Chapter 2

Math tools

Courtesy NASA

2.1 Why math is important

Math is a foundation for science, medicine, engineering, construction, and business. Math has ***concepts*** (pictures, words, ideas), ***calculations*** (operations, symbols, equations, definitions), and ***context*** (situations in which it is relevant and useful). More generally, math is a language and set of rules that helps us count, quantify, calculate, manipulate, relate, define, extrapolate, and abstract "stuff".[1] Advances in math depend on precise ***definitions***.[2] For example, consider the following ***definition*** of π.

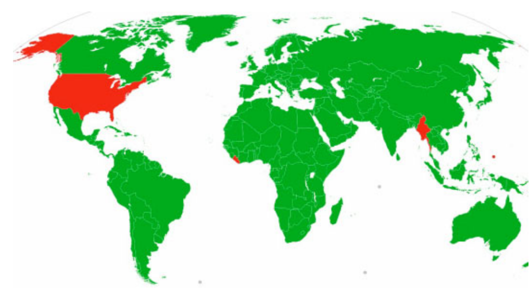
Object	Example	Approximate age of human comprehension
Picture		Toddlers
Spoken word	"circle"	Pre-school
Written word	"circle", "diameter", "circumference"	Elementary school
Symbol	d for diameter, c for circumference	Middle school
Equation	$c = \pi\, d$	Middle/high school
Definition	$\pi \triangleq \frac{c}{d}$ ($\triangleq$ means "***defined as***")	University

A 3-page math history for dynamic systems is at www.MotionGenesis.com $\Rightarrow$ Textbooks $\Rightarrow$ Dynamic Systems

2.2 Unit systems - SI and USA

Units quantify measurement. The ***SI*** system was first adopted by France in 1799 and is now used in all countries other than Liberia, Myanmar, and the United States.

The ***SI*** (metric) system uses a base-10 number system and decimals (not fractions) and has measures for length (meters), mass (kilogram), force (Newton), time (second), etc.



Countries using SI units (green) vs. USA units (red).

NIST (National Institute of Standards & Technology) defines physical constants and conversion factors.

Length	1 inch $\triangleq$ 2.54 cm		$g_{\text{SI}} \triangleq 9.80665\ \frac{\text{m}}{\text{s}^2}$
Mass	1 lbm $\triangleq$ 0.45359237 kg	1 slug $\triangleq$ g_{USA} lbm	$g_{\text{USA}} \approx 32.2\ \frac{\text{ft}}{\text{s}^2}$
Force	1 Newton $\triangleq 1\ \frac{\text{kg m}}{\text{s}^2}$	1 lbf $\triangleq 1\ \frac{\text{slug ft}}{\text{s}^2}$	1 lbf $\triangleq g_{\text{USA}}\ \frac{\text{lbm ft}}{\text{s}^2}$

Inaccurate unit conversions have caused ***many*** failures. In 1999, NASA lost a \$125,000,000 Mars orbiter because one engineering team used SI units while another used USA units. In 1983, an Air Canada Boeing 767 ran out of fuel mid-flight because of a kg to lbm unit conversion.[3]

[1] For example, the philosophy/idea of **value** (**"how much something is worth"**) is frequently quantified by money.

[2] Kurt Godel (1906-1978) demonstrated that any reasonably powerful mathematical system contains seemingly true statements that cannot be proven.

[3] Ironically, Thomas Jefferson helped the United States become the first country (in 1792) to use a monetary system with decimals and a base-10 number system. The historical origin of USA units trace to 2575 B.C. and through ancient Egypt, Greece, and Rome. The ***inch*** approximates the width of a man's thumb. The ***foot***foot (ft) approximates a foot with shoe and

2.3 Mathematical operations

1. ***Addition:*** One way to add real numbers is to with a number line (e.g., a football field). For $3 + 5 = 8$, stand on the 3 yard line (with larger numbers right and smaller numbers left), then **move right** 5 yards to the answer (8). For $8 + \text{-}2$, start on the 8 yard line and **move left** 2 yards (adding a negative number).

   The $+$ symbol has different operational significance in other contexts, e.g., adding vectors such as $\vec{\mathbf{a}} + \vec{\mathbf{b}}$ or adding two appropriately sized matrices as $A + B$. One representation[4] of a complex number with real part a and imaginary part $b\,i$ is $a + b\,i$. One way to add a real and imaginary number is with a field with two perpendicular axes (real and imaginary axes). To find $3 + 5 * i$, start at the intersection of the real and imaginary axes (0,0) and **move right** 3 and **move forward** 5.

2. ***Subtraction*** and ***negation:*** The subtraction dash used in $8 - 5$ "eight minus five" is called a ***binary minus*** whereas the negation dash used in -5 "negative five" is called a ***unary minus***. Overloading the dash symbol $-$ with contextually different meanings is confusing and there are still unresolved order-of-operations negation conventions (see Section 2.3.1).

   Note: Before 1800 AD, negative numbers were treated with great suspicion. For example, Pascal regarded $0 - 4$ as utter nonsense. Maseres and Frend wrote algebra texts renouncing both negative and imaginary numbers on the grounds that mathematicians were unable to explain their use except by analogy.

   Subtraction is negation and addition (in calculators, computers, spreadsheets, ...)
   To avoid burdensome **memorization**, subtraction can be taught as negation and addition, e.g.,

   $$\sin(a - b) = \sin(a + \text{-}b) \qquad \cos(a - b) = \cos(a + \text{-}b) \qquad \frac{d\,(u - v)}{dt} = \frac{d\,(u + \text{-}v)}{dt}$$

   The efficient method for teaching subtraction, called the ***addition method*** or ***Austrian method***:
   - Eliminates the unwieldy multi-column process of "***borrowing***".
   - Relies on a student's ability to ***add*** (eliminating memorization of subtraction tables).
   - Mimics the teaching of division (which relies on a student's ability to multiply).

3. ***Multiplication:*** Multiplication of two integers is a convenient way to represent "multiple additions".
   Example: $3 * 5$ represents adding 3 five times, as $3 * 5 = 3 + 3 + 3 + 3 + 3$
   The $*$ symbol has different operational significance in other contexts. For example, one may multiply two complex numbers, two matrices, or two vectors $\vec{\mathbf{a}} * \vec{\mathbf{b}}$.

4. ***Division:*** Both the horizontal fraction bar (e.g., in $\frac{3}{5}$) and diagonal fraction bar (in 3/5) denote division. The symbols and ideas in division are rooted in ***fractions*** and the ratio of two **integers** and date back to Egypt (3000 BC), Babylonia (2000 BC), and Greece (500 BC).

   Like subtraction, division is extraneous as **division is multiplication and exponentiation with -1**, e.g., as shown right. $\boxed{\frac{y}{x} = y * x^{\text{-}1}}$

   Concomitantly, there is **no need to memorize formulas involving division**, e.g., the formula for $\frac{d}{dt}\left(\frac{u}{v}\right)$ is easily calculated using product rules and exponent rules for differentiation.

