Chapter 2

Math review

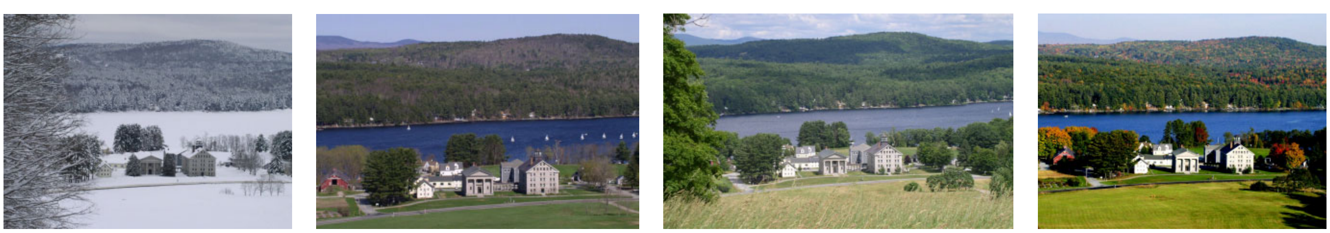
Courtesy NASA

2.1 Why math is important

Math is a foundation for science, medicine, engineering, construction, and business. Math provides ***concepts*** (pictures, words, ideas), ***calculations*** (mathematical operations, symbols, equations, definitions), and ***context*** (situations in which the concepts and calculations are relevant and useful). More generally, math is a language and set of rules that helps us count, quantify, calculate, manipulate, relate, define, extrapolate, and abstract "stuff".[1] Advances in math depend on pictures[2] words, symbols, equations, and precise ***definitions***.[3] For example, consider the following ***definition*** of π.

Object	Example	Approximate age of human comprehension
Picture	○	Toddlers
Spoken word	"circle"	Pre-school
Written word	"circle", "diameter", "circumference"	Elementary school
Symbol	d for diameter, c for circumference	Middle school
Equation	$c = \pi\, d$	Middle/high school
Definition	$\pi \triangleq \frac{c}{d}$	($\triangleq$ means "***defined as***") University

A 3-page math history for dynamic systems is at www.MotionGenesis.com ⇒ Textbooks ⇒ Dynamic Systems



Geometry and trigonometry help predict seasons. Courtesy Claude Rheaume LaSalette Enfield NH.

[1]For example, the "idea" of ***value*** (answering **"how much something is worth"**) is quantified through money.

[2]***Art*** is ***not*** reserved for the sophisticated and educated with knowledge and historical context for art. Appreciation for shapes, colors, and emotional expression in art is available to humans on a basic (primitive/subconscious) level.

[3]Kurt Godel (1906-1978) demonstrated that any reasonably powerful mathematical system contains seemingly true statements that cannot be proven.

2.2 Mathematical operations

1. ***Addition***: One way to add real numbers is to with a number line (e.g., a football field). For $3+5=8$, stand on the 3 yard line (with larger numbers right and smaller numbers left), then **move right** 5 yards to the answer (8). For $8 + {}^{-}2$, start on the 8 yard line and **move left** 2 yards (adding a negative number).

   The $+$ symbol has different operational significance in other contexts, e.g., adding vectors such as $\vec{\mathbf{a}} + \vec{\mathbf{b}}$ or adding two appropriately sized matrices as $A+B$. One representation[4] of a complex number with real part a and imaginary part $b\,i$ is $a + b\,i$. One way to add a real and imaginary number is with a field with two perpendicular axes (real and imaginary axes). To find $3 + 5*i$, start at the intersection of the real and imaginary axes (0,0) and **move right** 3 and **move forward** 5.

2. ***Subtraction*** and ***negation***: The subtraction dash used in $8-5$ "eight minus five" is called a ***binary minus*** whereas the negation dash used in ${}^{-}5$ "negative five" is called a ***unary minus***. Overloading the dash symbol $-$ with contextually different meanings is confusing and there are still unresolved order-of-operations negation conventions (see Section 2.2.1).

   Note: Before 1800 AD, negative numbers were treated with great suspicion. For example, Pascal regarded $0-4$ as utter nonsense. Maseres and Frend wrote algebra texts renouncing both negative and imaginary numbers on the grounds that mathematicians were unable to explain their use except by analogy.

