Show work – except for ♣ fill-in-blanks (print .pdf from www.MotionGenesis.com ⇒ Textbooks ⇒ Resources).

7.1 ♣ How many functions do you seek? (Section 7.3).

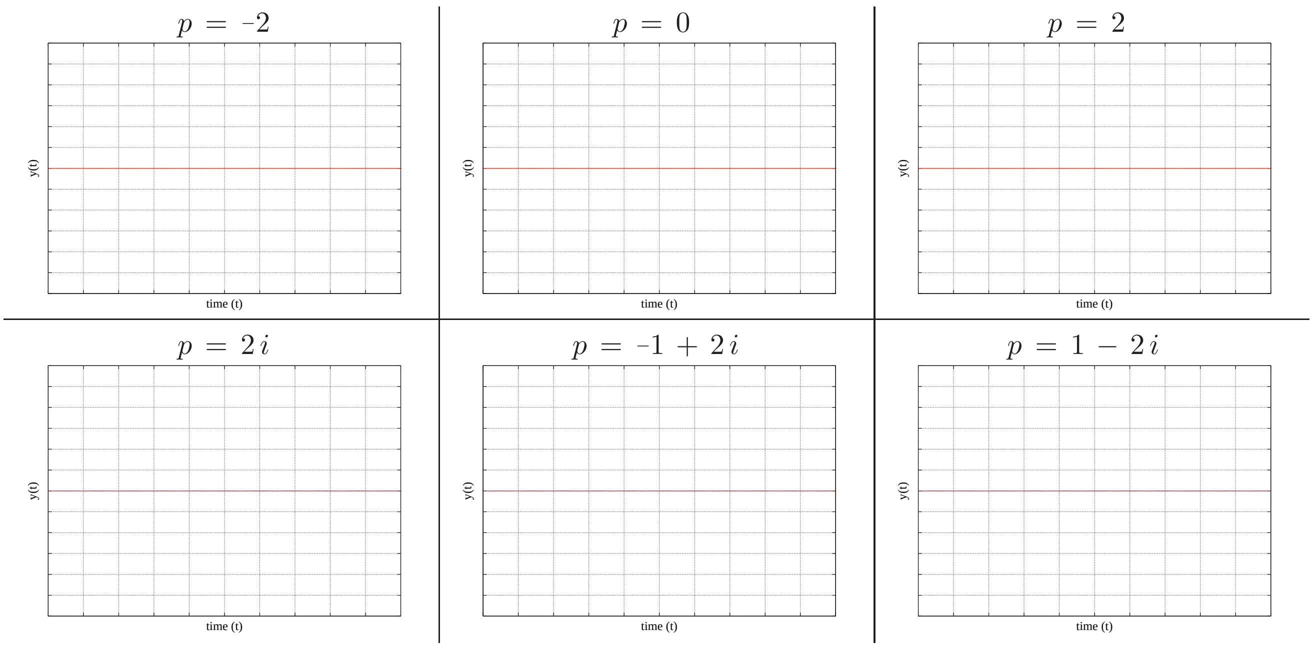
Consider a generic n^{th}-ordered linear, constant-coefficient, homogenous ODE of the form

$$\frac{d^n y}{dt^n} \;+\; \ldots \quad \frac{d^4 y}{dt^4} \;+\; \frac{d^3 y}{dt^3} \;+\; \frac{d^2 y}{dt^2} \;+\; \frac{dy}{dt} \;+\; y \;=\; 0$$

Determine how many independent functions there are in the solution for each order of ODE below.

Order of ODE	Number of independent functions in the solution (circle all that apply)								
1^{st}-order	0	1	2	3	4	5	6	7	∞	other
2^{nd}-order	0	1	2	3	4	5	6	7	∞	other
3^{rd}-order	0	1	2	3	4	5	6	7	∞	other
6^{th}-order	0	1	2	3	4	5	6	7	∞	other

7.2 ♣ Make a rough sketch ("finger graph") of the real part of $y(t) = e^{p\,t}$ for each value of p.



7.3 ♣ Response of the real part of $e^{p\,t}$ for various values of p. (Section 7.1).

Solutions to linear, constant-coefficient, ODEs involve the function $e^{p\,t}$ where p is a constant and t is the independent variable (e.g., time). Circle the most appropriate value of p for each of the following.