5. ***Exponents:*** An integer exponent is a convenient way to represent a series of multiplications:
   Example: 3^5 represents multiplying 3 five times, as $3^5 = 3 * 3 * 3 * 3 * 3$
   Exponents with special names include 2 (***squared***), $\frac{1}{2}$ (***square root***), 3 (***cubed***), and $\frac{1}{3}$ (***cube root***).
   Note: There are unresolved left/right order-of-operations conventions for exponents (see Section 2.3.1).

6. ***Logarithms: Exponents*** and ***logarithms*** are related. For example: $3^5 = 243$ and $\log_3 243 = 5$. A central issue of calculating logarithms is properly determining the ***logarithm's base***.

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was somewhat standardized in England to King Henry I. The ***mile*** "mille passus" is 1000 paces (2 steps) of a Roman soldier. An Australian study found that switching from British units to metric units freed $\frac{1}{2}$-year in science education. USA lawmakers have consistently failed to legislate changes in federal systems, e.g., in road signs and for NASA, DOD, and NSF.

[4] Complex numbers are sometimes represented as (a, b). The imaginary number $\sqrt{\text{-}1}$ is usually represented by i or j. Note: The complex number $a + b\,i$ is called a ***formal sum*** because the real quantity a to the left of the $+$ sign is not the same type as the imaginary quantity $b\,i$ to the right of the $+$ sign. Similarly, 3 apples + 5 oranges is a formal sum.

Properties of exponents and logarithms

Addition	$x^{a+b} = \boxed{x^a * x^b}$
Negation	$x^{-b} = 1/x^b$
Subtraction	$x^{a-b} = x^{a+-b} = x^a\, x^{-b} = \frac{x^a}{x^b}$ Subtraction properties can be deduced from addition and negation properties
Exponentiation	$(x^a)^b = (x^b)^a = x^{(a*b)}$ **Only valid for some values of x, a, b.** ***Invalid*** for negative x or non-integer a and b, e.g., $[(\text{-}4)^2]^{\frac{1}{2}} \neq [(\text{-}4)^{\frac{1}{2}}]^2$. Also, parentheses avoid confusion e.g., $2^{3^2} = (2^3)^2 = 8^2 = 64$ or $2^{3^2} = 2^{(3^2)} = 2^9 = 512$?
Multiplication	$\log(a*b) = \boxed{\log(a) + \log(b)}$
Exponentiation	$\log(a^n) = \boxed{n\ \log(a)}$
Fractions	$\log(\frac{a}{b}) = \log[\,(\frac{b}{a})^{-1}\,] = -\log(\frac{b}{a})$
Division	$\log(\frac{a}{b}) = \log(a*b^{-1}) = \log(a) + \log(b^{-1}) = \log(a) + -\log(b) = \log(a) - \log(b)$ Fraction and division properties can be deduced from multiplication and exponential properties.
Change in base	$\log_b(a) = \frac{\log_e(a)}{\log_e(b)}$. For example, $\log_{10}(a) = \frac{\log_e(a)}{\log_e(10)} \approx 0.4343 \log_e(a)$

2.3.1 Order of operations: Established mathematical convention?

PEMDAS (**P**arentheses, **E**xponents, **M**ultiplication/**D**ivision, **A**ddition/**S**ubtraction) is a useful mnemonic to remember the order in which calculations are performed and is useful in reverse to solve equations. However, PEMDAS fails to address ***negation*** and the left-right order of exponents. Modern calculators, math programs, spreadsheets, textbooks and compilers produce different values for some calculations.[5] [6]

Calculation	Google Calculator	Google Sheets	Microsoft Excel®	Open Office	MATLAB®	Python	Wolfram Alpha	Motion Genesis	Your Calculator
$\text{-}3^2 = \boxed{}$	-9	+9	+9	+9	-9	-9	-9	-9	??
$2^{3^2} = \boxed{}$	512	512	64	64	64	512	512	Requires ()	??

2.3.2 Patterns in mathematics

There are many patterns in mathematics that are most recognizable with **three or more items**. This applies to the product rule for differentiation (see Section 2.8.7), the calculation of a determinate, and the transpose rule for matrices, e.g., $(A\,B\,C)^T = C^T\,B^T\,A^T$. For example, a simple, extensible way to clear parentheses is to start with the 1^{st} term inside the first parentheses and multiply it by each successive term inside the second parentheses, then restart with the 2^{nd} term inside the first parentheses, etc., e.g.,

$$(\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c})\,(d + e + f) \;=\; \boldsymbol{a}\,d + \boldsymbol{a}\,e + \boldsymbol{a}\,f \;+\; \boldsymbol{b}\,d + \boldsymbol{b}\,e + \boldsymbol{b}\,f \;+\; \boldsymbol{c}\,d + \boldsymbol{c}\,e + \boldsymbol{c}\,f$$

It is unfortunate that many students learn the mnemonic ***FOIL*** to multiply parenthesized expressions containing **two** terms. The ***FOIL*** method is not an efficient way to multiply parenthesized expressions containing **three** or more terms.

[5] Advocates for -9 regard negation as a ***special case of multiplication*** so $\text{-}x \triangleq (\text{-}1) * x$ which parallels imaginary numbers such as $i\,x \triangleq (\sqrt{\text{-}1}) * x$. Alternatively, if $-$ is regarded as an inherent part of 3, $(\text{-}3)^2 = {+9}$.

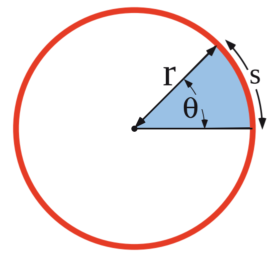
[6] In a poll of undergraduate and graduate Stanford mathematicians and engineers, 87 of 90 voted $2^{3^2} = 2^{(3^2)} = \mathbf{512}$ (right-to-left calculation) whereas only 3 of 90 voted $2^{3^2} = (2^3)^2 = \mathbf{64}$ (left-to-right calculation). Yet, all engineers agreed $10 - 5 + 3 = (10 - 5) + 3 = 8$ (left-to-right calculation) as opposed to $10 - (5 + 3) = 2$ (right-to-left calculation).

2.4 Geometry: Ancient Euclid and modern vectors

Geometry is the study of figures (e.g., lines, curves, surfaces, solids) and their properties (e.g., distance, area, volume, angles). Geometry plays a central role in construction, farming, engineering, medicine, science, etc.

Many students spend 2^+ years learning ancient ($\approx$300 BC) 2D Euclidean geometry and trigonometry (trigonometry translates to "triangle measurement"). The invention of ***vectors*** (Gibbs $\approx$ 1900 AD) and its easy-to-use vector addition, dot-products, and cross-products have **greatly simplified** 2D and 3D geometry. Unfortunately, few K-12 instructors teach geometry or trigonometry with vectors.

2.5 Circles and their properties



The ratio of ***any*** circle's ***circumference*** to its ***diameter*** is the number[a]

$$\boxed{\pi \;\triangleq\; \frac{\text{circumference}}{\text{diameter}} \;\approx\; 3.14159265358979323846264338\;...}$$

$\boldsymbol{\pi}$ is called an ***irrational number*** because it is not a whole number or fraction, nor does it terminate or repeat. It is chaotic, disorderly, and has no discernible pattern.

