

Amplitude-phase. ODEs and system identification with separation of variables.

2.1 Cosine addition formula to prove formulas for  $\sin^2(x)$  and  $\cos^2(x)$ . (Section 2.6.1).

Use the addition formula for  $\cos(x+x)$  in equation (2.8) together with equation (2.2) to show

$$\sin^2(x) = \frac{1 - \cos(2x)}{2} \qquad \cos^2(x) = \frac{1 + \cos(2x)}{2}$$

2.2 Optional: Proof of cosine addition theorem by sine addition theorem.

Use the sine addition formula  $\sin(x+y) = \sin(x)\cos(y) + \sin(y)\cos(x)$  to show

$$\sin(\underbrace{\alpha + \beta} + 90^\circ) = \cos(\alpha + \beta) \qquad \sin(\beta + 180^\circ) = -\sin(\beta)$$

Note: Setting  $\alpha = 0$  in the first expression produces  $\sin(\beta + 90^\circ) = \cos(\beta)$ .

Combining the first and second expression, one can see  $\sin(90^\circ + \beta + 90^\circ) = \cos(90^\circ + \beta) = -\sin(\beta)$ .

Use  $\sin(\alpha + \underbrace{\beta + 90^\circ}) = \cos(\alpha + \beta)$  and the previous facts to prove the cosine addition formula

$$\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$$

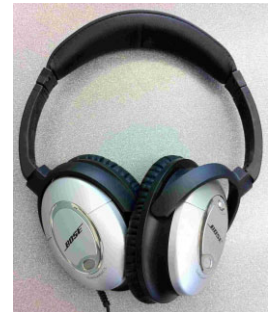
2.3 Amplitude-phase examples. [See equation (2.21), equation (2.22) in Section 2.6.3].

Use the *amplitude/phase formulas for sine or cosine* to complete the following blanks.

$$\begin{aligned} -\sin(2t) &= \sin(2t + \boxed{\phantom{00}}) = \cos(2t + \boxed{\phantom{00}}) \\ -\cos(2t) &= \sin(2t + \boxed{\phantom{00}}) = \cos(2t + \boxed{\phantom{00}}) \\ -3*\sin(2t) + 4*\cos(2t) &= 5*\sin(2t + \boxed{\phantom{00}}) \\ -3*\sin(2t + \frac{\pi}{3}) + 4*\cos(2t) &= \boxed{\phantom{00}}*\cos(2t + 0.82) \end{aligned}$$

2.4 Design of noise cancelling headphones. (Section 2.6.3).

Noise cancelling headphones use both passive (insulated earphones) and active (electronic “anti-noise”) methods to nullify ambient noise. One task of a sound engineer is to design low-energy anti-noise signals that help cancel ambient noise. Consider anti-noise that is to be combined (to cancel) ambient-noise.



Courtesy Betai Koffi

AmbientNoise =  $100 \sin(\omega t)$  Amplitude 100 and frequency  $\omega$  (graphed below).

AntiNoise =  $A \sin(\omega t + \phi)$  Amplitude  $A$  is a positive number.  $-\pi < \phi \leq \pi$

- Choose AntiNoise so the sum AmbientNoise + AntiNoise has a combined amplitude of 20 (much quieter than AmbientNoise). Guess/choose the phase  $\phi$  that minimizes  $A$  (minimum  $A$  decreases hearing fatigue and energy consumption). On the plot, sketch your anti-noise signal and combined signal.

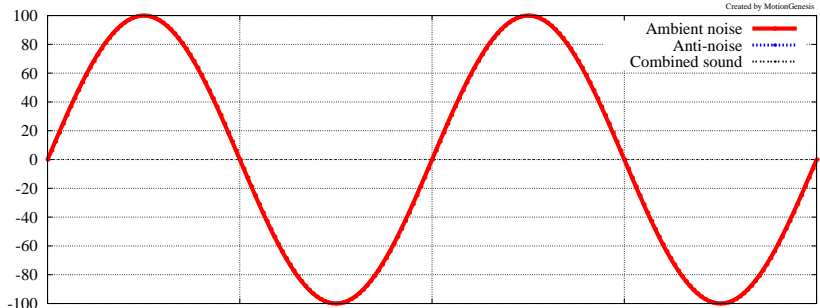
Result:

AmbientNoise =  $100 \sin(\omega t)$

AntiNoise =  $A \sin(\omega t + \phi)$

$A = \boxed{\phantom{00}}$   $\phi = \boxed{\phantom{00}}$  rad

CombinedSound =  $\boxed{\phantom{00}} \sin(\boxed{\phantom{00}})$



- It is difficult for AntiNoise to be perfectly out of phase with AmbientNoise (i.e., difficult for  $\phi$  to be

exactly  $\pi$ ). Consider  $\text{AntiNoise} = 100 \sin(\omega t + \pi + \delta)$ . Determine the maximum  $\delta$  between 0 and  $\pi$  to create a combined noise/anti-noise sound of amplitude 20, i.e.,

$$\text{CombinedSound} = 100 \sin(\omega t) + 100 \sin(\omega t + \pi + \delta) = 20 \sin(\omega t + \text{SomePhase})$$

Show  $\delta$  is governed by the following equation – and solve for  $\delta$ .

