} \dot{x}_n + \frac{\partial f}{\partial t} \end{aligned}} \tag{28}$$

Note: The ***interchange property of partial and ordinary derivatives*** is in Section 26.6.

$$\boxed{\frac{\partial}{\partial q_r}\left(\frac{df}{dt}\right) = \frac{d}{dt}\left(\frac{\partial f}{\partial q_r}\right)} \qquad \boxed{\frac{\partial}{\partial \dot{x}}\left(\frac{df}{dt}\right) = \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{x}}\right) + \frac{\partial f}{\partial x}}$$

1.6.5 Short table of derivatives frequently encountered in engineering

Function and its derivative			Function and its derivative		
$F(t) = \sin(t)$	$\frac{\partial F}{\partial t} = \boxed{\cos(t)}$		$F(t) = \cos(t)$	$\frac{\partial F}{\partial t} = \boxed{-\sin(t)}$	
$F(t) = t^n$	$\frac{\partial F}{\partial t} = \boxed{n * t^{n-1}}$	n = constant	$F(t) = \tan(t)$	$\frac{\partial F}{\partial t} = \frac{1}{\cos^2(t)}$	
$F(t) = \ln(t)$	$\frac{\partial F}{\partial t} = \boxed{t^{-1} = \frac{1}{t}}$		$F(t) = e^t$	$\frac{\partial F}{\partial t} = \boxed{e^t}$	important for ODEs $e = 2.71828\ldots$
$F(t) = \int_{x=t_0}^{t} f(x)\, dx$	$\frac{\partial F}{\partial t} = f(t)$	***Fundamental Theorem of Calculus***			
$F(t) = \int_{s=g(t)}^{h(t)} f(s,t)\, ds$	$\frac{\partial F}{\partial t} = \int_{s=g(t)}^{h(t)} \frac{\partial f(s,t)}{\partial t} ds - f[s=g(t),\, t] \frac{d[g(t)]}{dt} + f[s=h(t),\, t] \frac{d[h(t)]}{dt}$				

1.6.6 Example: Partial and ordinary differentiation

Example A: Consider a function f that only depends on **1** independent variable t (time), but which is expressed in terms of dependent variables x and y (both x and y depend on t). The function f can also be ***regarded*** as a function of **3** independent scalar quantities (x, y, t).

$$f(x, y, t) \;=\; \sin(x)\, y^2 \;+\; e^{3\,t}$$

Partial derivatives of $f(x, y, t)$ with respect to x, y, or t and the ordinary (total) derivative of f are

$$\frac{\partial f}{\partial x} = \cos(x)\, y^2 \qquad \frac{\partial f}{\partial y} = 2\,\sin(x)\, y \qquad \frac{\partial f}{\partial t} = 3\, e^{3\,t} \qquad \frac{df}{dt} = \cos(x)\, \dot{x}\, y^2 \;+\; 2\,\sin(x)\, y\, \dot{y} \;+\; 3\, e^{3\,t}$$

Example B: Consider a function g that depends on **1** independent variable t (time), but which is expressed in terms of a dependent variable x and its ordinary time-derivative $\dot{x}$. The function g can also be ***regarded*** as a function of **3** independent scalars $(x, \dot{x}, t)$ as

$$g(x, \dot{x}, t) \;=\; \sin(x)\, \dot{x}^2 \;+\; e^{3\,t}$$

Partial derivatives of $g(x, \dot{x}, t)$ with respect to x, $\dot{x}$, or t and the ordinary (total) derivative of g are

$$\frac{\partial g}{\partial x} = \cos(x)\, \dot{x}^2 \qquad \frac{\partial g}{\partial \dot{x}} = 2\,\sin(x)\, \dot{x} \qquad \frac{\partial g}{\partial t} = 3\, e^{3\,t} \qquad \frac{dg}{dt} = \cos(x)\, \dot{x}^3 \;+\; 2\,\sin(x)\, \dot{x}\, \ddot{x} \;+\; 3\, e^{3\,t}$$

1.6.7 Good product rule for differentiation (for scalars, $\vec{\mathbf{v}}$ectors, matrices, ...)

Good product rule:
$$\boxed{\frac{\partial(u * v * w)}{\partial t} \;=\; \frac{\partial u}{\partial t} * v * w \;+\; u * \frac{\partial v}{\partial t} * w \;+\; u * v * \frac{\partial w}{\partial t}} \qquad (29)$$