The ***arc-length*** of a portion of the circle's periphery and the ***area*** of a wedge of the circle can be calculated in terms of the circle's ***radius*** r and the ***angle*** θ as shown right.[6]

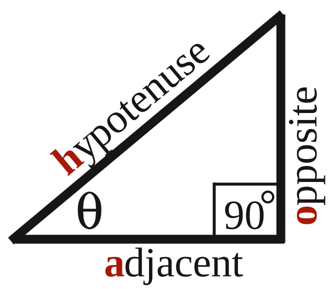
Arc-length $\quad s = \theta\, r$	Area of wedge $= \frac{\theta}{2}\, r^2$
Circumference $= 2\,\pi\, r$	Area of circle $= \pi\, r^2$

[a] The symbol $\boldsymbol{\pi}$ was popularized by Euler circa 1750, but the value $\pi \approx 3.14$ was known in Egypt circa 3000 BC.[7] In 2006, Akira Haraguchi memorized/recited 111,700 digits of π.

2.6 Triangles and ratios of their sides (sine, cosine, tangent)

A triangle ("three angles") is a 3-sided planar geometric shape widely used in construction, engineering, and science.

SohCahToa is a ***mnemonic*** for memorizing the definitions of ***Sine***, ***Cosine***, and ***Tangent*** (ratios of various sides of a right triangle).



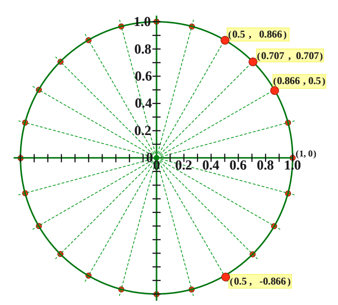
$$\boxed{\begin{aligned} \sin(\theta) &\triangleq \frac{\text{opposite}}{\text{hypotenuse}} \\ \cos(\theta) &\triangleq \frac{\text{adjacent}}{\text{hypotenuse}} \\ \tan(\theta) &\triangleq \frac{\text{opposite}}{\text{adjacent}} = \frac{\sin(\theta)}{\cos(\theta)} \end{aligned}} \qquad (1)$$

The ***Pythagorean theorem*** in equation (2) relates lengths of sides of a right triangle. Combining the definitions of $\sin(\theta)$ and $\cos(\theta)$ with the Pythagorean theorem gives the second relationship to the right.

$$\boxed{\begin{aligned} \text{hypotenuse}^2 &= \text{adjacent}^2 + \text{opposite}^2 \\ \sin^2(\theta) + \cos^2(\theta) &\underset{(1)}{=} 1 \end{aligned}} \qquad (2)$$

Note: Numbers under $=$ refer to equation numbers, e.g., $\underset{(1)}{=}$ means "refers to equation (1)".

2.6.1 Unit circle concept of sine and cosine



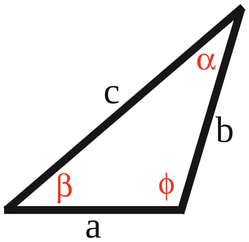
The ***unit circle*** expands the definition of sine and cosine from a 90° **triangle** and allows negative values for sine and cosine which provides tabulated for Euler's function concept of sine and cosine ($\approx$ 1730 AD).

Triangle	$0° < \theta < 90°$	$0 < \sin(\theta) < 1$	$0 < \cos(\theta) < 1$
Unit circle	$0° \le \theta \le 360°$	$-1 \le \sin(\theta) \le 1$	$-1 \le \cos(\theta) \le 1$

Note: Negative numbers were invented $\approx$ 650 AD and widely adopted 1500 AD.

[7] An ***angle*** involves two lines (or vectors) and is measured in radians or degrees. A radian is the ratio of the arc-length of part of a circle's perimeter to its radius. A degree is an archaic unit of angle measurement based on the ancient Babylonian year which had 360 days (12 months $*$ 30 days). Each degree represents one day of Earth's travel about the sun and the degree symbol's circular appearance $^\circ$ is a reminder that 360° measures the Earth's quasi-circular travel around the sun.

2.6.2 Formulas involving sine and cosine



$$\boxed{\begin{array}{lllll} c^2 & = a^2 + b^2 & (\text{when } \phi = 90°) & \textit{Pythagorean theorem} & \text{(Unknown, } \approx 500 \text{ BC)} \\ c^2 & = a^2 + b^2 - 2\,a\,b\,\cos(\phi) & & \textit{Law of cosines} & \text{(Euclid, Egypt, 300 BC)} \\ \frac{\sin(\alpha)}{a} & = \frac{\sin(\beta)}{b} = \frac{\sin(\phi)}{c} & & \textit{Law of sines} & \text{(Ptolemy, Egypt, 100 AD)} \\ \text{Area} & = \frac{1}{2}\,\text{base} * \text{height} = \frac{1}{2}\,a\,c\,\sin(\beta) & & & \end{array}} \qquad (3)$$

$$\boxed{\sin(\text{-}\alpha) = \text{-}\sin(\alpha)} \qquad \boxed{\sin(\alpha+\beta) = \sin(\alpha)\,\cos(\beta) + \sin(\beta)\,\cos(\alpha) \quad \textit{Addition formula for sine} \text{ (Ptolemy)}} \qquad (4)$$

$$\boxed{\cos(\text{-}\alpha) = \cos(\alpha)} \qquad \boxed{\cos(\alpha+\beta) \underset{(4)}{=} \cos(\alpha)\,\cos(\beta) - \sin(\alpha)\,\sin(\beta) \quad \textit{Addition formula for cosine}} \qquad (5)$$

$$\text{Sine \& cosine are periodic} \quad \sin(x) \underset{(4)}{=} \sin(x \pm 2\,\pi\,n) \qquad \cos(x) = \cos(x \pm 2\,\pi\,n) \quad (n = 0, 1, 2, 3\,...)$$

$$\sin(\text{-}x) \underset{(4)}{=} \text{-}\sin(x \pm 2\,\pi\,n) \qquad \cos(\text{-}x) \underset{(5)}{=} \cos(x \pm 2\,\pi\,n) \quad (n = 0, 1, 2, 3\,...)$$

$$\text{Sine \& cosine phased by } \frac{\pi}{2} \quad \sin(x) \underset{(5)}{=} \cos(x - \frac{\pi}{2}) = \cos(\text{-}x + \frac{\pi}{2}) \qquad \cos(x) \underset{(4)}{=} \sin(x + \frac{\pi}{2}) = \sin(\text{-}x + \frac{\pi}{2})$$

$$\text{Sine double-angle formula} \quad \sin(2\,x) \underset{(4)}{=} 2\,\sin(x)\,\cos(x) \qquad \sin(x) = 2\,\sin(\frac{x}{2})\,\cos(\frac{x}{2}) \qquad (6)$$

$$\sin^2(x) \underset{(5)}{=} \frac{1-\cos(2\,x)}{2} \qquad \sin^2(\frac{x}{2}) = \frac{1-\cos(x)}{2} \quad \Rightarrow \quad \cos(x) = 1 - 2\sin^2(\frac{x}{2}) \quad \text{Half-angle formula.} \qquad (7)$$