Solution behavior	Value of p						
Grows fastest	-3	$-1+i$	i	0	1	$2+2\,i$	3
Decays fastest	-3	$-1+i$	i	0	1	$2+2\,i$	3
Oscillates fastest (most jiggles per second)	-3	$-1+i$	i	0	1	$2+2\,i$	3
Pure oscillation (no growth or decay)	-3	$-1+i$	i	0	1	$2+2\,i$	3
Stays constant (no oscillation/growth/decay)	-3	$-1+i$	i	0	1	$2+2\,i$	3
Grows and oscillates	-3	$-1+i$	i	0	1	$2+2\,i$	3
Decays and oscillates	-3	$-1+i$	i	0	1	$2+2\,i$	3

7.4 ♣ Pole location and dynamic response. (Section 7.1).

Circle the phrase that ***best*** describes the behavior associated with the various pole locations.

(a) A pole in the right half plane is **stable/neutrally stable/unstable**

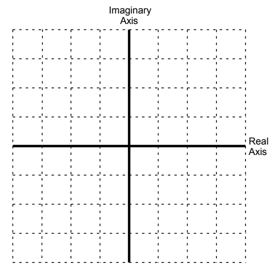
(b) A pole in the left half plane is **stable/neutrally stable/unstable**

(c) A pole on the imaginary axis is **stable/neutrally stable/unstable**

(d) As a pole moves up or down from the real axis, it is more **stable/unstable/oscillatory/damped/fast**

(e) As a pole moves to the left, it is more **stable/unstable/oscillatory/damped/fast**

(f) As a pole moves away from the origin (in any direction), it is more **stable/unstable/oscillatory/damped/fast**



7.5 ♣ The complex plane, pole locations, and root locus versus ζ. (Section 7.1 and Chapter 7).

Consider a dynamic system governed by a homogeneous, constant-coefficient, linear, 2^{nd}-order ODE.

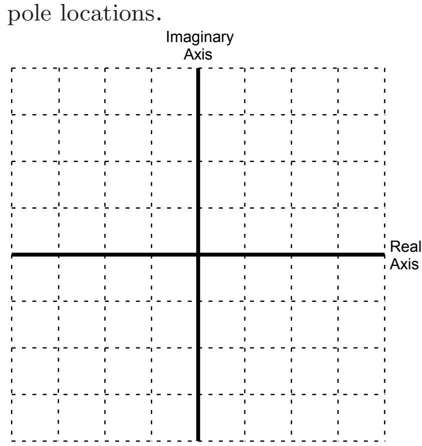
The constant ζ (called "***damping ratio***") is to-be-determined. $\quad \ddot{y} + 2\zeta\omega_n\dot{y} + \omega_n^2 y = 0$

The constant ω_n (called "***natural frequency***") is known to be 1. $\quad \ddot{y} + 2\zeta\dot{y} + y = 0$

Solutions of this ODE have the form $y(t) = e^{p\,t}$ where p is a constant (called a ***pole***).
This problem acquaints you with various values of p and their locations in the complex plane.

Tabulate the values of p_1 and p_2 for each value of damping ratio ζ.
Under "Damping", write **undamped**, **underdamped**, **critically-damped**, or **overdamped**.
Under "Stability", write **stable**, **unstable**, or **neutrally stable**.
Note: For this problem, damping and stability can be determined solely from pole locations.

ζ	Pole p_1	Pole p_2	Damping	Stability
0.0	i	$-i$		
0.2	-0.2 + 0.98i	-0.2 - 0.98i		
0.4	-0.4 + 0.92i	-0.4 - 0.92i	underdamped	
0.6	-0.6 + 0.8i	-0.6 - 0.8i		
0.8				
1.0				
1.2	−0.54	−1.9		
1.5	−0.38	−2.62		
2.0	−0.27	−3.7		stable



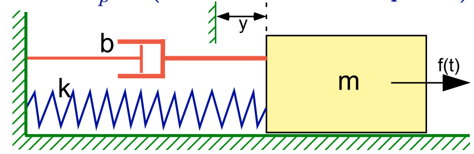
Root locus versus ζ

Complete the root locus (above right) by drawing a complex plane with a real horizontal axis and an imaginary vertical axis as shown. For each value of ζ, locate the pole p_1 in the complex plane and then connect each value of p_1 with an arrow that shows the direction of increasing ζ. Subsequently locate each value of p_2 and connect them with arrows that show increasing ζ. The resulting graph is called a ***root locus versus*** ζ for $0 \le \zeta \le 2$.

The data and root locus show the value of ζ that causes $y(t) \to 0$ most quickly is $\zeta =$ ____.
Explain your answer.