   Subtraction is negation and addition (in calculators, computers, spreadsheets, ...)
   To avoid burdensome **memorization**, subtraction can be taught as negation and addition, e.g.,

   $$\sin(a-b) \;=\; \sin(a + {}^{-}b) \qquad \cos(a-b) \;=\; \cos(a + {}^{-}b) \qquad \frac{d\,(u-v)}{dt} = \frac{d\,(u + {}^{-}v)}{dt}$$

   The efficient method for teaching subtraction, called the ***addition method*** or ***Austrian method***:
   - Eliminates the unwieldy multi-column process of "***borrowing***".
   - Relies on a student's ability to ***add*** (eliminating memorization of subtraction tables).
   - Mimics the teaching of division (which relies on a student's ability to multiply).

3. ***Multiplication***: Multiplication of two integers is a convenient way to represent "multiple additions".
   Example: $3*5$ represents adding 3 five times, as $3*5 \;=\; 3+3+3+3+3$
   The $*$ symbol has different operational significance in other contexts. For example, one may multiply two complex numbers, two matrices, or two vectors $\vec{\mathbf{a}} * \vec{\mathbf{b}}$.

4. ***Division***: Both the horizontal fraction bar (e.g., in $\frac{3}{5}$) and diagonal fraction bar (in 3/5) denote division. The symbols and ideas in division are rooted in ***fractions*** and the ratio of two **integers** and date back to Egypt (3000 BC), Babylonia (2000 BC), and Greece (500 BC).

   Like subtraction, division is extraneous as **division is multiplication and exponentiation with ${}^{-}1$**, e.g., as shown right.

   $$\boxed{\frac{y}{x} \;=\; y * x^{-1}}$$

   Concomitantly, there is **no need to memorize formulas involving division**, e.g., the formula for $\frac{d}{dt}\left(\frac{u}{v}\right)$ is easily calculated using product rules and exponent rules for differentiation.

5. ***Exponents***: An integer exponent is a convenient way to represent a series of multiplications:
   Example: 3^5 represents multiplying 3 five times, as $3^5 \;=\; 3*3*3*3*3$
   Exponents with special names include 2 (***squared***), $\frac{1}{2}$ (***square root***), 3 (***cubed***), and $\frac{1}{3}$ (***cube root***).
   Note: There are unresolved left/right order-of-operations conventions for exponents (see Section 2.2.1).

6. ***Logarithms***: ***Exponents*** and ***logarithms*** are related. For example: $3^5 = 243$ and $\log_3 243 = 5$. A central issue of calculating logarithms is properly determining the ***logarithm's base***.

[4]Complex numbers are sometimes represented as (a, b). The imaginary number $\sqrt{-1}$ is usually represented by i or j.
Note: The complex number $a + b\,i$ is called a "***formal sum***" because the real quantity a to the left of the $+$ sign is not the same type as the imaginary quantity $b\,i$ to the right of the $+$ sign. Similarly, 3 apples + 5 oranges is a formal sum.

Properties of exponents and logarithms

Addition	$x^{a+b} = \boxed{x^a * x^b}$
Negation	$x^{-b} = 1/x^b$
Subtraction	$x^{a-b} = x^{a+-b} = x^a\,x^{-b} = \frac{x^a}{x^b}$ Subtraction properties can be deduced from addition and negation properties
Exponentiation	$(x^a)^b = (x^b)^a = x^{(a*b)}$ **Valid for some values of x, a, and b.** ***Invalid*** for negative x or non-integer a and b, e.g., $[(-4)^2]^{\frac{1}{2}} \neq [(-4)^{\frac{1}{2}}]^2$. Also, parentheses avoid confusion e.g., $2^{3^2} = (2^3)^2 = 8^2 = 64$ or $2^{3^2} = 2^{(3^2)} = 2^9 = 512$?
Multiplication	$\log(a*b) = \boxed{\log(a) + \log(b)}$
Exponentiation	$\log(a^n) = \boxed{n\,\log(a)}$
Fractions	$\log(\frac{a}{b}) = \log[\,(\frac{b}{a})^{-1}\,] = -\log(\frac{b}{a})$
Division	$\log(\frac{a}{b}) = \log(a*b^{-1}) = \log(a) + \log(b^{-1}) = \log(a) + -\log(b) = \log(a) - \log(b)$ Fraction and division properties can be deduced from multiplication and exponential properties.
Change in base	$\log_b(a) = \frac{\log_e(a)}{\log_e(b)}$. For example, $\log_{10}(a) = \frac{\log_e(a)}{\log_e(10)} \approx 0.4343\,\log_e(a)$