Example:
$$\frac{\partial\big[\, t^2 * \sin(t) * e^t \,\big]}{\partial t} \;=\; 2\,t\, \sin(t)\, e^t \;+\; t^2\, \cos(t)\, e^t \;+\; t^2\, \sin(t)\, e^t$$

Unfortunately, many calculus books use the "**bad**" ***product rule for differentiation*** $\frac{d\,(u * v)}{dt} = \mathrm{u} * \frac{dv}{dt} + \mathrm{v} * \frac{du}{dt}$, which fails if u and v are $\vec{\mathbf{v}}$ectors or matrices, and is inefficient for differentiating 3^+ scalars (e.g., $u * v * w$).

1.6.8 Quotient rule for derivatives: Use exponents and the product rule

Since the quotient $\frac{u}{v}$ is equivalent to $u\, v^{-1}$, the derivative of $\frac{u}{v}$ with respect to t can be implemented with the ***product rule*** and exponents (without memorizing special ***quotient-rule*** formulas).

$$\boxed{\frac{\partial}{\partial t}\left(\frac{u}{v}\right) \;=\; \frac{\partial u}{\partial t}\, v^{-1} \;-\; u\, v^{-2}\, \frac{\partial v}{\partial t}} \qquad (30)$$

1.6.9 Chain rule for derivatives

When the variable x depends on the variable t, the derivative of the function $f(x)$ with respect to t can be written via the ***chain rule for differentiation*** as shown in equation (31).

$$\boxed{\frac{\partial\, f(x)}{\partial t} \;=\; \frac{\partial\, f(x)}{\partial x}\, \frac{\partial x}{\partial t}} \qquad (31)$$

1.6.10 Implicit differentiation: A useful tool for calculating derivatives

Example: In general, it is difficult to solve the nonlinear equation below to find y explicitly in terms of t. However, ***implicit differentiation*** calculates $\frac{dy}{dt}$ **without** first solving for y, e.g.,

$$y^2 \;+\; \sin(y) \;=\; \cos(t) \qquad \Rightarrow \qquad 2\,y\, \frac{dy}{dt} \;+\; \cos(y)\, \frac{dy}{dt} \;=\; \text{-}\sin(t) \qquad \Rightarrow \qquad \frac{dy}{dt} \;=\; \frac{\text{-}\sin(t)}{2\,y + \cos(y)}$$

Example: The use of implicit differentiation in conjunction with ***natural logarithms*** is useful for calculating the ordinary time-derivative of $y = c^t$ (c is a constant and t is time), as shown below.

$$y \;=\; c^t \qquad \Rightarrow \qquad \ln(y) \;=\; t\;\ln(c) \qquad \Rightarrow \qquad d\,[\ln(y)] \;=\; \ln(c)\;dt \qquad \Rightarrow \qquad \frac{1}{y}\,dy \;=\; \ln(c)\;dt$$

$$\frac{dy}{dt} \;=\; \ln(c)\;y \;=\; \ln(c)\;c^t$$

Note: When $c = e = 2.718281828$, $\frac{dy}{dt} = y$.
This plays a **central role** in solving ordinary differential equations.

Example: Implicit differentiation is useful for proof of derivative formula for `atan2` (from Section 1.4.4).

$$\begin{aligned}
\sin(\phi) &\underset{(23)}{=} \frac{y}{+\sqrt{x^2+y^2}} &\Rightarrow\quad& \left[\cos(\phi)\,\dot{\phi} \;=\; \frac{\dot{y}}{\sqrt{x^2+y^2}} - \frac{y\,(x\,\dot{x}+y\,\dot{y})}{(x^2+y^2)^{1.5}}\right][\,\cos(\phi)] \\
\cos(\phi) &\underset{(23)}{=} \frac{x}{+\sqrt{x^2+y^2}} &\Rightarrow\quad& \left[-\sin(\phi)\,\dot{\phi} \;=\; \frac{\dot{x}}{\sqrt{x^2+y^2}} - \frac{x\,(x\,\dot{x}+y\,\dot{y})}{(x^2+y^2)^{1.5}}\right][-\sin(\phi)]
\end{aligned}
\;\Bigg\}\;\underset{\substack{\text{add}\\\text{equations,}\\\text{simplify.}}}{\Rightarrow}\;\boxed{\dot{\phi} \;=\; \frac{x\,\dot{y} - y\,\dot{x}}{x^2+y^2}}$$

1.7 Integration and a short table of integrals

An ***integral*** can be regarded as either an ***anti-derivative*** or as a ***sum*** (e.g., **area under a curve**).