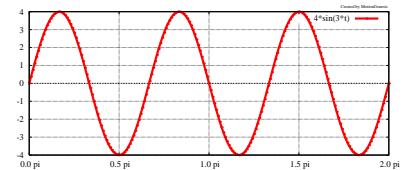
**Result:**  $\sqrt{2 - 2 \cos(\delta)} = 0.2 \qquad \delta \approx 0.2 \text{ rad} \approx 11.5^\circ$

**2.5 Manipulating sine and cosine functions into amplitude-phase form.** (See Section 2.6.3).

An engineer is designing a wave-pool for a water park. The waves are created by two wave generators, and the amplitude, frequency, and phase of each wave at the center of the pool is measured separately and determined as shown below. With both wave generators operating, the resultant wave height  $y(t)$  at the center of the pool, is approximately the sum of the two waves.



$$\begin{aligned} \text{wave}_A(t) &= 4 * \sin(3 * t + \phi_A) & -\pi < \phi_A \leq \pi \\ \text{wave}_B(t) &= 5 * \cos(3 * t + \phi_B) & -\pi < \phi_B \leq \pi \\ y(t) &\approx \text{wave}_A(t) + \text{wave}_B(t) \end{aligned}$$



Assuming  $\phi_A = 0$ , determine the minimum and maximum amplitude of the sum of the two waves and the associated value of  $\phi_B$ .

**Result:** The minimum wave height is:  $\min(|y(t)|) = \square$  which occurs when  $\phi_B = \square$   
 The maximum wave height is:  $\max(|y(t)|) = \square$  which occurs when  $\phi_B = \square$

**First** using intuition (guessing) and **then** using mathematics, determine  $y(t)$  when it is expressed in terms of the (non-negative) amplitude  $C$ , frequency  $\omega$ , and phase  $\phi$  as  $y(t) = C \sin(\omega * t + \phi)$ . Comment on the accuracy of your intuition in predicting the wave's amplitude  $C$ .

**Result:**

Use these values for this question:

$$\begin{aligned} \phi_A &= \frac{\pi}{9} = 20^\circ \\ \phi_B &= \frac{\pi}{3} = 60^\circ \end{aligned}$$

|                                      |  |
|--------------------------------------|--|
| Intuition ( <b>guess</b> ):          | $y(t) = \square * \sin(\square * t + \square)$ |
| Mathematics ( <b>analysis</b> ):     | $y(t) = \square * \sin(\square * t + \square)$ |
| Mathematics ( <b>analysis</b> ):     | $y(t) = \square * \cos(\square * t + 0.15)$    |
| Accuracy in guessing wave amplitude: | <b>Good/Bad</b>                                |



**2.6 Is Newton's law  $\vec{F} = \frac{d(m\vec{v})}{dt}$  or  $\vec{F} = m\frac{d\vec{v}}{dt}$  ? (Separate variables & integrate – Chapter 4).**

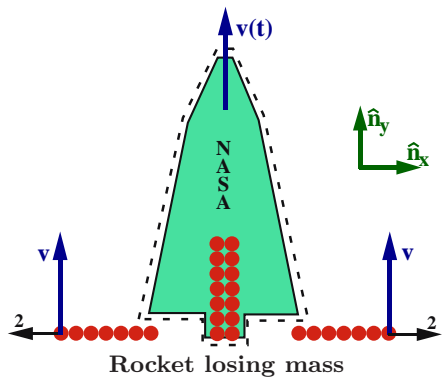
The figure below shows a rocket moving with velocity  $v\hat{n}_y$ . The rocket's propulsion system is malfunctioning and it ejects equal amounts of propellant left and right. The velocity of each propellant particle on the right is  $2\hat{n}_x + v\hat{n}_y$  and each particle on the left is  $-2\hat{n}_x + v\hat{n}_y$  (the  $\hat{n}_y$  component of the ejected propellant's velocity is **equal** to the rocket's velocity).

The two candidate motion laws shown below are suggested for analyzing the motion of the enclosed rocket system (the matter inside the dotted line in the figure below).<sup>1</sup>

$$\text{Is } \vec{F} = \frac{d(m\vec{v})}{dt} \quad \text{or} \quad \vec{F} = m\frac{d\vec{v}}{dt}$$

Using each candidate equation, the speed  $v(t)$  and kinetic energy  $K$  of the rocket at time  $t$  can be expressed in terms of the rocket's initial mass  $m_o$  and initial speed  $v_o$ . In the limit as  $m \rightarrow 0$ , these laws produce radically different values of  $v$ . Decide which provide more sensible results.<sup>2 3</sup>

Note: There are no gravitational, electromagnetic, or other external forces on the system. Hence  $\vec{F} = \vec{0}$ .

|   |   |  |
|---|---|--|
| $\vec{F} = \frac{d(m\vec{v})}{dt}$ $\vec{0} = \frac{d(m\vec{v})}{dt}$ $m\vec{v} = \text{constant} = m_o v_o \hat{n}_y$ $v(t) = \frac{\text{yellow}}{\text{yellow}}$ $K = \frac{1}{2} \frac{(m_o v_o)^2}{m}$ |  | $\vec{F} = m\frac{d\vec{v}}{dt}$ $\vec{0} = m\frac{d\vec{v}}{dt}$ $\vec{0} = \frac{d\vec{v}}{dt}$ $v(t) = \text{yellow}$ $K = \frac{1}{2} m v_o^2$ |
|---|---|--|

|   |  |
|---|--|
| Law that results in $v$ increasing as $m$ decreases.                                    | $\vec{F} =$ <span style="background-color: yellow; border: 1px solid black; display: inline-block; width: 50px; height: 15px;"></span> |
| Law that results in constant $v$ .  | $\vec{F} =$ <span style="background-color: yellow; border: 1px solid black; display: inline-block; width: 50px; height: 15px;"></span> |
| Law that conserves linear momentum of combined system (inside and outside dotted line). | $\vec{F} =$ <span style="background-color: yellow; border: 1px solid black; display: inline-block; width: 50px; height: 15px;"></span> |
| Based on this thought-experiment, the more plausible fundamental law.                   | $\vec{F} =$ <span style="background-color: yellow; border: 1px solid black; display: inline-block; width: 50px; height: 15px;"></span> |