$$\cos^2(x) \underset{(5)}{=} \frac{1+\cos(2\,x)}{2} \qquad \cos^2(\frac{x}{2}) = \frac{1+\cos(x)}{2} \quad \Rightarrow \quad \cos(x) = 2\cos^2(\frac{x}{2}) - 1 \quad \text{Half-angle formula.} \qquad (8)$$

$$\cos(b) - \cos(a) \underset{(4)}{=} 2\,\sin(\frac{a+b}{2})\,\sin(\frac{a-b}{2}) \quad \text{Useful for beat phenomenon analysis} \qquad (9)$$

$$\cos(\omega_2\,t + \phi_2) - \cos(\omega_1\,t + \phi_1) \underset{(9)}{=} 2\,\sin[\,(\frac{\omega_1+\omega_2}{2})\,t + \frac{\phi_1+\phi_2}{2}\,]\,\sin[\,(\frac{\omega_1-\omega_2}{2})\,t + \frac{\phi_1-\phi_2}{2}\,] \qquad (10)$$

2.6.3 Sine and cosine as functions (Euler, circa 1730)

Euler's interpretation of ***cosine*** and ***sine*** as ***functions*** (not just ratios of sides of a triangle) was a major breakthrough for trigonometry and functions.[8]

Triangle: $\boxed{\cos(\theta) \triangleq \frac{\textbf{a}\text{djacent}}{\textbf{h}\text{ypotenuse}}}$

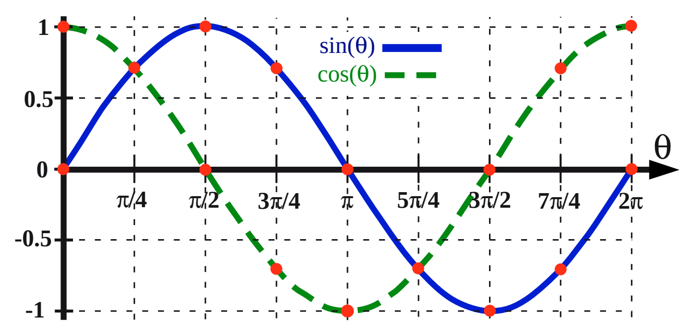
whereas the **cosine function** is shown right

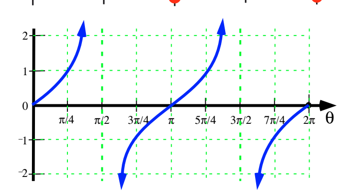
Triangle: $\boxed{\sin(\theta) \triangleq \frac{\textbf{o}\text{pposite}}{\textbf{h}\text{ypotenuse}}}$

whereas the **sine function** is shown right

$\boxed{\tan(\theta) \triangleq \frac{\textbf{o}\text{pposite}}{\textbf{a}\text{djacent}} = \frac{\sin(\theta)}{\cos(\theta)}}$

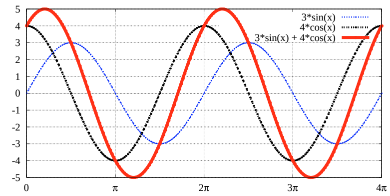
Tangent function





[8]The Babylonians and others used right triangle formulas for thousands of years before their proofs by Pythagoras of Samos [≈ 500 BC]. The definitions of ***sine***, ***cosine***, and ***tangent*** as ratios of sides of a right triangle predate 140 BC when the Greek Hipparchus made sine, cosine, and tangent tables. Euler's interpretation of sine, cosine, and tangent as ***functions*** was a breakthrough for math. Gibb's invention of vectors (≈1900 AD) significantly simplified 3D geometry and trigonometry and proofs of ***law of cosines***, ***law of sines***, and ***sine addition formula***, from which other trigonometric formulas are derived (***cosine addition formula***, ***half or double-angle formulas***, etc). The trig identities to prove eqn (9) include $\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b)$, $\sin^2(x) + \cos^2(x) = 1$, and $\cos^2(\frac{x}{2}) = \frac{1+\cos(x)}{2}$.

2.6.4 The amplitude-phase formulas for sine and cosine



Two trigonometric identities that are particularly helpful in dynamic systems are the ***amplitude/phase formulas for sine and cosine***.

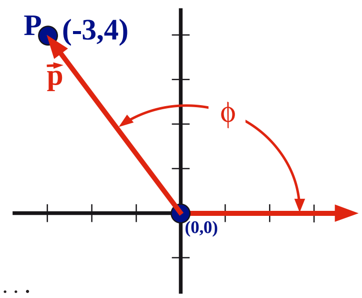
Note: These amplitude-phase formulas are used extensively in vibration analysis. These formulas use `atan2` because A and B may be **positive**, **negative**, or **zero**. These formulas are proved in Sections 2.13.2 and 2.13.3.

$$\boxed{A\sin(x) + B\cos(x) \;=\; C\sin(x+\phi_s) \qquad \texttt{where}\;\; C \;=\; {}^{+}\!\sqrt{A^2+B^2} \qquad \texttt{and} \quad \phi_s \;=\; \texttt{atan2}(\,B, A)} \tag{11}$$

$$\boxed{A\sin(x) + B\cos(x) \;=\; C\cos(x+\phi_c) \qquad \texttt{where}\;\; C \;=\; {}^{+}\!\sqrt{A^2+B^2} \qquad \texttt{and} \quad \phi_c \;=\; \texttt{atan2}(\text{-}A, B)} \tag{12}$$

2.6.5 The function atan2(y, x)

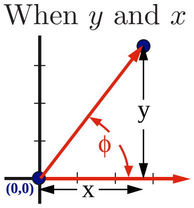
The **`atan2`** function is named because: it is similar to the atan (arc-tangent) function; it takes two arguments; it calculates an angle ϕ with range $\text{-}\pi < \phi \leq \pi$ (2 times atan function's range of $\frac{\text{-}\pi}{2} < \phi \leq \frac{\pi}{2}$).



To determine the angle $\phi = \texttt{atan2}(\,y,\, x\,)$

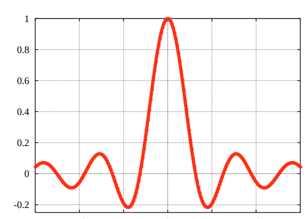
- Draw horizontal and vertical axes as shown right.
- Draw a point P located at the designated y and x values.
  Example: For `atan2(4,-3)`, draw point P at $y = 4$ and $x = \text{-}3$.
- Draw a vector $\vec{\mathbf{p}}$ from $(0,0)$ to point P.
- Draw angle ϕ from the ${}^{+}x$-axis to $\vec{\mathbf{p}}$, with +counter-clockwise sense.
- Using trigonometry, calculate the value of ϕ, e.g., $\phi \;=\; {}^{+}2.21$ rads.
- Alternatively, calculate `atan2`(y, x) with MATLAB®, **M**otion**G**enesis, Java, C, C++, ...