Function	Integral of $F(t)$	
$F(t) = t^n$	$\int F(t)\,dt = \frac{t^{n+1}}{n+1} + C$	(n is a number other than -1)
$F(t) = t^{-1}$	$\int F(t)\,dt = \ln(t) + C$	
$F(t) = e^t$	$\int F(t)\,dt = e^t + C$	
$F(t) = \sin(t)$	$\int F(t)\,dt = -\cos(t) + C$	
$F(t) = \cos(t)$	$\int F(t)\,dt = \sin(t) + C$	

The website www.WolframResearch.com is a valuable resource for calculating integrals.

History: In 1675, Leibniz invented the integral notation $\int$ (Latin abbreviation for summa - sum) and its natural extension to double and triple integrals. Newton's integral notation was so defective, it was never popular – even in England. Euler was the first to use a symbol for an integral's limits, and its modern notation, e.g., $\int_a^b x\,dx$, was invented by Fourier in 1820.

Optional: Short history of differentiation

The modern differential notation $\frac{df}{dt}$ was introduced by Gottfried Leibniz in **1675** and relates to the ratio of differentials df and dt. The dot-notation $\dot{f}$ was introduced by Newton in his "***method of fluxions***" around **1675** and relates to his idea of flux (time-rates of change) of "***fluents***" (now called ***variables***). The prime notation f' was introduced by Lagrange in 1797 in his *Théorie des fonctiones analytiques*. Lagrange called $f'(t)$ the "derived function" of $f(t)$, from which the modern term ***derivative*** comes [55, pgs. 95-97]. An important concept introduced by Euler and Lagrange was that the derivative was a ***function*** which itself could be differentiated. The limit notation was introduced by Cauchy in **1823** and refined in the 1840s and 1850s by Cauchy, Seidel, Stokes, and Weierstrauss.

Although ***Newton and Leibniz*** share the discovery of ***calculus***, their relationship was contentious - with Newton and Leibniz and their respective supporters alleging plagiarism and undermining each other's credibility. As President of the Royal Society, Newton appointed an "impartial" committee to decide whether he or Leibniz invented calculus. He wrote the committee's official published report (although not under his name) and then wrote a review (again anonymously) which appeared in the Philosophical Transactions of the Royal Society. Ironically, the introverted Newton died at 80-years old a national hero of England with a state funeral of the highest honors whereas the more sociable Leibniz's died at 70-years old, almost completely forgotten, with a funeral attended by only his secretary. Newton's daunting reputation intimidated British mathematicians. England did not produce another first-rate mathematician for over a century. Undaunted and unintimidated by their English neighbors, the rest of Europe, lead by the Bernoulli family, Leonard Euler, D'Alembert, Lagrange, Laplace, Fourier, and many others, quickly expanded analytical analysis through differential equations, the calculus of variations, etc.

The order in which certain mathematics are taught can be non-intuitive as evidenced by history. For example, derivatives are usually taught starting with limits whereas the derivative was first ***used*** (Fermat and Descartes, 1637), then ***discovered*** (Newton and Leibniz, 1669-1684), then ***explored*** and ***developed*** (Taylor, Euler, Maclaurin, Lagrange, 1755-1797) and finally ***defined*** (Cauchy and Wiestrass, 1823-1861) [30]. Similarly, the ***Pythagorean theorem*** was used for thousands of years before its proof by Pythagoras ≈ 500 BC.

1.8 Minimization and maximization (optimization)

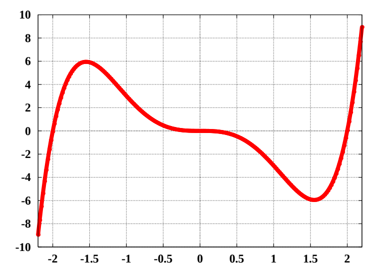
Consider a single scalar function $f(x_1, ..., x_n)$ that depends on n **independent** scalar variables $x_1, ..., x_n$ (no constraints between $x_1, ..., x_n$).