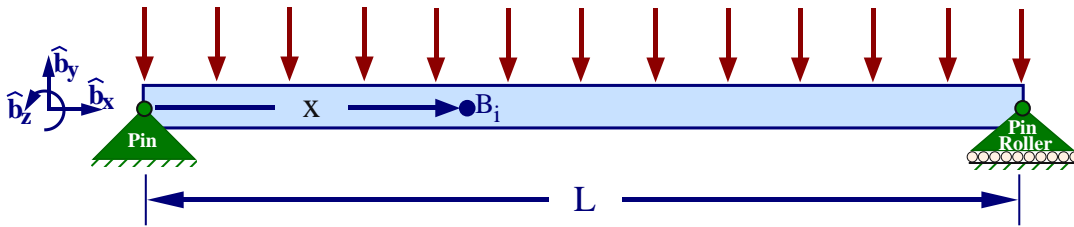
<sup>1</sup>Relativity is not relevant for this rocket. The fastest human object is the unmanned spacecraft Helios 2 (1976). Its peak speeds of  $252,792 \frac{\text{km}}{\text{hr}}$  (157,078 mph) are a mere  $\frac{1}{4269}$  (0.0002) of the speed of light.

<sup>2</sup>A senior Lockheed-Martin dynamics engineer recently notified NASA of their incorrect version of Newton's second law on several NASA's websites, including <http://exploration.grc.nasa.gov/education/rocket/newton2r.html> and <http://www.grc.nasa.gov/WWW/BGH/newton2.html>. The correct law (according to Kane and others) is  $\vec{F} = m\frac{d\vec{v}}{dt}$ .

<sup>3</sup>There are other ways to properly account for the inflow and outflow of linear momentum.

## 2.7 Beam bending moment diagram via 2<sup>nd</sup>-order ODE with boundary-values.

The following shows a long horizontal uniform elastic beam whose left end is connected to ground by an pin and whose right end is connected to ground by an pin-roller.



Note:  $x$  is the  $\hat{b}_x$  measure of  $B_i$ 's position from the left-pin and is regarded as an independent variable.

Note:  $M_z(x)$  is the internal "bending-moment" in the beam and is regarded as a dependent variable.

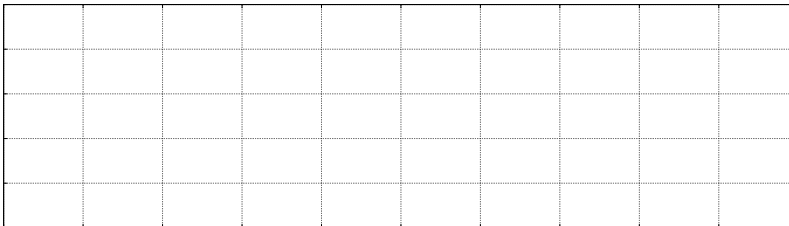
- (a) One can use statics and *free-body diagrams* to relate shear force  $F_y$  via an integral (or the differential equation shown below) to a **downward**-vertical load per unit length  $w(x)$  applied to the beam. Similarly, the bending moment  $M_z$  is related via an integral (or differential equation) to shear force  $F_y$ . Concatenate these relationships to relate  $M_z$  to  $w(x)$ .

**Result:**  $\frac{dF_y}{dx} = w(x)$        $\frac{dM_z}{dx} = -F_y(x)$        $\Rightarrow$        $\frac{d^2 M_z}{dx^2} =$   

- (b) **Solve the ODE for  $M_z$  with separation of variables and integration.** (See Section 4.1) When the pin and pin-roller are ideal frictionless connections, the *boundary values* are  $M_z(0) = 0$  and  $M_z(L) = 0$  ( $L$  is the distance between the pin and pin-roller). Use these boundary values, a *constant*  $w$ , and the previous ODE to find an analytical expression for  $M_z(x)$ .

**Result:** (in terms of  $L$ ,  $w$ , and  $x$ )       $M_z(x) =$   

- (c) **Sketch the bending-moment diagram** [i.e., sketch  $M_z(x)$  for  $0 \leq x \leq L$ ].

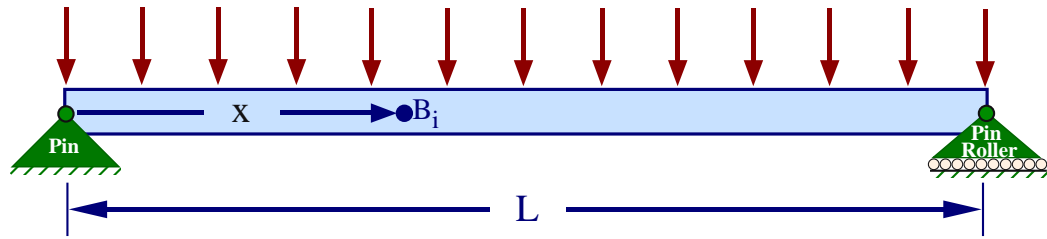


- (d) The maximum value of  $M_z$  and where the beam is most likely to break is at  $x =$   .



**2.8 Beam deflection boundary-value problem - classification.** (Section 3.5).

The following shows a long horizontal uniform elastic beam whose left end is connected to ground by a pin and whose right end is connected to ground by a pin-roller. The beam's shear, moment, slope, and deflection at an arbitrary point  $B_i$  on the beam's centerline depend on load, geometry, and material – and can be regarded as a function of two independent variables, namely time  $t$  and  $x$ .



Note:  $t$  (time) and  $x$  (horizontally-right measure of  $B_i$ 's position along the beam) are regarded as independent variables.