7.6 ♣ Root locus versus proportional feedback control constant k_p. (Section 7.1 and Chapter 7).

A root locus is a visual indication of how system response changes when a constant in the governing ODE is varied. Consider the system shown right whose governing ODE and ***proportional feedback control law*** $\widehat{f}(t)$ are given below.



$m = 1 \qquad b = 1 \qquad k = 1$

$$\vec{\mathbf{F}} = m\,\vec{\mathbf{a}} \quad \Rightarrow \quad m\,\ddot{y} + b\,\dot{y} + k\,y = \widehat{f}(t) \qquad \widehat{f}(t) = -k_p\,y \quad \Rightarrow \quad \ddot{y} + \dot{y} + (k_p + 1)\,y = 0$$

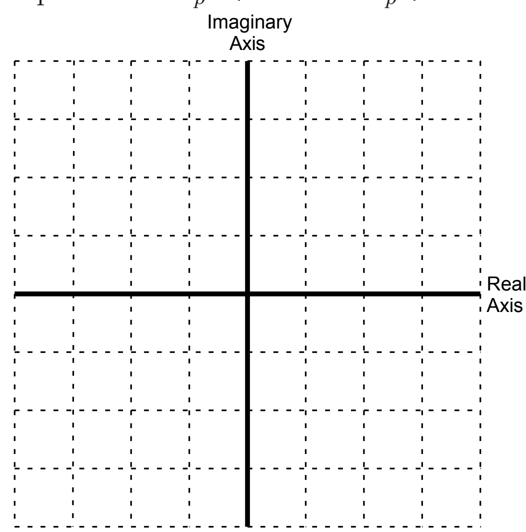
(a) Determine the range of values of the feedback control constant k_p that stabilize $y(t)$.
Result: $k_p >$ stabilizes the response

(b) Express the poles p_1 and p_2 in terms of k_p. Sketch the ***root locus versus*** k_p for $-2 \le k_p \le 2$ with arrows showing the direction of increasing k_p. Extrapolate for $k_p < -2$ and $k_p > +2$.

Result:

$p_{1,2} =$

k_p	p_1	p_2
-2.0	0.618	-1.618
-1.5	0.366	-1.366
-1.0		
-0.75		
-0.5	$-0.5 + 0.5\,i$	$-0.5 - 0.5\,i$
0.0	$-0.5 + 0.866\,i$	$-0.5 - 0.866\,i$
+0.75	$-0.5 + 1.225\,i$	$-0.5 - 1.225\,i$
+1.0	$-0.5 + 1.323\,i$	$-0.5 - 1.323\,i$
+1.5	$-0.5 + 1.5\,i$	$-0.5 - 1.5\,i$
+2.0	$-0.5 + 1.658\,i$	$-0.5 - 1.658\,i$



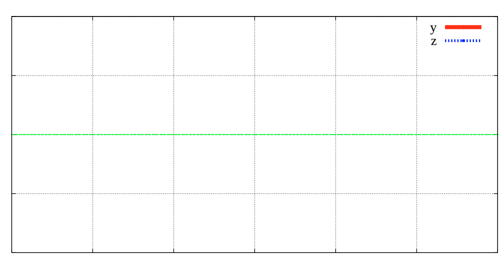
Root locus versus k_p

(c) When $k_p > 0$, increasing k_p:

decreases/has no effect on/increases magnitude of net force on the block when $y = 1$ and $\dot{y} = 0$
decreases/has no effect on/increases stability
decreases/has no effect on/increases decay ratio
decreases/has no effect on/increases damping ratio ζ
decreases/has no effect on/increases natural frequency ω_n
decreases/has no effect on/increases time it takes for $y(t) \rightarrow 0$
decreases/has no effect on/increases setting time t_{settling}
decreases/has no effect on/increases peak time t_{peak}; and seems to
decreases/has no effect on/increases rise time t_{rise}.

The data and root locus show $y(t) \rightarrow 0$ quickly with little/no oscillation when $k_p \approx$.

Make a rough sketch of $y(t)$ for $k_p = 0$ and $k_p = 2$.
Use $y(0) = 1$ and $\dot{y}(0) = 0$.



7.7 ♣ Root locus versus proportional feedback control constant k_d. (Section 7.1 and Chapter 7).