Math humor: Two adders (snakes) were sad because they had no offspring. A park ranger noticed their sorrows and built them a log table. Now they multiply by adding.

2.2.1 Order of operations: Established mathematical convention?

Order of operations is sometimes taught with the mnemonic ***PEMDAS*** (**P**arentheses, **E**xponents, **M**ultiplication, **D**ivision, **A**ddition, **S**ubtraction). PEMDAS is helpful for many calculations and is useful in reverse to solve equations. However, PEMDAS fails to address ***negation*** and the left-right order of exponents. and modern textbooks, compilers, calculators, math programs, spreadsheets, produce different values.[5] [6]

Calculation	MATLAB®	Python	Wolfram Alpha	Microsoft Excel®	Open Office	Google Calculator	**M**otion**G**enesis
$-\mathbf{3^2}$ =	-9	-9	-9	9	9	-9	-9
$\mathbf{2^{3^2}}$ =	64	512	512	64	64	512	Requires ()

2.2.2 Patterns in mathematics

There are many patterns in mathematics that are most recognizable with **three or more items**. This applies to the product rule for differentiation (see Section 2.8.7), the calculation of a determinate, and the transpose rule for matrices, e.g., $(A\,B\,C)^T = C^T\,B^T\,A^T$. For example, a simple, extensible way to clear parentheses is to start with the 1^{st} term inside the first parentheses and multiply it by each successive term inside the second parentheses, then restart with the 2^{nd} term inside the first parentheses, etc., e.g.,

$$(\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c})\,(d + e + f) \;=\; \boldsymbol{a}\,d + \boldsymbol{a}\,e + \boldsymbol{a}\,f \;+\; \boldsymbol{b}\,d + \boldsymbol{b}\,e + \boldsymbol{b}\,f \;+\; \boldsymbol{c}\,d + \boldsymbol{c}\,e + \boldsymbol{c}\,f$$

It is unfortunate that many students learn the mnemonic ***FOIL*** to multiply parenthesized expressions containing **two** terms. The ***FOIL*** method is not an efficient way to multiply parenthesized expressions containing **three** or more terms.

[5] MATLAB® and **M**otion**G**enesis report $-3^2 = -9$ whereas Microsoft Excel® and some compilers report $-3^2 = +9$. Advocates for -9 regard negation as a ***special case of multiplication*** so $-x \triangleq (-1) * x$ parallels imaginary numbers such as $i\,x \triangleq (\sqrt{-1}) * x$. Alternately, if - is regarded as an inherent part of 3, $(-3)^2 = 9$.

[6] In a 2013 poll of undergraduate and graduate Stanford engineers, 87 of 90 voted $2^{3^2} = 2^{(3^2)} = \mathbf{512}$ (right-to-left calculation) whereas only 3 of 90 voted $2^{3^2} = (2^3)^2 = \mathbf{64}$ (left-to-right calculation). Yet, all engineers agreed $10 - 5 + 3 = (10 - 5) + 3 = 8$ (left-to-right calculation) as opposed to $10 - (5 + 3) = 2$ (right-to-left calculation).

2.3 Unit systems - SI and U.S.

Units quantify the measurement of “stuff”. The ***SI*** system was first adopted by France on December 10, 1799 and is now used in all countries other than Liberia, Myanmar, and the United States.

Countries using SI units (green) vs. U.S. units (grey).

The ***SI*** (metric) system uses a base-10 number system and decimals (not fractions) and has measures for length, mass, force, temperature, time, etc.

NIST (National Institute of Standards & Technology) defines physical constants and conversion factors (e.g., conversion from U.S. to SI units).