Note:  $v(x,t)$  is the vertically-upward deflection of the beam's elastic curve and is regarded as a dependent variable.

(a) The **dynamic** behavior of  $B_i$ 's deflection is approximated by **Euler beam theory** with

$$\frac{\partial^2}{\partial x^2} \left( E I \frac{\partial^2 v}{\partial x^2} \right) + \rho \frac{\partial^2 v}{\partial t^2} = -w(x,t)$$

|           |  |
|-----------|--|
| $w(x,t)$  | Downward-vertical load per unit length |
| $E(x)$    | Elastic modulus (material property)    |
| $I(x)$    | Centroidal area moment of inertia      |
| $\rho(x)$ | Mass density per unit length           |

Note:  $w, E(x), I(x), \rho(x)$  are **known** (specified/prescribed) functions, whereas  $v(x,t)$  is **unknown**.

Classify the previous equation by picking the relevant qualifiers from the following list.

|           |               |                      |                        |          |              |
|-----------|---------------|----------------------|------------------------|----------|--------------|
| Linear    | Homogeneous   | Constant-coefficient | 2 <sup>nd</sup> -order | Ordinary | Algebraic    |
| Nonlinear | Inhomogeneous | Variable-coefficient | 4 <sup>th</sup> -order | Partial  | Differential |

(b) Simplify the previous equation to form an equation that govern the beam's **static** deflection  $v(x)$  (**static** means the beam is stationary – it does not move).

Next, classify the simplified equation by picking the relevant qualifiers from the list that follows.

**Result:**

$$\boxed{\phantom{E I v''''}} = \boxed{\phantom{-w}}$$

|           |               |                      |                        |          |              |
|-----------|---------------|----------------------|------------------------|----------|--------------|
| Linear    | Homogeneous   | Constant-coefficient | 2 <sup>nd</sup> -order | Ordinary | Algebraic    |
| Nonlinear | Inhomogeneous | Variable-coefficient | 4 <sup>th</sup> -order | Partial  | Differential |

(c) Further simplify the previous equation to a uniform beam (**constant**  $E, I,$  and  $\rho$ ).

Next, classify the simplified equation by picking the relevant qualifiers from the list that follows.

**Result:**

$$\boxed{\phantom{E I v''''}} = \boxed{\phantom{-w}}$$

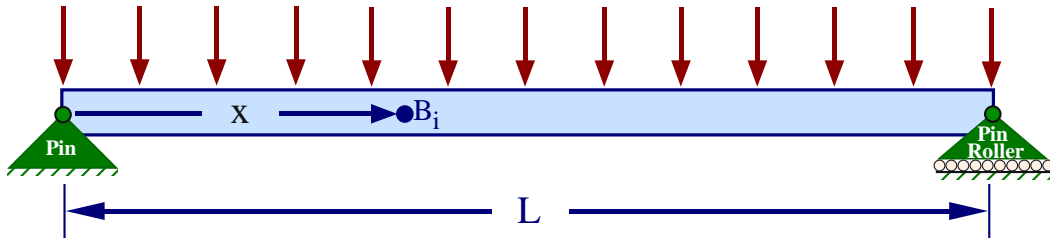
|           |               |                      |                        |          |              |
|-----------|---------------|----------------------|------------------------|----------|--------------|
| Linear    | Homogeneous   | Constant-coefficient | 2 <sup>nd</sup> -order | Ordinary | Algebraic    |
| Nonlinear | Inhomogeneous | Variable-coefficient | 4 <sup>th</sup> -order | Partial  | Differential |

(d) Using engineering intuition (guessing), I expect the maximum beam deflection and where the beam is most likely to break (under uniform load, i.e.,  $w$  is **constant**) to be at  $x = \boxed{\phantom{L/2}}$ .



2.9 Beam deflection via 4<sup>th</sup>-order ODE with boundary-values (separate variables & integrate).

The following shows a long horizontal uniform elastic beam of length  $L$  whose left end is connected to ground by an ideal frictionless pin and whose right end is connected to ground by an ideal frictionless pin-roller. The beam's shear, moment, slope, and deflection at an arbitrary point  $B_i$  on the beam's centerline depend on load, geometry, and material and are a function of  $x$ .



Note:  $x$  is a horizontally-right measure of  $B_i$ 's position along the beam and is regarded as an independent variable.

Note:  $v(x)$  is the vertically-upward deflection of the beam's elastic curve and is regarded as a dependent variable.

This problem calculates **static** deflection of  $v(x)$  (vertically-downward measure of  $B_i$ 's deflection) of a beam whose elastic modulus  $E$  and centroidal area moment of inertia  $I$  are **known constants**. A **known constant** downward-vertical force per unit length  $w$  is applied to the beam.

- (a) Using *separation of variables and integration*, solve the following 4<sup>th</sup>-order ODE for  $v(x)$ .