One way to find a **local** minimum or maximum of this function is by setting to 0 each partial derivative of f with respect to x_i $(i = 1, ..., n)$, i.e.,

Unconstrained optimization

$$\boxed{\frac{\partial f(x_1, ..., x_n)}{\partial x_1} = 0 \quad \dots \quad \frac{\partial f(x_1, ..., x_n)}{\partial x_n} = 0} \tag{32}$$

Example: Minimum and maximum of a function of one variable



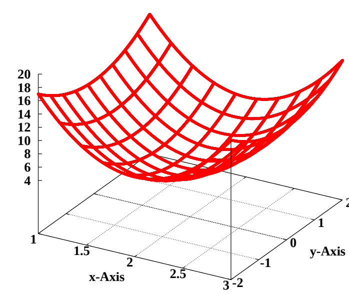
The plot to the right shows the function $f(x) = x^5 - 4\,x^3$.
In view of equation (32), local minimum and maximum are calculated via

$$\frac{\partial f(x)}{\partial x} = 5\,x^4 - 12\,x^2 = 0$$

whose multiple solutions are $x \approx \pm 1.55$ and $x = 0$.

Local minimum: $f(x = 1.55) \approx -5.95$	Local maximum: $f(x = -1.55) \approx 5.95$	Saddle point: $f(x{=}0) = 0$
Global minimum: $f(x \Rightarrow -\infty) \Rightarrow -\infty$	Global maximum: $f(x \Rightarrow +\infty) \Rightarrow +\infty$	

Example: Minimum of a function of two variables



The plot to the right shows the function $f(x, y) = (3\,x - 6)^2 + y^2 + 4$.
In view of equation (32), local minimum or maximum are calculated via

$$\frac{\partial f(x,y)}{\partial x} = 6\,(3\,x - 6) = 0 \qquad\qquad \frac{\partial f(x,y)}{\partial y} = 2\,y = 0$$

whose only solution is $x = 2$ and $y = 0$.
The function's local (and global) minimum is $f(x = 2,\ y = 0) = 4$.

Minimization and maximization with constraints (constrained optimization)

Consider a single scalar function $f(x_1, ..., x_n)$ that depends on n scalar variables $x_1, ..., x_n$ and "***equality constraints***" of the form $g(x_1, ..., x_n) = 0$ that interrelate $x_1, ..., x_n$. One problem is to:

Find a local minimum or maximum of $f(x_1, ..., x_n)$
Subject to m equality constraints $g_1(x_1, ..., x_n) = 0 \quad \dots \quad g_m(x_1, ..., x_n) = 0$

To transform (trick) this "***constrained optimization***" problem in $f(x_1, ..., x_n)$ and $g(x_1, ..., x_n)$ into an easier "***unconstrained optimization***" problem in a single scalar function L, introduce m ***Lagrange multipliers*** λ_j (one for each equality constraint) and define a single scalar function L (called a ***Lagrangian***) that is regarded as depending on $n + m$ **independent** scalar variables $x_1, ..., x_n,\ \lambda_1, ..., \lambda_m$ as

$$\boxed{L(\,x_1, ..., x_n,\ \ \lambda_1, ..., \lambda_m\,) \ \triangleq\ f(x_1, ..., x_n) \ + \ \sum_{j=1}^{m} \lambda_j\ g_j(x_1, ..., x_n)} \tag{33}$$

Using equation (32), the solution is found by solving equation (34), whose first n equations depend on both $x_1, ..., x_n$ and $\lambda_1, ..., \lambda_m$ and whose last m equations depend only on $x_1, ..., x_n$ (and guarantee the constraints are satisfied).