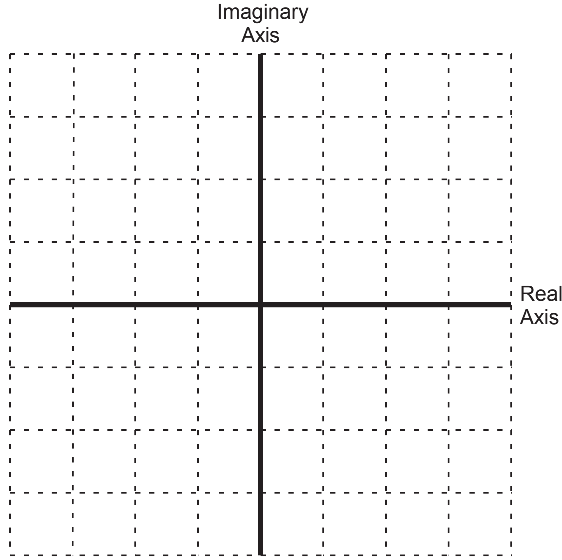
Form the characteristic polynomial equation governing the poles p of the following ODE.

ODE: $\ddot{y} \; + \; (k_d - 4)\dot{y} \; + \; 4\,y \; = \; 0$

Polynomial: $p^2 \; + \;$ ______ $\; + \;$ ___ $\; = \; 0$

Draw a root locus versus k_d.
Show the direction of increasing k_d.

k_d	Pole location p_1	Pole location p_2
0	2.0	2.0
2	$1 + 1.73\,i$	$1 - 1.73\,i$
4	$2\,i$	$-2\,i$
6	$-1 + 1.73\,i$	$-1 + 1.73\,i$
8	-2	-2
10	-0.76	-5.24



Note: The root locus may not fit completely on the graph.

(a) The range of values of k_d that stabilize $y(t)$ is ______

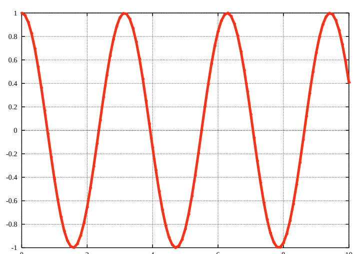
(b) The value of k_d that results in a purely oscillatory solution for $y(t)$ is $k_d =$ ___.

(c) The value of k_d that makes $y(t) \to 0$ most quickly is $k_d =$ ___.

(d) The range of values of k_d that result in some oscillatory behavior of $y(t)$ is ______.

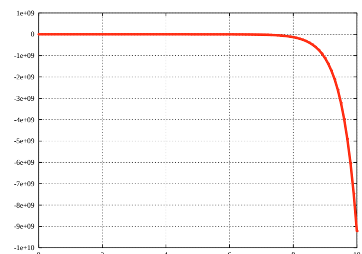
(e) The range of values of k_d that result in an overdamped solution for $y(t)$ is ______.

(f) Each graph below shows $y(t)$ versus t and uses $y(0) = 1$ and $\dot{y}(0) = 0$.
Below each graph, fill in the correct value of k_d, which is 0, 2, 4, 6, 8, or 10.

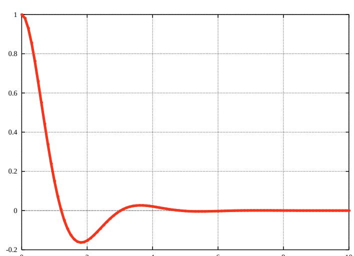


$k_d =$ ___

$k_d =$ ___

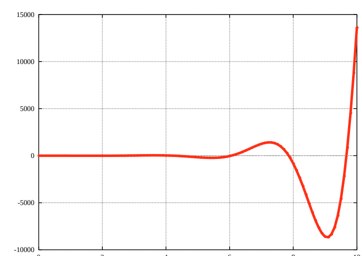


$k_d =$ ___



$k_d =$ ___

$k_d =$ ___



$k_d =$ ___

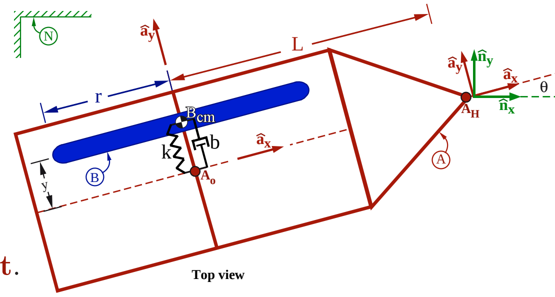
7.8 ♣ Dynamic response of a single-wheel trailer. (Section 7.1 and Chapter 7).

Although it is known that single-wheel trailers sometimes behave poorly, it is not always clear why they do so. One possibility is tire flexibility and loose wheel mounting give rise to unstable behavior.