*Euler beam equation:* 
$$E I \frac{d^4 v}{dx^4} = -w$$

**Result:** (In terms of  $x$ , the **constants**  $E, I, w$ , and arbitrary constants  $c_0, c_1, c_2, c_3$ )

$$v(x) = \text{[Yellow Box]}$$

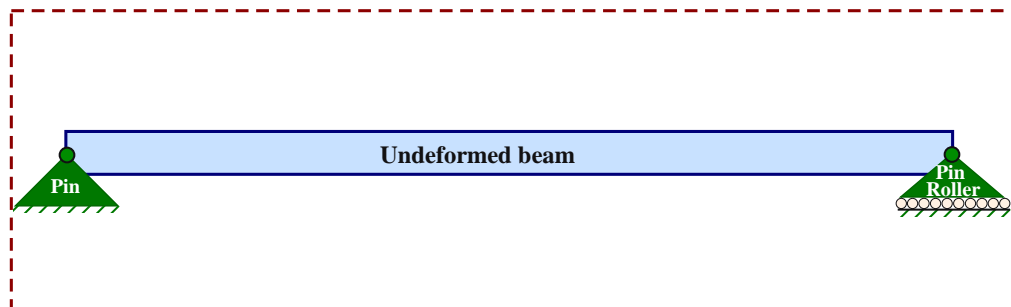
- (b) The values of the constants  $c_0, c_1, c_2, c_3$  depend on **boundary values**. The displacement boundary values are  $v(0) = 0$  and  $v(L) = 0$ . When the pin and pin-roller are ideal frictionless connections, they cannot support moments, hence two **boundary values** are  $M(0) = 0$  and  $M(L) = 0$ . Using these moment boundary values, the relationship  $M = EI \frac{d^2 v}{dx^2}$ , find an analytical expression for  $v(x)$  in terms of  $L, E, I, w$ , and  $x$ .

**Result:**

$$v(x) = \frac{-w}{24EI} * \left( \text{[Yellow Box]} - 2 \text{[Yellow Box]} x^3 + \text{[Yellow Box]} x \right)$$

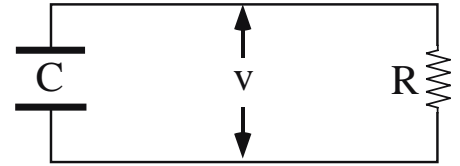
- (c) Sketch the shape of the **deformed** beam for  $0 \leq x \leq L$ .

**Result:**



**2.10 Solve an electrical circuit's 1<sup>st</sup>-order ODE (separate variables & integrate).**

The following "R-C" circuit has constant linear resistance  $R$  and constant linear capacitance  $C$ . Equating the electrical current  $i$  through the resistor and capacitor yields an ODE for voltage  $v$  across the resistor/capacitor.



$$i = \frac{v}{R} = C \frac{dv}{dt} \quad \Rightarrow \quad C \frac{dv}{dt} + \frac{1}{R} v = 0$$

(a) Classify the ODE for  $v$  by picking the relevant qualifiers from the following list.

|           |               |                      |                        |              |
|-----------|---------------|----------------------|------------------------|--------------|
| Linear    | Homogeneous   | Constant-coefficient | 1 <sup>st</sup> -order | Algebraic    |
| Nonlinear | Inhomogeneous | Variable-coefficient | 2 <sup>nd</sup> -order | Differential |

(b) Using *separation of variables and integration*, find the solution for  $v(t)$ .

**Result:** [in terms of  $C$ ,  $R$ ,  $t$ , and  $v_0$ , the *initial value* of  $v$ , i.e.,  $v_0 \triangleq v(t=0)$ ].

$$v(t) = \text{[Yellow Box]}$$

(c) Design  $R$  and  $C$  so  $i = 2.0$  milliAmps and  $v = 40$  volts at  $t = 2.0$  sec. Use  $v_0 = 120$  volts.

**Result:**  $R = 20$  kOhms (20,000 Ohms)  $C = 91$  microFarads ( $9.1 \times 10^{-5}$  Farads)

**2.11 System identification for a 1<sup>st</sup>-order dynamic system (Coulomb friction).**

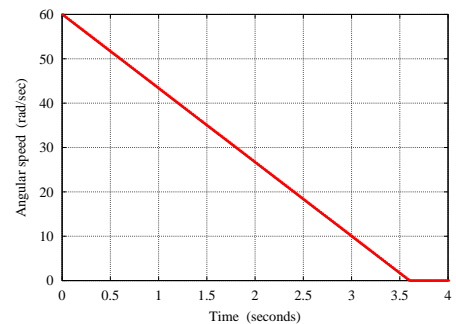
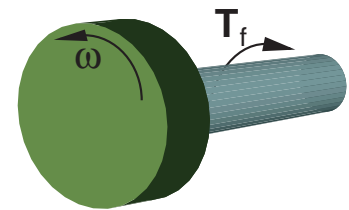
The figure to the right shows a free-spinning (non-driven) motor whose laboratory data for spin rate  $\omega$  correlates with the mathematical function

$$\omega(t) = \omega_0 + slope * t$$

where *slope* is the slope of the line. and  $\omega_0$  is the *initial spin rate* (i.e,  $\omega_0$  is the value of  $\omega$  at  $t = 0$ ). This mathematical function suggests a 1<sup>st</sup>-order ODE of the form<sup>a</sup>

$$I \frac{d\omega}{dt} = -T_f$$

where  $I$  is a constant (the moment of inertia of the motor and its rotor and attachments about the motor's axis) and  $T_f$  is the **constant** torque exerted by Coulomb friction.



<sup>a</sup>With knowledge of ODEs, an engineer can view sensor data and decide if the motor is dominated by Coulomb friction, viscous friction, etc.

(a) Classify the previous equation by picking the relevant qualifiers from the following list.

|           |               |                      |                        |              |
|-----------|---------------|----------------------|------------------------|--------------|
| Linear    | Homogeneous   | Constant-coefficient | 1 <sup>st</sup> -order | Algebraic    |
| Nonlinear | Inhomogeneous | Variable-coefficient | 2 <sup>nd</sup> -order | Differential |

(b) Using *separation of variables and integration*, find  $\omega(t)$  in terms of  $I$ ,  $T_f$ ,  $t$ , and  $\omega_0$ .

**Result:**  $\omega(t) = \text{[Yellow Box]}$

(c) A motor having an inertia (with attached rotor) of  $I = 300$  g cm<sup>2</sup> is spinning with  $\omega = 60$  rad/sec. The power is then shut off and the motor spins down with the time-history shown in the previous graph. Approximate  $T_f$ , the Coulomb friction torque constant.