Length	1 inch $\triangleq$ 2.54 cm		
Mass	1 lbm $\approx$ 0.45359237 kg	1 slug $\approx g_{\text{US}}$ lbm	$g_{\text{US}} \approx 32.17404855643044$
Force	1 Newton $\triangleq 1 \, \frac{\text{kg m}}{\text{s}^2}$	1 lbf $\triangleq 1 \, \frac{\text{slug ft}}{\text{s}^2}$	1 lbf $\triangleq g_{\text{US}} \, \frac{\text{lbm ft}}{\text{s}^2}$

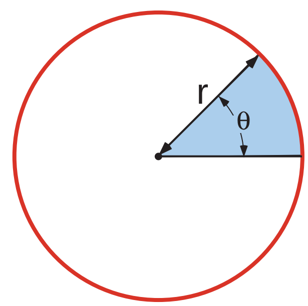
Inaccurate unit conversions have caused ***many*** failures. In 1999, NASA lost a \$125,000,000 Mars orbiter because one engineering team used SI units while another used U.S. units. In 1983, an Air Canada Boeing 767 ran out of fuel mid-flight because of a kg to lbm unit conversion.[7]

2.4 Geometry: Ancient Euclid and modern vectors

Geometry is the study of figures (e.g., lines, curves, surfaces, solids) and their properties (e.g., distance, area, and volume). Geometry plays a central role in construction, farming, engineering, medicine, science, etc.

Many students spend 2^+ years learning ancient ($\approx$300 BC) 2D Euclidean geometry and trigonometry (trigonometry translates to “triangle measurement”). The invention of ***vectors*** (Gibbs $\approx$ 1900 AD) and its easy-to-use vector addition, dot-products, and cross-products have **greatly simplified** 2D and 3D geometry. Unfortunately, few instructors teach geometry or trigonometry with vectors.

2.5 Circles and their properties



The ratio of ***any*** circle’s ***circumference*** to its ***diameter*** is the number[a]

$$\pi \;=\; 3.14159265358979323846264338327950288419716939937510582\ldots$$

$\boldsymbol{\pi}$ is called an “***irrational number***” because it is not a whole number or fraction, nor does it terminate or repeat. It is chaotic, disorderly, and has no discernible pattern (π has been memorized to $67,890^+$ digits).

The ***arc-length*** of a portion of the circle’s periphery and the ***area*** of a wedge of the circle can be calculated in terms of the circle’s ***radius*** r and the ***angle*** θ as shown right.[6]

Arc-length $= \theta\, r$	Area of wedge $= \frac{\theta}{2}\, r^2$
Circumference $= 2\,\pi\, r$	Area of circle $= \pi\, r^2$

[a] The symbol $\boldsymbol{\pi}$ was popularized by Euler circa 1750, but the value $\pi \approx 3.14$ was known in Egypt circa 3000 BC.[8]

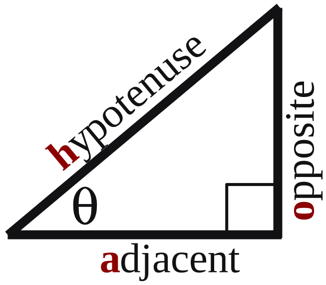
[7] Ironically, Thomas Jefferson helped United States become the first country (in 1792) to use a monetary system with decimals and a base-10 number system. The historical origin of U.S. units trace to 2575 B.C. and through ancient Egypt, Greece, and Rome. The ***inch*** approximates the width of a man’s thumb. The ***foot*** approximates a foot with shoe and was somewhat standardized in England to King Henry I. The ***mile*** “mille passus” is 1000 paces (2 steps) of a Roman soldier. An Australian study found that switching from British units to metric units freed $\frac{1}{2}$-year in science education. U.S. lawmakers have consistently failed to legislate changes in federal systems, e.g., road signs, NASA, DOD, and NSF.

[8] An ***angle*** involves two lines (or vectors) and is measured in radians or degrees. A radian is the ratio of the arc-length of part of a circle’s perimeter to its radius. A degree is an archaic unit of angle measurement based on the ancient Babylonian year which had 360 days (12 months $*$ 30 days). Each degree represents one day of Earth’s travel about the sun and the degree symbol’s circular appearance $^\circ$ is a reminder that 360° measures the Earth’s quasi-circular travel around the sun.