The function $\phi = \texttt{atan2}(\,y,\, x\,)$ returns an angle that satisfies both $\sin(\phi)$ and $\cos(\phi)$ as shown below. When y and x are continuous, ordinary or partial derivatives of `atan2` can be calculated (proof in Section 2.8.10).



$$\left.\begin{aligned} \sin(\phi) \;&=\; \frac{y}{{}^{+}\!\sqrt{x^2 \,+\, y^2}} \\ \cos(\phi) \;&=\; \frac{x}{{}^{+}\!\sqrt{x^2 \,+\, y^2}} \end{aligned}\right\} \;\Rightarrow\; \begin{array}{ll} \boxed{\phi \;=\; \texttt{atan2}(\,y,\; x\,)} & \text{-}\pi < \phi \leq \pi \\ \boxed{\dot{\phi} \;=\; \dfrac{x\,\dot{y} \;-\; y\,\dot{x}}{x^2 \,+\, y^2}} & \boxed{\dfrac{\partial\theta}{\partial s} \;=\; \dfrac{x\,\frac{\partial y}{\partial s} \;-\; y\,\frac{\partial x}{\partial s}}{x^2 \,+\, y^2}} \end{array} \tag{13}$$

2.6.6 Optional: The sinc function



The sinc ***function*** (also call the ***sine cardinal*** or ***sampling function***) arises frequently in ***Fourier transforms*** and signal processing and is defined as

$$\text{sinc}(x) \;\triangleq\; \begin{cases} \frac{\sin(x)}{x} & \text{for} \quad x \neq 0 \\ 1 & \text{for} \quad x \;=\; 0 \end{cases}$$

2.7 Types of scalars: Variable, specified, constant

- An ***independent variable*** is a quantity that varies independently, i.e., it does not depend on other variables. Many dynamic systems have one independent variable, namely ***time*** t.
- A ***dependent variable*** is a quantity whose value depends on the independent variable and its dependence is considered to be **unknown**, e.g., governed by an algebraic or differential equation.
- A ***specified variable*** is a quantity that varies in a **known** way, e.g., it is ***prescribed*** as a function of constants, time, and other variables, such as $x = \sin(t)$.
- A ***constant*** is a quantity whose value does not change (a constant may be **known** or **unknown**).

2.8 Differentiation

2.8.1 Definition of an ordinary derivative of a scalar function

When a function f depends on **1** scalar variable t, it is denoted $f(t)$. The ordinary ***1st-derivative*** of f with respect to t is denoted in various ways as shown in eqn (14).[a]

$$\boxed{f' \;=\; \dot{f} \;=\; \frac{df}{dt} \;=\; \lim_{h \to 0} \frac{f(t+h) - f(t)}{h}} \qquad (14)$$

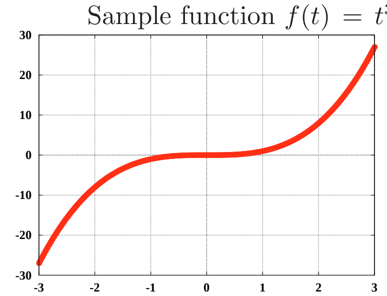
[a] The derivative notation using a ratio of differentials $\frac{df}{dt}$ was invented by Leibniz in 1675, the dot-notation $\dot{f}$ by Newton $\approx$ 1675, the prime notation f' by Lagrange in 1797, and the limit notation by Cauchy and Weierstrauss in 1850.

The ***2nd-derivative*** of $f(t)$ with respect to t is the derivative with respect to t of the derivative of $f(t)$ with respect to t.

$$\boxed{f'' \;=\; \ddot{f} \;=\; \frac{d^2 f}{dt^2} \;\triangleq\; \frac{d}{dt}\Big(\frac{df}{dt}\Big)}$$

There are various ways to denote a 2^{nd}-derivative.

Sample function $f(t) \;=\; t^3$



From a geometric (Newton's) perspective, the 1^{st}-derivative is ***slope*** and the 2^{nd}-derivative is ***curvature***. From a function (Euler's) perspective, the derivative of a function is a function.

2.8.2 Definition of a partial derivative of a scalar function

When a function f depends on n independent scalar variables $t_1 \ldots t_n$, it is denoted $f(t_1 \ldots t_n)$.[9] There are n quantities $\frac{\partial f}{\partial t_i}$ called first ***partial derivatives*** of f with respect to t_i, defined as

$$\frac{\partial f}{\partial t_i} \;\triangleq\; \lim_{h \to 0} \frac{f(t_1, \;\ldots\; t_i + h, \;\ldots\; t_n) \;-\; f(t_1, \;\ldots\; t_i, \;\ldots\; t_n)}{h} \qquad (i = 1 \ldots n) \qquad (15)$$

The definition of the ***partial derivative*** of f with respect to t in equation (15) reduces to the ***ordinary derivative*** of f with respect to t when f is a function of ***one*** independent variable,[10] i.e., $\frac{df}{dt} = \frac{\partial f}{\partial t}$.

Since $\frac{\partial f}{\partial t_i}$ is defined as a limit and is not a ratio of differentials, one cannot cancel the ∂t_i in the denominator by multiplying through by ∂t_i. In other words ∂t_i is not an entity in its own right.

2.8.3 Definition of the differential of an independent variable and scalar function

The ***differentials*** of the independent scalar variables $t_1 \ldots t_n$, are denoted $dt_1 \ldots dt_n$, and defined as arbitrary **non-zero** scalar quantities having the same dimension (units) as $t_1 \ldots t_n$. When a scalar variable f is regarded as a function of n independent scalar variables $t_1 \ldots t_n$, one may define the:

Differential of the function f

$$\boxed{df \;\triangleq\; \frac{\partial f}{\partial t_1} * dt_1 \;+\; \frac{\partial f}{\partial t_2} * dt_2 \;+\; \ldots \; \frac{\partial f}{\partial t_n} * dt_n} \qquad (16)$$

Leibniz regarded differentials as "infinitesimal" whereas Cauchy (tutored by Laplace and Lagrange, inventor of limits) did not.

When f is regarded as a function of ***one*** scalar variable t, equation (16) simplifies as shown below-left.

Since the ***differential*** dt is defined as a **non-zero** scalar quantity, divide by dt to produce the **ratio** of df to dt, i.e.,

$$df \underset{(16)}{=} \frac{\partial f}{\partial t} * dt \qquad \Rightarrow \qquad \frac{df}{dt} \;=\; \frac{\partial f}{\partial t} \qquad (17)$$

Hence, when f is a function of ***one*** independent variable t, the symbol $\frac{df}{dt}$ can means both a **ratio** of the differential df to the differential dt and as a **limit** (or ***ordinary derivative***) in the sense of equation (14). Although "overloading" the symbol $\frac{df}{dt}$ may be confusing, it is useful - particular for integration.

[9] Euler invented the function notation, e.g., $f(t)$, $f(x, y)$, circa 1730.

[10] Synonyms for ordinary (as in ordinary derivative) are "plain" and "boring" because f is a function of only ***one*** variable, whereas a "hot and spicy" partial derivative is a function of ***two or more variables***.