Constrained optimization

$$\boxed{\begin{array}{cc} \frac{\partial L}{\partial x_1} = 0 \quad \dots \quad \frac{\partial L}{\partial x_n} = 0 \\ \frac{\partial L}{\partial \lambda_1} = g_1(\mathbf{x}) = 0 \quad \dots \quad \frac{\partial L}{\partial \lambda_m} = g_m(\mathbf{x}) = 0 \end{array}} \tag{34}$$

Solving the $n+m$ equations in (34) for the $n+m$ unknowns $x_1, ..., x_n$ and $\lambda_1, ..., \lambda_m$ provides local optimal solutions (maxima or minima) of $f(x_1, ..., x_n)$ subject to the m constraints. In general, equations (34) are **nonlinear**. One way to solve this set of equations is:

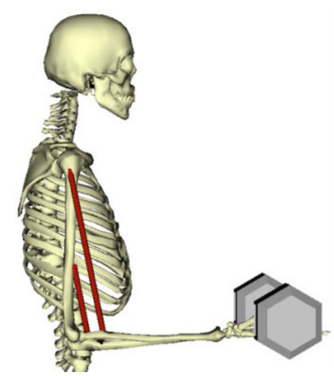
- Using intuition, insight, or **guessing**, provide an initial numerical value for $x_1, ..., x_n$.
- Substitute these numerical guesses into the first n of equations (34) to form n **linear** equations that provide guidance for initial numerical values of the m Lagrange multipliers $\lambda_1, ..., \lambda_m$.
- Solve these **n** linear equations for the **m** initial guesses of $\lambda_1, ..., \lambda_m$, [e.g., if $n = m$ and non-singular equations, use ***Gauss elimination***, alternately use ***singular value decomposition (SVD)***].
- Iterate towards the solution using standard computational techniques (e.g., gradient methods).

Example: Minimum of a function of two variables with a constraint

The figure to the right shows an athlete curling a dumbbell. For given data, the equation governing biceps and brachialis muscles tensions is

$$0.05\, T_{\text{Biceps}} \;+\; 0.02\, T_{\text{Brachialis}} \;=\; 34.6$$

Since this **one** linear equation has **two** unknowns, there is insufficient information to solve for the muscle tensions. One way to find a solution is to postulate that a certain "***cost function***" must be optimized. There are many choices of cost functions and debates on their merits. One choice is a cost-function that minimizes the sum-square of muscle tensions, i.e.,



OpenSim muscular-skeletal model
Courtesy Apoorva Rajagopal

$$\text{Minimize:} \quad f(\,T_{\text{Biceps}},\, T_{\text{Brachialis}}\,) \;=\; T^2_{\text{Biceps}} \;+\; T^2_{\text{Brachialis}}$$

In view of equation (33), the Lagrangian is

$$L(\,T_{\text{Biceps}},\, T_{\text{Brachialis}},\, \lambda_1\,) \underset{(33)}{=} T^2_{\text{Biceps}} \;+\; T^2_{\text{Brachialis}} \;+\; \lambda_1\,(\,0.05\, T_{\text{Biceps}} \;+\; 0.02\, T_{\text{Brachialis}} \;-\; 34.6\,)$$

Performing the calculations in equation (34) yields the three equations shown below-left. These equations happen to be linear, so they can be cast into matrix form, whose solution is shown below-right.

$$\frac{\partial\, L}{\partial\, T_{\text{Biceps}}} = 2\, T_{\text{Biceps}} + 0.05\, \lambda_1 = 0$$
$$\frac{\partial\, L}{\partial\, T_{\text{Brachialis}}} = 2\, T_{\text{Brachialis}} + 0.02\, \lambda_1 = 0$$
$$\frac{\partial\, L}{\partial\, \lambda_1} = 0.05\, T_{\text{Biceps}} + 0.02\, T_{\text{Brachialis}} - 34.6 = 0$$

$$\begin{bmatrix} 2 & 0 & 0.05 \\ 0 & 2 & 0.02 \\ 0.05 & 0.02 & 0 \end{bmatrix} \begin{bmatrix} T_{\text{Biceps}} \\ T_{\text{Brachialis}} \\ \lambda_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 34.6 \end{bmatrix}$$

$$T_{\text{Biceps}} = 596.6 \qquad T_{\text{Brachialis}} = 238.6 \qquad \lambda_1 \;=\; \text{-}23862$$

Note: Since the choice of cost function can be arbitrary, it is advisable to check the muscle tensions to ensure they are plausible (e.g., muscle tension is positive since muscles pull not push) and are less than physiological maximums.