2.6 Triangles and ratios of their sides (sine, cosine, tangent)

A triangle ("three angles") is a 3-sided planar geometric shape widely used in construction, engineering, and science.

SohCahToa is a ***mnemonic*** for memorizing the definitions of ***Sine***, ***Cosine***, and ***Tangent*** (ratios of various sides of a right triangle).



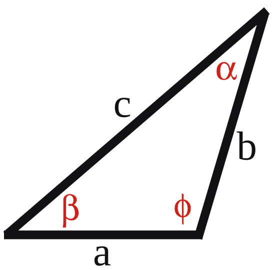
$$\sin(\theta) \triangleq \frac{\text{opposite}}{\text{hypotenuse}} \qquad \cos(\theta) \triangleq \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \tan(\theta) \triangleq \frac{opposite}{adjacent} = \frac{\sin(\theta)}{\cos(\theta)} \tag{1}$$

The ***Pythagorean theorem*** in equation (2) relates lengths of sides of a right triangle. Combining the definitions of $\sin(\theta)$ and $\cos(\theta)$ with the Pythagorean theorem gives the second relationship to the right.

$$\begin{aligned} \text{hypotenuse}^2 &= \text{adjacent}^2 + \text{opposite}^2 \\ \sin^2(\theta) + \cos^2(\theta) &\underset{(1)}{=} 1 \end{aligned} \tag{2}$$

Note: Numbers under = refer to equation numbers, e.g., (1) under = means "refers to equation (1)".

2.6.1 Properties of sine and cosine and useful trigonometric formulas



Law of cosines	Euclid of Alexandria Egypt, 300 BC
Law of sines	Ptolemy of Alexandria Egypt, 100 AD
Addition formula for sine	Ptolemy of Alexandria Egypt, 100 AD

$$c^2 = a^2 + b^2 - 2\,a\,b\cos(\phi) \qquad \textit{Law of cosines} \tag{3}$$

$$\frac{\sin(\alpha)}{a} = \frac{\sin(\beta)}{b} = \frac{\sin(\phi)}{c} \qquad \textit{Law of sines} \tag{4}$$

$$\sin(-\alpha) = -\sin(\alpha) \tag{5}$$

$$\cos(-\alpha) = \cos(\alpha) \tag{6}$$

$$\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \sin(\beta)\cos(\alpha) \qquad \textit{Addition formula for sine} \tag{7}$$

$$\cos(\alpha+\beta) \underset{(7)}{=} \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta) \qquad \textit{Addition formula for cosine} \tag{8}$$

$$\sin(x) = \sin(x + 2\,\pi\,n) \qquad \text{n} = 1, 2, 3, \ldots \qquad \text{Sine is periodic} \tag{9}$$

$$-\sin(x) \underset{(7)}{=} \sin(x \pm \pi\,n) \qquad \text{n} = 1, 3, 5, \ldots \tag{10}$$

$$\cos(x) = \cos(x + 2\,\pi\,n) \qquad \text{n} = 1, 2, 3, \ldots \qquad \text{Cosine is periodic} \tag{11}$$

$$-\cos(x) \underset{(8)}{=} \cos(x \pm \pi\,n) \qquad \text{n} = 1, 3, 5, \ldots \tag{12}$$

$$\sin(x) \underset{(8)}{=} \cos(x - \frac{\pi}{2}) = \cos(-x + \frac{\pi}{2}) \tag{13}$$

$$\cos(x) \underset{(7)}{=} \sin(x + \frac{\pi}{2}) = \sin(-x + \frac{\pi}{2}) \tag{14}$$

$$\sin(x) \underset{(7)}{=} 2\sin(\frac{x}{2})\cos(\frac{x}{2}) \quad \textbf{or} \quad \sin(2\,x) = 2\sin(x)\cos(x) \tag{15}$$