To explore this concept, the linearized equation of motion for the system depicted to the right was formed, and after assuming a solution of the form $C\,e^{p\,t}$, the equation governing values of p was found to be

$$p^3 \;+\; 5.06\,p^2 \;+\; (50.6 + 3.4\,v)\,p \;+\; 33.7\,v \;=\; 0$$

where v (the $\hat{\mathbf{n}}_x$ measure of hitch-point velocity) is a **constant**.



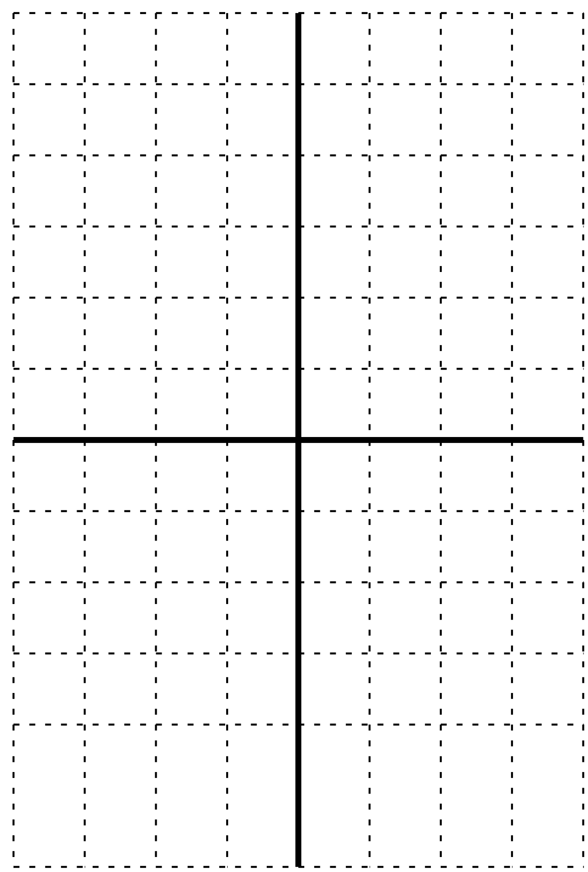
(a) Classify the equation governing p by picking the relevant qualifiers from the list.

Linear	**Homogeneous**	**Algebraic**
Nonlinear	**Inhomogeneous**	**Differential**

(b) By inspecting the coefficients of p, it is possible to know certain facts about p_1, p_2, and p_3 (the values of p that satisfy the equation), and to know that the solution $C_1\,e^{p_1\,t} + C_2\,e^{p_2\,t} + C_3\,e^{p_3\,t}$ is **stable/unstable** (circle one) when $v < 0$ because:
Explain:

Calculate the missing values of p_1, p_2, and p_3 in the table below. Sketch the root locus for $0 \le v \le 30$ and extrapolate for $v < 0$ and $v > 30$. Use arrows to show the direction of increasing v.

v	p_1	p_2, p_3
0		
5	-2.75	$-1.15 \pm 7.74\,i$
10	-4.17	$-0.45 \pm 8.98\,i$
15	-4.99	$-0.03 \pm 10.06\,i$
20	-5.55	$+0.25 \pm 11.01\,i$
25	-5.97	$+0.46 \pm 11.87\,i$

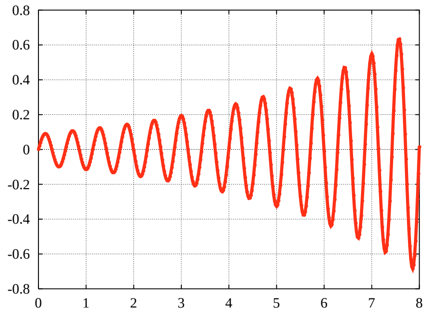


(c) Complete the following with a value of v or range of values of v from the previous table.

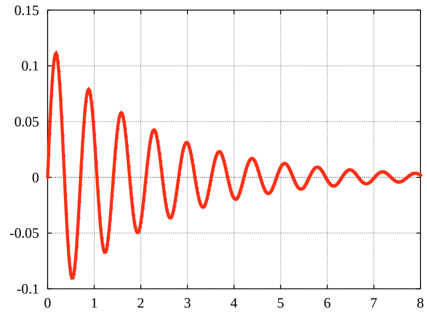
Statement	Value(s) of v
Solution is stable	
Solution is neutrally stable	$v =$ ___ and $v \approx$ ___
Solution is unstable	$v <$ ___ and $v \ge$ ___
Solution is most oscillatory	$v =$
Solution is least oscillatory	$v =$
Solution is most stable	$v =$
Solution is least stable	$v =$
Fastest speed of response	$v =$
Slowest speed of response	$v =$
$e^{p\,t} \rightarrow 0$ most quickly	$v =$