Note: An easier (albeit less extensible) solution process is to use the original equation to find:

$$T_{\text{Brachialis}} \;=\; \frac{34.6}{0.02} \;-\; \frac{0.05}{0.02}\, T_{\text{Biceps}} \;=\; 1730 \;-\; 2.5\, T_{\text{Biceps}}$$

Substituting for $T_{\text{Brachialis}}$ in the cost function f yields a minimization problem in **one** variable whose solution is found by setting the ordinary derivative of f with respect to T_{Biceps} to 0. This produces the same solution.

$$\text{Minimize:} \quad f(\,T_{\text{Biceps}}\,) \;=\; T^2_{\text{Biceps}} \;+\; (1730 \;-\; 2.5\, T_{\text{Biceps}})^2 \qquad \Rightarrow \qquad \frac{d\, f}{d\, T_{\text{Biceps}}} \;=\; 0 \qquad \Rightarrow \qquad T_{\text{Biceps}} \;=\; 596.6$$

1.9 Solutions of *polynomial* equations (roots)

Polynomial equations are a special class of nonlinear algebraic equations. A special polynomial equation is the ***quadratic equation***, which is a polynomial equation of degree **2**. Shown below is a quadratic equation in x and its **2** ***roots*** (solutions).

Quadratic equation

$$\boxed{a\,x^2 \;+\; b\,x \;+\; c \;=\; 0}$$

Solution to quadratic equation

$$\boxed{x \;=\; \frac{-b \;+\; \sqrt{b^2 \;-\; 4\,a\,c}}{2\,a} \qquad \text{and} \qquad x \;=\; \frac{-b \;-\; \sqrt{b^2 \;-\; 4\,a\,c}}{2\,a}}$$

Two other polynomial equations with "closed-form solutions" are the ***cubic*** and ***quartic*** equations

$$x^3 \;+\; c_2\,x^2 \;+\; c_1\,x \;+\; c_0 \;=\; 0 \qquad \text{and} \qquad x^4 \;+\; c_3\,x^3 \;+\; c_2\,x^2 \;+\; c_1\,x \;+\; c_0 \;=\; 0$$

The ***Fundamental Theorem of Algebra***, states that any polynomial of degree n with complex coefficients has n complex roots.[9] In 1824, Abel proved that no general closed-form solution for 5^{th}-order (or higher) polynomials exist. Numerical methods are useful for calculating roots of polynomials of any order.

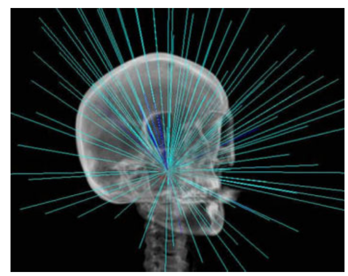
1.10 Continuous solutions of *nonlinear* algebraic equations

One way to find a continuous solution for x in the range $0 \le t \le 8$ for

$$x^2 \;-\; \cos^2(x) \;=\; 0.3\,\sin(t)$$

is to differentiate this ***nonlinear*** equation with respect to t and then solve the derivative equation that is ***linear*** in $\dot{x}$ as

$$2\,x\,\dot{x} + 2\,\cos(x)\,\sin(x)\,\dot{x} \;=\; 0.3\,\sin(t) \qquad \Rightarrow \qquad \dot{x} \;=\; \frac{0.3\,\cos(t)}{2\,x \;+\; 2\,\cos(x)\,\sin(x)}$$



Courtesy Accuray Inc.

Solving the nonlinear equation **once** at $t = 0$ gives $x(t=0) \approx 0.74$. With this initial value for x and continuous formula for $\dot{x}$, ODE techniques can numerically integrate $\dot{x}(t)$ to solve for $x(t)$.

Math helps predicts the weather and saves lives: Weather satellite and massive hurricane

[9] The proof of the ***Fundamental Theorem of Algebra*** is difficult and was presented with various rigor between 1608 and 1981 by great mathematicians including, Rothe(1608) Girard (1629), Leibniz (1702), Bernoulli (1742), d'Alembert (1746), Euler (1749), Lagrange (1772), Laplace (1795), Gauss (1799), Argand (**1806**), Gauss (again in 1816 and 1849), Cauchy (1821), Weierstrauss (1891), Hellmuth Kneser (1940), and his son Martin Kneser (**1981**).