$$\sin^2(x) \underset{(8)}{=} \frac{1 - \cos(2\,x)}{2} \qquad \cos^2(x) \underset{(8)}{=} \frac{1 + \cos(2\,x)}{2} \tag{16}$$

$$\sin^2(\frac{x}{2}) \underset{(16)}{=} \frac{1 - \cos(x)}{2} \quad \textbf{or} \quad \cos(x) = 1 - 2\sin^2(\frac{x}{2}) \tag{17}$$

$$\cos^2(\frac{x}{2}) \underset{(16)}{=} \frac{1 + \cos(x)}{2} \quad \textbf{or} \quad \cos(x) = 2\cos^2(\frac{x}{2}) - 1 \tag{18}$$

$$\cos(b) - \cos(a) \underset{(7)}{=} 2\sin(\frac{a+b}{2})\sin(\frac{a-b}{2}) \qquad \text{Useful for beat phenomenon analysis} \tag{19}$$

$$\cos(\omega_2\, t + \phi_2) - \cos(\omega_1\, t + \phi_1) \underset{(19)}{=} 2 \sin\left[(\frac{\omega_1 + \omega_2}{2})\, t + \frac{\phi_1 + \phi_2}{2} \right] \sin\left[(\frac{\omega_1 - \omega_2}{2})\, t + \frac{\phi_1 - \phi_2}{2} \right] \quad (20)$$

2.6.2 Sine and cosine as functions (Euler, circa 1730)

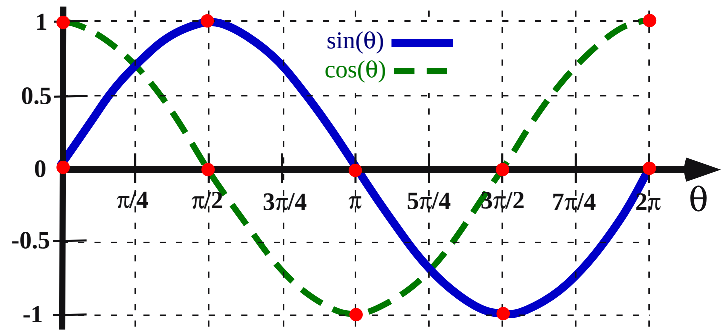
Euler's interpretation of ***cosine*** and ***sine*** as ***functions*** (not just ratios of sides of a triangle) was a major advance for trigonometry and functions.[9]

$$\boxed{\cos(\theta) \triangleq \frac{\textbf{a}\text{djacent}}{\textbf{h}\text{ypotenuse}}}$$

Cosine function

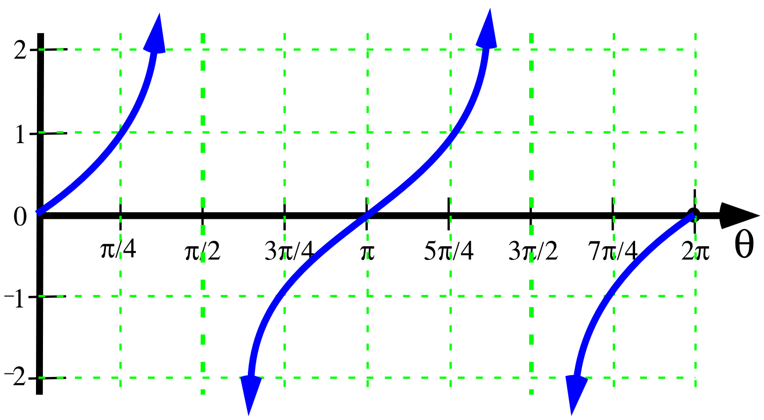
$$\boxed{\sin(\theta) \triangleq \frac{\textbf{o}\text{pposite}}{\textbf{h}\text{ypotenuse}}}$$

Sine function



$$\boxed{\tan(\theta) \triangleq \frac{\textbf{o}\text{pposite}}{\textbf{a}\text{djacent}} = \frac{\sin(\theta)}{\cos(\theta)}}$$