**Result:**  $T_f \approx 0.5$  milliNewtons \* m

## 2.12 System identification for a 1<sup>st</sup>-order dynamic system (viscous damping).

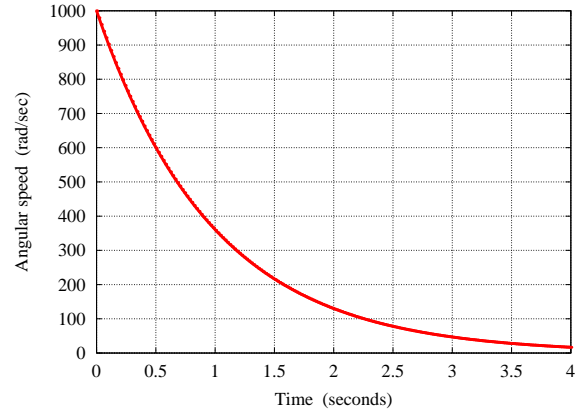
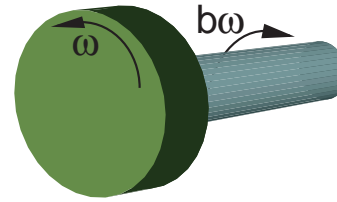
The figure to the right shows a freely-spinning (non-driven) motor whose laboratory data correlates with the mathematical function

$$\omega(t) = \omega_0 e^{-\frac{t}{\tau_c}}$$

where  $\omega_0$  is the initial angular spin rate and  $\tau_c$  is called the **time constant**. This mathematical function suggests a 1<sup>st</sup>-order ODE of the form

$$I \frac{d\omega}{dt} + b\omega = 0$$

where  $I$  is a constant (the moment of inertia of the motor and its rotor and attachments about the motor's axis) and  $b$  is the linear viscous damping constant.



- (a) Classify the previous equation by picking the relevant qualifiers from the following list.

|           |               |                      |                        |              |
|-----------|---------------|----------------------|------------------------|--------------|
| Linear    | Homogeneous   | Constant-coefficient | 1 <sup>st</sup> -order | Algebraic    |
| Nonlinear | Inhomogeneous | Variable-coefficient | 2 <sup>nd</sup> -order | Differential |

- (b) A motor is spinning with  $\omega = 1000$  rad/sec. The power is shut off and the motor spins down with the time-history shown. Use the laboratory data to approximate the **time-constant**  $\tau_c$ .

**Result:**

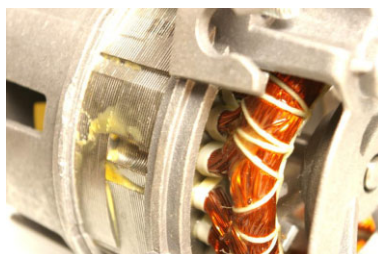
$$\tau_c \approx 0.98 \text{ sec}$$

Note: **time constant**  $\tau_c$  is defined in Section 4.2.

- (c) A motor having inertia (with attached rotor) of  $I = 300 \text{ g cm}^2$  is spinning with  $\omega = 1000$  rad/sec. The power is then shut off and the motor spins down. By observing the time-history of the angular speed of the motor, an engineer approximates the time-constant to be  $\tau_c = 4$  sec. Assuming that linear viscous damping is the only relevant factor in the motor's spin-down, determine the linear viscous damping constant  $b$ .<sup>4</sup>

**Result:**

$$b = 75 \frac{\text{g} \cdot \text{cm}^2}{\text{sec}}$$



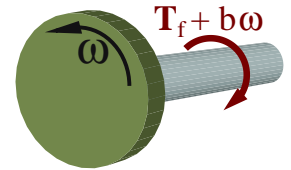
<sup>4</sup>Hint: Use **separation of variables and integration**, to solve for  $\omega(t)$  in terms of  $I$ ,  $b$ , and  $\omega_0$  (the **initial value** of  $\omega$ ).



### 2.13 System identification for a 1<sup>st</sup>-order dynamic system (Coulomb friction and viscous damping).

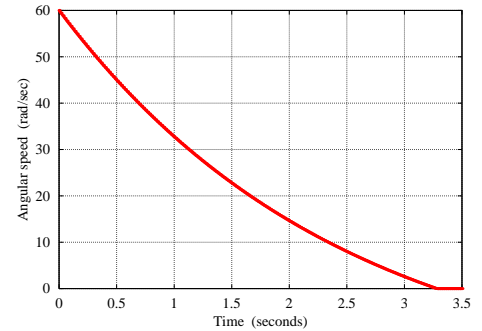
The figure to the right shows a freely-spinning (non-driven) motor. The equation governing  $\omega$  (the motor's angular speed) is

$$I\dot{\omega} + b\omega = -T_f$$



- $I$  is the moment of inertia of the motor and its attachments about the motor's axis
- $b$  is the linear viscous damping constant
- $T_f$  is the constant torque exerted by Coulomb friction

Laboratory data (shown right) suggests that both Coulomb friction and linear viscous damping are present. At large values of  $\omega$ , viscous damping dominates and motor speed decreases exponentially whereas at small values of  $\omega$ , Coulomb friction dominates and motor speed decreases linearly.



- (a) Classify the previous equation by picking the relevant qualifiers from the following list.

|           |               |                      |                        |              |
|-----------|---------------|----------------------|------------------------|--------------|
| Linear    | Homogeneous   | Constant-coefficient | 1 <sup>st</sup> -order | Algebraic    |
| Nonlinear | Inhomogeneous | Variable-coefficient | 2 <sup>nd</sup> -order | Differential |

- (b) Use *separation of variables* to show the ODE can be written as shown (below left).