2.8.4 Definition of the total derivative of a scalar function (a better chain rule)

- At times, a function f is regarded as depending on **1** scalar quantity t and denoted $f(t)$.
- Alternatively, f may be denoted $f(x_1(t), \dots x_n(t),\, t)$ to signify that f is regarded as a function of ***n*+1** scalar variables $x_1 \dots x_n$ and t, where $x_1 \dots x_n$ are themselves functions of t.

When f is denoted $f(x_1(t), \dots x_n(t),\, t)$, the ordinary derivative of f with respect to t is called the ***total derivative*** of f with respect to t and can be calculated as

$$\boxed{\frac{df}{dt} \;=\; \frac{\partial f}{\partial x_1}\underbrace{\frac{dx_1}{dt}}_{\dot{x}_1} \;+\; \frac{\partial f}{\partial x_2}\underbrace{\frac{dx_2}{dt}}_{\dot{x}_2} \;+\; \dots \;\frac{\partial f}{\partial x_n}\underbrace{\frac{dx_n}{dt}}_{\dot{x}_n} \;+\; \frac{\partial f}{\partial t}} \qquad (18)$$

2.8.5 Short table of derivatives frequently encountered in engineering

Function and its derivative		Function and its derivative	
$f(t) \;=\; t^n$	$\frac{df}{dt} \;=\; \boxed{n * t^{n-1}}$ n = constant	$f(t) \;=\; \ln(t)$	$\frac{df}{dt} \;=\; \boxed{\frac{1}{t}}$ Note: $\frac{1}{t} \;=\; t^{-1}$
$f(t) \;=\; \sin(t)$	$\frac{df}{dt} \;=\; \boxed{\cos(t)}$	$f(t) \;=\; e^t$	$\frac{df}{dt} \;=\; \boxed{e^t}$ important for ODEs $e \;=\; 2.71828\dots$
$f(t) \;=\; \cos(t)$	$\frac{df}{dt} \;=\; \boxed{\text{-}\sin(t)}$	$f(t) \;=\; \tan(t)$	$\frac{df}{dt} \;=\; \frac{1}{\cos^2(t)}$
$f(t) \;=\; \int\limits_{x=t_0}^{t} f(x)\,dx$	$\frac{df}{dt} \;=\; f(t)$ ***Fundamental Theorem of Calculus***		
$f(t) \;=\; \int\limits_{s=g(t)}^{h(t)} f(s,t)\,ds$	$\frac{\partial f}{\partial t} \;=\; \int\limits_{s=g(t)}^{h(t)} \frac{\partial f(s,t)}{\partial t}\,ds \;+\; f[s\!=\!h(t),\, t]\,\frac{d[h(t)]}{dt} \;-\; f[s\!=\!g(t),\, t]\,\frac{d[g(t)]}{dt}$ ***Leibniz's integral rule***		
$\frac{\partial}{\partial q_r}\big(\frac{df}{dt}\big) \;=\; \frac{d}{dt}\big(\frac{\partial f}{\partial q_r}\big)$	$\frac{\partial}{\partial \dot{x}}\big(\frac{df}{dt}\big) \;=\; \frac{d}{dt}\big(\frac{\partial f}{\partial \dot{x}}\big) \;+\; \frac{\partial f}{\partial x}$ ***Interchange property of partial and ordinary derivatives***		

2.8.6 Example: Partial and ordinary differentiation

Example A: Consider a function $f(t)$ that only depends on **1** independent variable t (time), but which is expressed in terms of dependent variables x and y.

$f \;=\; \sin(x)\, y^2 \;+\; e^{3\,t}$
Both x and y depend on t.

Function f can also be ***regarded*** as a function $f(x, y, t)$ of **3** independent scalar quantities. In that context, partial derivatives of $f(x, y, t)$ with respect to x, y, t and the ordinary (total) derivative of f are

$$\frac{\partial f}{\partial x} = \cos(x)\, y^2 \qquad \frac{\partial f}{\partial y} = 2\, \sin(x)\, y \qquad \frac{\partial f}{\partial t} = 3\, e^{3\,t} \qquad \frac{df}{dt} = \cos(x)\, \dot{x}\, y^2 \;+\; 2\, \sin(x)\, y\, \dot{y} \;+\; 3\, e^{3\,t}$$

Example B: Consider a function $g(t)$ that depends on **1** independent variable t (time), but which is expressed in terms of a dependent variables x and $\dot{x}$.

$g \;=\; \sin(x)\, \dot{x}^2 \;+\; e^{3\,t}$
Both x and $\dot{x}$ depend on t.

Function g can also be ***regarded*** as a function $g(x, \dot{x}, t)$ of **3** independent scalar quantities. In that context, partial derivatives of $g(x, \dot{x}, t)$ with respect to x, $\dot{x}$, t and the ordinary (total) derivative of g are

$$\frac{\partial g}{\partial x} = \cos(x)\, \dot{x}^2 \qquad \frac{\partial g}{\partial \dot{x}} = 2\, \sin(x)\, \dot{x} \qquad \frac{\partial g}{\partial t} = 3\, e^{3\,t} \qquad \frac{dg}{dt} = \cos(x)\, \dot{x}^3 \;+\; 2\, \sin(x)\, \dot{x}\, \ddot{x} \;+\; 3\, e^{3\,t}$$

2.8.7 Good product rule for differentiation (for scalars, $\vec{\text{v}}$ectors, matrices, . . .)

Good product rule:
$$\boxed{\frac{\partial(u * v * w)}{\partial t} \;=\; \frac{\partial u}{\partial t} * v * w \;+\; u * \frac{\partial v}{\partial t} * w \;+\; u * v * \frac{\partial w}{\partial t}} \qquad (19)$$

Example:
$$\frac{\partial\big[\,t^2 * \sin(t) * e^t\,\big]}{\partial t} \;=\; 2\,t\, \sin(t)\, e^t \;+\; t^2\, \cos(t)\, e^t \;+\; t^2\, \sin(t)\, e^t$$

Unfortunately, many calculus books use the "**bad**" ***product rule for differentiation*** $\frac{d\,(u * v)}{dt} \;=\; \text{u} \;*\; \frac{dv}{dt} \;+\; \text{v} \;*\; \frac{du}{dt}$, which fails if u and v are $\vec{\text{v}}$ectors or matrices and is inefficient for differentiating 3^+ scalars (e.g., $u * v * w$). See Hw 1.17, 1.18.