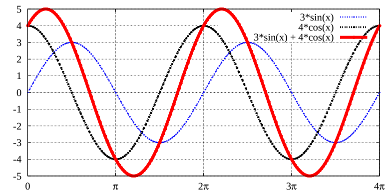
Tangent function



Sine and cosine are two of the most important functions in dynamics

[9]The Babylonians used properties of right triangles for thousands of years before their proofs by Pythagoras of Samos [≈500 BC]. The definitions of ***sine***, ***cosine***, and ***tangent*** as ratios of sides of a right triangle predate 140 BC when the Greek Hipparchus made sine, cosine, and tangent tables. Euler's interpretation of sine, cosine, and tangent as ***functions*** was a breakthrough for math. Gibb's invention of vectors (≈1900 AD) significantly simplified 3D geometry and trigonometry and proofs of ***law of cosines***, ***law of sines***, and ***sine addition formula***, from which other trigonometric formulas are derived [***cosine addition formula*** (Homework 2.2), ***half-angle formulas***, ***double-angle formulas***, etc.]. The trigonometric identities to prove equation (19) include $\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b)$, $\sin^2(x) + \cos^2(x) = 1$, and $\cos^2(\frac{x}{2}) = \frac{1+\cos(x)}{2}$.

2.6.3 The amplitude-phase formulas for sine and cosine



Two trigonometric identities that are particularly helpful in dynamic systems are the ***amplitude/phase formulas for sine and cosine.***[a]

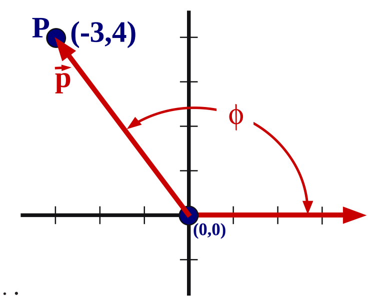
[a] These amplitude-phase formulas are used extensively in vibration analysis. These formulas use `atan2` because A and B may be **positive**, **negative**, or **zero**. These formulas are proved in Sections 2.12.2 and 2.12.3.

$$\boxed{A\sin(x) + B\cos(x) \;=\; C\sin(x+\phi_s) \quad \texttt{where}\;\; C \;=\; {}^{+}\sqrt{A^2+B^2} \quad \texttt{and}\;\; \phi_s \;=\; \texttt{atan2}(\,B, A)} \tag{21}$$

$$\boxed{A\sin(x) + B\cos(x) \;=\; C\cos(x+\phi_c) \quad \texttt{where}\;\; C \;=\; {}^{+}\sqrt{A^2+B^2} \quad \texttt{and}\;\; \phi_c \;=\; \texttt{atan2}(-A, B)} \tag{22}$$

2.6.4 The function atan2(y, x)

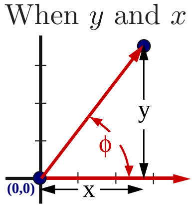
The ***atan2*** function is named because: it is similar to the atan (arc-tangent) function; it takes two arguments; it calculates an angle ϕ with range $-\pi < \phi \le \pi$ (2 times atan function's range of $\frac{-\pi}{2} < \phi \le \frac{\pi}{2}$).



To determine the angle $\phi = \texttt{atan2}(\,y,\,x\,)$

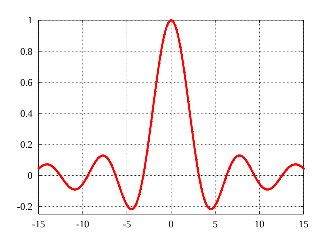
- Draw horizontal and vertical axes as shown right.
- Draw a point P located at the designated y and x values.
  Example: For $\texttt{atan2}(4, -3)$, draw point P at $y = 4$ and $x = -3$.
- Draw a vector $\vec{\mathbf{p}}$ from $(0, 0)$ to point P.
- Draw angle ϕ from the ^{+}x-axis to $\vec{\mathbf{p}}$, with $^{+}$counter-clockwise sense.
- Using trigonometry, calculate the value of ϕ, e.g., $\phi \;=\; {}^{+}2.21$ rads.
- Alternately, calculate $\texttt{atan2}(\,y,\,x\,)$ with MATLAB®, **M**otion**G**enesis, Java, C, C^{++}, ...