(d) Your younger sister is driving the car with attached trailer. What do you tell her about safe speeds/direction when driving with the trailer?
Result:

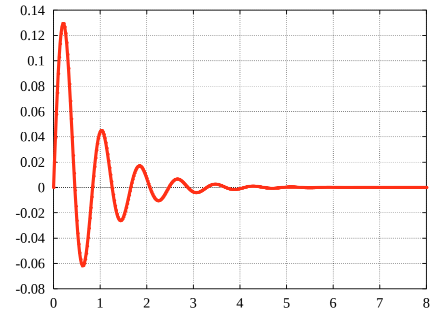
(e) For each graph shown below, circle the associated value of v.



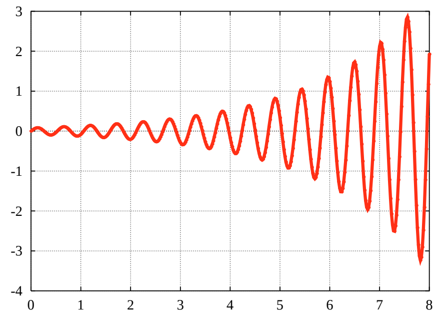
$v = \mathbf{0/5/10/15/20/25}$



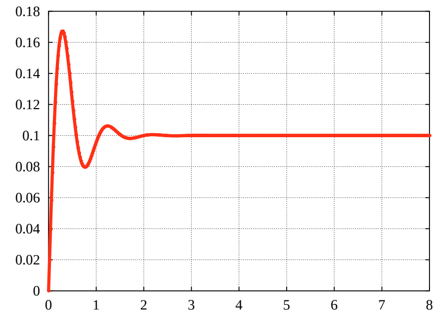
$v = \mathbf{0/5/10/15/20/25}$



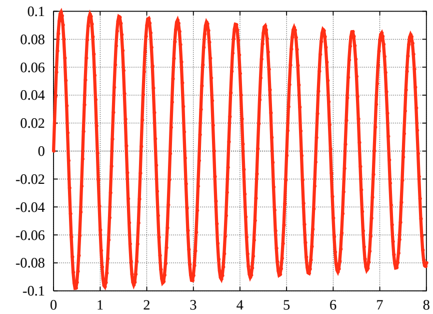
$v = \mathbf{0/5/10/15/20/25}$



$v = \mathbf{0/5/10/15/20/25}$



$v = \mathbf{0/5/10/15/20/25}$



$v = \mathbf{0/5/10/15/20/25}$

7.9 ♣ Exploring the function e^{pt}. (Section 7.1).

Consider the function $y(t) = e^{pt}$ where p is a constant and t is the independent variable time.

Value of p	For each value of p, circle ***all*** relevant words for the ***real*** part of e^{pt}					
-2	Constant	Oscillate	Decay	Grow	Stable	Neutrally stable	Unstable
-1	Constant	Oscillate	Decay	Grow	Stable	Neutrally stable	Unstable
0	Constant	Oscillate	Decay	Grow	Stable	Neutrally stable	Unstable
1	Constant	Oscillate	Decay	Grow	Stable	Neutrally stable	Unstable
2	Constant	Oscillate	Decay	Grow	Stable	Neutrally stable	Unstable
$-2\,i$	Constant	Oscillate	Decay	Grow	Stable	Neutrally stable	Unstable
$-1\,i$	Constant	Oscillate	Decay	Grow	Stable	Neutrally stable	Unstable
$0\,i$	Constant	Oscillate	Decay	Grow	Stable	Neutrally stable	Unstable
$1\,i$	Constant	Oscillate	Decay	Grow	Stable	Neutrally stable	Unstable
$2\,i$	Constant	Oscillate	Decay	Grow	Stable	Neutrally stable	Unstable
$-1 - 2\,i$	Constant	Oscillate	Decay	Grow	Stable	Neutrally stable	Unstable
$-1 + 2\,i$	Constant	Oscillate	Decay	Grow	Stable	Neutrally stable	Unstable
$1 - 2\,i$	Constant	Oscillate	Decay	Grow	Stable	Neutrally stable	Unstable
$2 + 2\,i$	Constant	Oscillate	Decay	Grow	Stable	Neutrally stable	Unstable