2.8.8 Quotient rule for derivatives (or remember $\frac{u}{v} = u\,v^{-1}$ and use exponent and product rules)

The derivative of $\frac{u}{v}$ with respect to t can be calculated with the ***quotient-rule*** or with the ***product rule*** and exponents (hence no need to memorize the ***quotient-rule***).

$$\boxed{\frac{d}{dt}\left(\frac{u}{v}\right) \;=\; \frac{d\,(u\,v^{-1})}{dt} \;=\; \dot{u}\,v^{-1} \;-\; u\,v^{-2}\,\dot{v} \;=\; \frac{\dot{u}\,v \;-\; u\,\dot{v}}{v^2}} \qquad (20)$$

2.8.9 Chain rule for ordinary derivatives

When the variable x depends on the independent variable t, the ordinary derivative of the function $f(x)$ with respect to t can be calculated via the ***chain rule for differentiation*** in eqn (21).

$$\boxed{\frac{d\,f(x(t))}{dt} \;=\; \frac{d\,f(x)}{dx}\,\frac{dx}{dt}} \qquad (21)$$

2.8.10 Implicit differentiation: A useful tool for calculating derivatives

Example: In general, it is difficult to analytically solve a nonlinear equation to find y explicitly in terms of t. However, ***implicit differentiation*** calculates $\frac{dy}{dt}$ **without** first solving for y, e.g.,

$$y^2 \;+\; \sin(y) \;=\; \cos(t) \qquad \Rightarrow \qquad 2\,y\,\frac{dy}{dt} \;+\; \cos(y)\,\frac{dy}{dt} \;=\; \text{-}\sin(t) \qquad \Rightarrow \qquad \frac{dy}{dt} \;=\; \frac{\text{-}\sin(t)}{2\,y + \cos(y)}$$

Example: Implicit differentiation and ***natural logarithms*** to differentiate $y = c^t$ (c is a constant, t is time).
Note: When $c = e \approx 2.718$, $\frac{dy}{dt} = y$. This plays a **central role** in solving ordinary differential equations.

$$y = c^t \;\;\Rightarrow\;\; \ln(y) = t\,\ln(c) \;\;\Rightarrow\;\; d[\ln(y)] = \ln(c)\,dt \;\;\Rightarrow\;\; \tfrac{1}{y}\,dy = \ln(c)\,dt \;\;\Rightarrow\;\; \frac{dy}{dt} \;=\; \ln(c)\,y = \ln(c)\,c^t$$

2.9 Integration and a short table of integrals

Function	Integral of $f(t)$	
$f(t) = t^n$	$\int f(t)\,dt \;=\; \frac{t^{(n+1)}}{n+1} \;+\; C$	n is a number e.g., $n = 0.5$ but $n \neq$ -1.
$f(t) = t^{-1}$	$\int f(t)\,dt \;=\; \ln(t) \;+\; C$	
$f(t) = e^t$	$\int f(t)\,dt \;=\; e^t \;+\; C$	
$f(t) = \sin(t)$	$\int f(t)\,dt \;=\; \text{-}\cos(t) \;+\; C$	
$f(t) = \cos(t)$	$\int f(t)\,dt \;=\; \sin(t) \;+\; C$	

An ***integral*** can be regarded as a ***sum*** or as an ***anti-derivative***. From a geometric (Newton's) perspective, a ***definite integral*** can describe area under a curve, displacement, or volume. From a function (Euler's) perspective, the integral of a function is a function.

The website www.WolframResearch.com calculates integrals.

History: In 1675, Leibniz invented the integral notation $\int$ (Latin abbreviation for summa - sum) and its natural extension to double and triple integrals. Newton's integral notation was so defective, it was never popular – even in England. Euler was the first to use a symbol for an integral's limits, and its modern notation, e.g., $\int_a^b x\,dx$, was invented by Fourier in 1820.

2.10 Solutions of *polynomial* equations (roots) quadratic equation

Polynomial equations are a special class of nonlinear algebraic equations. A special polynomial equation is the ***quadratic equation***, which is a polynomial equation of degree **2**. Shown below is a quadratic equation in x and its **2** solutions ***(roots)***. Note: The two solutions for x are imaginary or complex if $b^2 - 4\,a\,c < 0$.

Quadratic equation

$$\boxed{a\,x^2 \;+\; b\,x \;+\; c \;=\; 0}$$

Solution to quadratic equation

$$\boxed{x \;=\; \frac{\text{-}b \;+\; \sqrt{b^2 - 4\,a\,c}}{2\,a} \qquad \text{and} \qquad x \;=\; \frac{\text{-}b \;-\; \sqrt{b^2 - 4\,a\,c}}{2\,a}}$$

Two other polynomial equations with "closed-form solutions" are the ***cubic*** and ***quartic*** equations:

$$x^3 \;+\; c_2\,x^2 \;+\; c_1\,x \;+\; c_0 \;=\; 0$$
$$x^4 \;+\; c_3\,x^3 \;+\; c_2\,x^2 \;+\; c_1\,x \;+\; c_0 \;=\; 0$$

The ***Fundamental Theorem of Algebra*** states that any polynomial of degree n with complex coefficients has n complex roots.[11] In 1824, Abel proved that no general closed-form solution for 5^{th}-order (or higher) polynomials exist. Numerical methods are useful for calculating roots of polynomials of any order.

2.11 Computer solutions of algebraic and differential equations

The invention of computers radically changed our ability to <u>form</u> and <u>solve</u> equations governing space, matter, and time – and <u>visualize</u> their results (plots, animation, virtual reality, etc). Shown right are equations that are usually solved with a computer.

Equation type	Example	Unknown
Linear algebraic	$A\,x = B$	x
Polynomial	$x^6 + 3\,x^3 + x + 9 = 0$	x
Eigen	$A\,v = \lambda v$	$\lambda,\ v$
Nonlinear algebraic	$x^2 - \cos^2(x) = 0$	x
Differential	$\ddot{\theta} + 4\,\sin(\theta) = \cos(t)$	$\theta(t)$

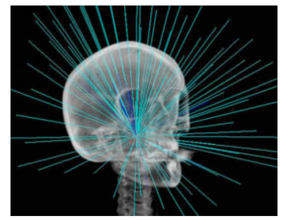
2.12 Optional: Continuous solutions of *nonlinear* algebraic equations

One way to find a continuous solution for x in the range $0 \le t \le 8$ for

$$x^2 - \cos^2(x) = 0.3\,\sin(t)$$

is to differentiate this ***nonlinear*** equation with respect to t and then solve the derivative equation that is ***linear*** in $\dot{x}$ as

$$2\,x\,\dot{x} + 2\,\cos(x)\,\sin(x)\,\dot{x} = 0.3\,\sin(t) \qquad \Rightarrow \qquad \dot{x} = \frac{0.3\,\cos(t)}{2\,x + 2\,\cos(x)\,\sin(x)}$$



Courtesy Accuray Inc.

Solving the nonlinear equation <u>**once**</u> at $t = 0$ gives $x(t=0) \approx 0.74$. With this initial value for x and continuous formula for $\dot{x}$, ODE techniques can numerically integrate $\dot{x}(t)$ to solve for $x(t)$.

Optional: Short history of differentiation

The modern differential notation $\frac{df}{dt}$ was introduced by Gottfried Leibniz in **1675** and relates to the ratio of differentials df and dt. The dot-notation $\dot{f}$ was introduced by Newton in his "***method of fluxions***" around **1675** and relates to his idea of flux (time-rates of change) of "***fluents***" (now called ***variables***). The prime notation f' was introduced by Lagrange in 1797 in his *Théorie des fonctiones analytiques*. Lagrange called $f'(t)$ the "derived function" of $f(t)$, from which the modern term ***derivative*** comes [9, pgs. 95-97]. An important concept introduced by Euler and Lagrange was that the derivative was a ***function*** which itself could be differentiated. Limit notation and its $\varepsilon - \delta$ definition were developed by Cauchy in **1823** and Seidel, Stokes, Bolzano, Weierstrauss in 1840^+.