Next, *integrate* to solve for  $\omega(t)$  in terms of  $I$ ,  $b$ ,  $T_f$ , and  $\omega_0$  (the initial value of  $\omega$ ).

**Result:** Note: It may be helpful to make a substitution ( $u = \dots$ ) to do the integral.

$$\frac{I d\omega}{b\omega + T_f} = -dt \quad \Rightarrow \quad \omega(t) = \text{[Yellow Box]}$$

Note: Substituting  $T_f = 0$  gives  $\omega(t) = \omega_0 e^{-\frac{b}{I}t}$  which is useful for Homework 2.12. Doing a Taylor-series expansion about  $b=0$  in the solution for  $\omega(t)$  gives  $\omega(t) = \omega_0 - \frac{T_f}{I}t$  which verifies the solution in Homework 2.11.

- (c) A motor having inertia (with attached rotor) of  $I = 1.0 \text{ kg m}^2$  is spinning with initial angular speed  $\omega_0 = 60 \text{ rad/sec}$ . The power is then shut off and the motor spins down with the time-history shown above. Using the laboratory data, form two equations that suffice to approximate  $b$  and  $T_f$ . Two convenient points for accurate results are  $\omega(t = 0.5) \approx 45$  and  $\omega(t = 2.5) \approx 8$ .<sup>5</sup> Next, classify those two equations by picking the relevant qualifiers from the list that follows.

**Result:**

$$\begin{aligned} \left(60 + \frac{T_f}{b}\right) * e^{-0.5b} - \frac{T_f}{b} - 45 &= 0 \\ \left(60 + \frac{T_f}{b}\right) * e^{-2.5b} - \frac{T_f}{b} - 8 &= 0 \end{aligned}$$

|           |           |               |              |
|-----------|-----------|---------------|--------------|
| Uncoupled | Linear    | Homogeneous   | Algebraic    |
| Coupled   | Nonlinear | Inhomogeneous | Differential |

With a pencil and paper, can *you* solve these equations for  $b$  and  $T_f$ ? **Yes/No.**

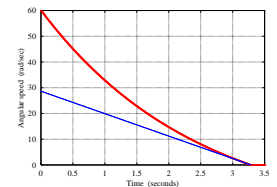
- (d) Calculate  $b$  and  $T_f$  (e.g., use MATLAB<sup>®</sup>, WolframAlpha, or MotionGenesis with the function Exp() for  $e$  and submit a print-out of your file or a screen shot).

Hint: [www.MotionGenesis.com](http://www.MotionGenesis.com)  $\Rightarrow$  [Get Started](#)  $\Rightarrow$  Solve nonlinear algebraic equations.

**Result:**

$$b = 0.41 \frac{\text{kg m}^2}{\text{sec}} \quad T_f = 8.72 \text{ N m}$$

Note: An estimate of  $T_f$  is the slope of the tangent line as  $\omega \rightarrow 0$ .



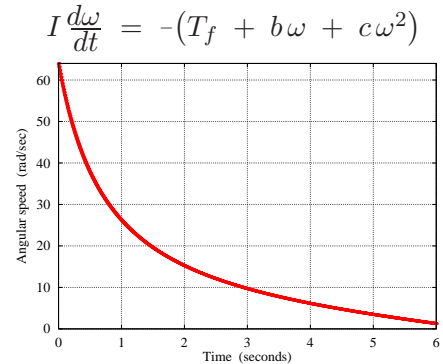
<sup>5</sup>Two data points are needed to form two equations to approximate  $b$  and  $T_f$ . Since resisting torque is modeled  $T_f + b\omega$ , use a first data point when the motor spins quickly ( $\omega$  is large so linear viscous damping  $b\omega$  dominates), and a second data point when the motor spins slowly ( $\omega$  is small so Coulomb friction  $T_f$  dominates).

### 2.14 Optional: Determining damping and friction constants from experimental data.

The following ODE (below-right) governs the angular speed  $\omega$  of a motor whose motor and attached rotor have a moment of inertia of  $I = 1.0 \text{ kg m}^2$ . The associated graph shows the time-history of  $\omega$  for a spin-down test corresponding to an initial value of  $\omega(t = 0) = 64 \text{ rad/sec}$ .

Laboratory data for  $\omega(t)$

| Time (sec) | Angular speed $\omega$ (rad/sec) |
|------------|----------------------------------|
| 0.5        | 37.74139                         |
| 2.0        | 15.23263                         |
| 4.0        | 6.147114                         |



Solve the ODE for  $\omega(t)$  with separation of variables. Next, use the laboratory data to approximate  $T_f$ ,  $b$ ,  $c$ .

**Result:**  $\omega(t) = \square$

$T_f = 2.0 \text{ N m}$        $b = 0.04 \text{ N m s}$        $c = 0.02 \text{ N m s}^2$

### 2.15 System dynamics for a parachutist.

The figure to the right shows a parachutist of mass  $m = 100 \text{ kg}$  falling at speed  $v$  near Earth whose gravitational acceleration is  $g = 9.8 \frac{\text{m}}{\text{s}^2}$ . The parachute provides upward air-resistance force of magnitude  $bv$  ( $b$  is a constant). An equation for  $v$  is formed as:



Courtesy Jorge Cham

$$\vec{F} = m\vec{a} \quad \Rightarrow \quad m\dot{v} + bv = mg$$

Pertinent questions for a parachutist are:

- “How fast will I hit the ground?”.
- “How much time will it take to hit ground?”.