Although ***Newton and Leibniz*** share the discovery of ***calculus***, their relationship was contentious - with Newton and Leibniz and their respective supporters alleging plagiarism and undermining each other's credibility. As President of the Royal Society, Newton appointed an "impartial" committee to decide whether he or Leibniz invented calculus. He wrote the committee's official published report (although not under his name) and then wrote a review (again anonymously) which appeared in the Philosophical Transactions of the Royal Society. Ironically, the introverted Newton died at 80-years old a national hero of England with a state funeral of highest honors whereas the sociable Leibniz's died at 70-years old, almost completely forgotten, with a funeral attended by only his secretary. Newton's daunting reputation intimidated British mathematicians. England did not produce another first-rate mathematician for over a century. Undaunted by their English neighbors, the rest of Europe, lead by the Bernoulli family, Leonard Euler, D'Alembert, Lagrange, Laplace, Fourier, and others, quickly expanded analytical analysis through differential equations, the calculus of variations, etc.

The order in which calculus is taught is misaligned with its history. Derivatives are usually taught starting with Cauchy's limits whereas derivatives were first ***used*** (Fermat and Descartes, 1637), then ***discovered*** (Newton and Leibniz, 1669-1684), then ***explored*** and ***developed*** (Taylor, Euler, Maclaurin, Lagrange, 1755-1797) and finally ***defined*** (Cauchy and Wiestrass, 1823-1861) [6]. Related: the ***Pythagorean theorem*** was used for thousands of years before its proof by Pythagoras $\approx$ 500 BC.

[11] The proof of the ***Fundamental Theorem of Algebra*** is difficult and was presented with various rigor between 1608 and 1981 by great mathematicians including, Roth (1608) Girard (1629), Leibniz (1702), Bernoulli (1742), d'Alembert (1746), Euler (1749), Lagrange (1772), Laplace (1795), Gauss (1799), Argand (**1806**), Gauss (again in 1816 and 1849), Cauchy (1821), Weierstrauss (1891), Hellmuth Kneser (1940), and his son Martin Kneser (**1981**).

2.13 Optional: Proofs of various trigonometric formulas

Trigonometric formulas can be tedious or difficult to prove. For example, a proof of the law of cosines can be found in the book Engineering Mechanics OnLine, by Thomas R. Kane and David A. Levinson, 1999.

2.13.1 Proof of the addition formula for the sine function

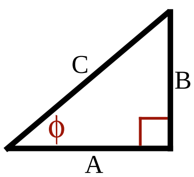
One way to prove eqns (4, 5) is via Euler's formula:
$$e^{i\,(\alpha+\beta)} \;=\; \cos(\alpha+\beta) \;+\; i\,\sin(\alpha+\beta) \qquad (22)$$

A second way to write $e^{i\,(\alpha+\beta)}$ uses the addition property of exponents as:
$$\begin{aligned} e^{i\,(\alpha+\beta)} \;&=\; e^{i\,\alpha} * e^{i\,\beta} \;=\; (\cos\alpha \,+\, i\,\sin\alpha) * (\cos\beta \,+\, i\,\sin\beta) \\ &=\; \cos\alpha\,\cos\beta - \sin\alpha\,\sin\beta \;+\; i\;(\sin\alpha\,\cos\beta + \sin\beta\,\cos\alpha) \qquad (23) \end{aligned}$$

Equating the real parts of the right hand-sides of eqns (22) and (23) leads directly to eqn (5). Similarly, equating the imaginary parts of the eqns (22) and (23) leads directly to eqn (4).

2.13.2 Geometrical proof of eqn (11), the amplitude-phase trigonometric identity

Multiplying $A\sin(x) + B\cos(x)$ [the left-hand side of eqn (11)] by $\frac{+\sqrt{A^2+B^2}}{+\sqrt{A^2+B^2}}$ gives the 1st expression below. Next, a right-triangle with sides A, B, C is drawn so its geometry forms fortuitous expressions.



$$A\,\sin(x) \,+\, B\,\cos(x) \;=\; \underbrace{+\sqrt{A^2+B^2}}_{C}\;\Big[\, \underbrace{\tfrac{A}{+\sqrt{A^2+B^2}}}_{\cos(\phi)}\,\sin(x) \;+\; \underbrace{\tfrac{B}{+\sqrt{A^2+B^2}}}_{\sin(\phi)}\,\cos(x)\,\Big]$$

$$\underset{(4)}{=}\; C\,\big[\,\underbrace{\cos(\phi)\,\sin(x) \;+\; \sin(\phi)\,\cos(x)}_{\sin(x+\phi)}\,\big] \qquad \text{Rewritten using eqn (4).}$$

2.13.3 Trigonometric proof of eqn (11), the amplitude-phase trigonometric identity

An alternative proof of eqn (11) begins with a guess and check that $C\cos(x+\phi)$ is equal to $A\sin(x) + B\cos(x)$.

Start of guess and check.
$$C\;\cos(x+\phi) \;\overset{?}{=}\; A\,\sin(x) \,+\, B\,\cos(x)$$

Expand via eqn (5).
$$C[\cos(x)\,\cos(\phi) - \sin(x)\,\sin(\phi)] \;\overset{?}{=}\; A\,\sin(x) \,+\, B\,\cos(x)$$

Bring all terms to left-hand-side.
$$\underbrace{[C\cos(\phi) - B]}_{0\,?}\,\cos(x) \;-\; \underbrace{[C\sin(\phi) + A]}_{0\,?}\,\sin(x) \;\overset{?}{=}\; 0$$

Since x may be assigned any value and $\cos(x)$ and $\sin(x)$ are linearly independent functions[12], hence the coefficients of both $\cos(x)$ and $\sin(x)$ must be zero (which means the **?** has been checked/answered as true). These two coefficients can be used to solve for the two unknowns C and ϕ.

$$\left.\begin{aligned} C\cos(\phi) \;&=\; B \\ C\sin(\phi) \;&=\; \text{-}A \end{aligned}\right\} \;\Rightarrow\; \begin{aligned} &C^2[\sin^2(\phi) \,+\, \cos^2(\phi)] \;=\; A^2 \,+\, B^2 &&\Rightarrow\quad C \underset{(2)}{=} +\sqrt{A^2+B^2} \\ &\cos(\phi) \;=\; \tfrac{B}{C} \quad\text{and}\quad \sin(\phi) \;=\; \tfrac{\text{-}A}{C} &&\Rightarrow\quad \phi \underset{(13)}{=} \texttt{atan2}(\text{-}A, B) \end{aligned}$$

Math helps design structures, robots, and satellites and predicts weather and saves lives.

[12] In other words, it is ***not*** possible to write $\cos(x) \;=\; a\,\sin(x) + b$ where a and b are constants (not functions of x). Another way to see that the coefficients must be zero is to recognize that x may be assigned ***any*** value. Choosing $x = 0$ results in $\cos(x) = 1$ and $\sin(x) = 0$, whereas choosing $x = \pi/2$ results in $\cos(x) = 0$ and $\sin(x) = 1$.