(a) Classify the previous equation by picking the relevant qualifiers from the following list.

|           |               |                      |                        |              |
|-----------|---------------|----------------------|------------------------|--------------|
| Linear    | Homogeneous   | Constant-coefficient | 1 <sup>st</sup> -order | Algebraic    |
| Nonlinear | Inhomogeneous | Variable-coefficient | 2 <sup>nd</sup> -order | Differential |

(b) **Separation of variables and integration** (and use of initial value).

**Separate:** Determine functions  $f(v)$  and  $g(t)$  so the ODE can be written as  $f(v) dv = g(t) dt$ .

**Integrate:** Solve the previous equation for  $v(t)$  in terms of  $m$ ,  $b$ ,  $g$ , and  $t$  when  $v(t=0) = 0$ .

**Result:**  $\square dv = \square dt \quad \Rightarrow \quad v(t) = \square$

(c) Referring to Section 12.5, determine the **steady-state** part and **transient part** of  $v(t)$  or explain why you are unable to find them or what additional information is needed.

**Result:**  $v_{ss}(t) \stackrel{\Delta}{=} \lim_{t \rightarrow \infty} v(t) = \square$        $v_{\text{transient}}(t) \stackrel{\Delta}{=} v(t) - v_{ss}(t) = \square$

(d) **Separation of variables and integration** (and use of initial value).

The vertical downward distance that the parachutist falls is denoted  $y(t)$ .

Find  $y(t)$  in terms of  $m$ ,  $b$ ,  $g$ , and  $t$  when  $y(t=0) = 0$ .

**Result:**  $y(t) = \square + \square + \frac{m^2 g}{b^2} e^{\frac{-b}{m} t}$

(e) Referring to Section 12.5, determine the **steady-state** part and **transient part** of  $y(t)$  or explain why you are unable to find them or what additional information is needed.

**Result:**

$$y_{ss}(t) \stackrel{\Delta}{=} \lim_{t \rightarrow \infty} y(t) = \text{[redacted]}$$

$$y_{\text{transient}}(t) \stackrel{\Delta}{=} y(t) - y_{ss}(t) = \text{[redacted]}$$

(f) Knowing the parachutist's **terminal velocity** is  $4.9 \frac{m}{s}$  ( $v = 4.9 \frac{m}{s}$  after the parachute has been open enough time), form an exact equation for  $t_f$  (the time for a 100 kg parachutist to fall a distance  $y_f$ ).

**Result:**  $b = 200 \frac{Ns}{m}$   $y_f = -2.45 + 4.9t_f + \text{[redacted]} e^{-2t_f}$

(g) Classify the previous equation by picking the relevant qualifiers from the following list.

|           |               |              |
|-----------|---------------|--------------|
| Linear    | Homogeneous   | Algebraic    |
| Nonlinear | Inhomogeneous | Differential |

In **general**, equations classified like this are solved with [redacted].

(h) Use engineering/mathematical insights to approximate  $t_f$  in terms of  $y_f$ .  
What conditions make your approximation better?

**Result:**  $t_f \approx \text{[redacted]}$  [redacted]

(i) **Separation of variables for 2<sup>nd</sup>-order ODEs?**

Separation of variables is useful for solving 1<sup>st</sup>-order ODEs. Is it possible to find an analytical solution for  $y(t)$  for the 2<sup>nd</sup>-order ODE to the right via separation of variables. **Yes/No.**

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} = 9.8$$

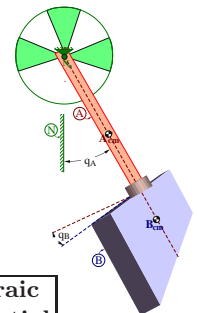
**Explain:** [redacted]

**2.16 Classify a set of equations (and optionally solve them).**

The equations of motion for the system described in Section 3.1 are shown below. Classify this set of equations by picking the relevant qualifiers from the list below.

$$\ddot{q}_A = \frac{2 [508.89 \sin(q_A) - \sin(q_B) \cos(q_B) \dot{q}_A \dot{q}_B]}{-21.556 + \sin^2(q_B)}$$

$$\ddot{q}_B = -\sin(q_B) \cos(q_B) \dot{q}_A^2$$



|           |           |               |                      |                        |              |
|-----------|-----------|---------------|----------------------|------------------------|--------------|
| Uncoupled | Linear    | Homogeneous   | Constant-coefficient | 1 <sup>st</sup> -order | Algebraic    |
| Coupled   | Nonlinear | Inhomogeneous | Variable-coefficient | 2 <sup>nd</sup> -order | Differential |

**Optional:** Find the maximum absolute value of  $q_B(t)$  for  $0 \leq t \leq 10$ . Use  $q_B(0) = 1.0^\circ$  and two different initial values of  $q_A$ , namely  $q_A(0) = 45^\circ$  and  $q_A(0) = 90^\circ$ . Submit a printed copy of your MotionGenesis or MATLAB<sup>®</sup> file. Hint: [www.MotionGenesis.com](http://www.MotionGenesis.com) ⇒ **Get Started** ⇒ **Solve ODEs**.

**Result:** • Maximum absolute value of  $q_B(t)$  when  $q_A(0) = 45^\circ$ :  $q_B(t=1.66) = \text{[redacted]} 1.26^\circ$   
• Maximum absolute value of  $q_B(t)$  when  $q_A(0) = 90^\circ$ :  $q_B(t=8.18) = \text{[redacted]} 1324.57^\circ$

**2.17 Solve a nonlinear 1<sup>st</sup>-order ODE (Submit a computer file or screen-shot).**

The function  $y(t)$  is governed by the ODE  $\frac{dy}{dt} = \sin(t * y)$ .  
Knowing  $y(t=0) = 3$  meters, show  $y(t = 4 \text{ sec}) \approx \text{[redacted]} 2.5$  meters.