The function $\phi = \texttt{atan2}(\,y,\,x\,)$ returns an angle that satisfies both $\sin(\phi)$ and $\cos(\phi)$ as shown below. When y and x are continuous, ordinary or partial derivatives of `atan2` can be calculated (proof in Section 2.8.10).



$$\left.\begin{aligned} \sin(\phi) \;&=\; \frac{y}{{}^{+}\sqrt{x^2 \,+\, y^2}} \\ \cos(\phi) \;&=\; \frac{x}{{}^{+}\sqrt{x^2 \,+\, y^2}} \end{aligned}\right\} \;\Rightarrow\; \begin{aligned} &\boxed{\phi \;=\; \texttt{atan2}(\,y,\,x\,)} \quad -\pi < \phi \le \pi \\ &\boxed{\dot{\phi} \;=\; \frac{x\,\dot{y} \,-\, y\,\dot{x}}{x^2 \,+\, y^2}} \quad \boxed{\frac{\partial\theta}{\partial s} \;=\; \frac{x\,\frac{\partial y}{\partial s} \,-\, y\,\frac{\partial x}{\partial s}}{x^2 \,+\, y^2}} \end{aligned} \tag{23}$$

2.6.5 Optional: The sinc function



The sinc ***function*** (also call the ***sine cardinal*** or ***sampling function***) arises frequently in ***Fourier transforms*** and signal processing and is defined as

2.7 Types of scalars: Variable, Specified, Constant

- An ***independent variable*** is a quantity that varies independently, i.e., it does not depend on other variables. Many dynamic systems have one independent variable, namely ***time*** t.
- A ***dependent variable*** is a quantity whose value depends on the independent variable and its dependence is considered to be **unknown**, e.g., governed by an algebraic or differential equation.
- A ***specified variable*** is a quantity that varies in a **known** way, e.g., it is ***prescribed*** as a function of constants, time, and other variables, such as $x = \sin(t)$.
- A ***constant*** is a quantity whose value does not change (a constant may be **known** or **unknown**).

2.8 Differentiation

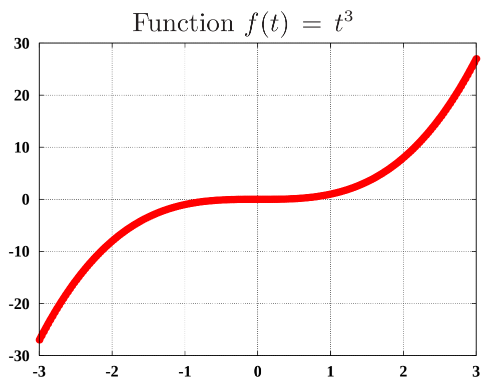
2.8.1 Definition of an ordinary derivative of a scalar function

When a function f is regarded to depend on **1** scalar variable t, it is denoted $f(t)$.

The ordinary ***1st-derivative*** of f with respect to t" is denoted in various ways as shown in equation (24).[a]

$$\boxed{f' \;=\; \dot{f} \;=\; \frac{df}{dt} \;=\; \lim_{h \to 0} \frac{f(t+h) \;-\; f(t)}{h}} \qquad (24)$$

[a]The notation using a ratio (fraction) of differentials $\frac{df}{dt}$ was invented by Leibniz in 1675, the dot-notation $\dot{f}$ by Newton ≈ 1675, the prime notation f' by Lagrange in 1797, and the limit notation by Cauchy and Weierstrauss in 1850.



Geometrically, the 1^{st}-derivative is ***slope*** (e.g., the ***slope*** of t^3 is $3\,t^2$).

The derivative of the derivative with respect to t is called the "***2nd-derivative*** of $f(t)$ with respect to t", and is denoted in various ways as shown below.

$$\boxed{f'' \;=\; \ddot{f} \;=\; \frac{d^2 f}{dt^2} \;\triangleq\; \frac{d}{dt}\left(\frac{df}{dt}\right)}$$

Geometrically, the second derivative is ***curvature***.

For example, the ***curvature*** of $f(t) = t^3$ is $\frac{d^2 f}{dt^2} = 6\,t$.

2.8.2 Definition of a partial derivative of a scalar function

When a function f depends on n independent scalar variables $t_1, \ldots, t_n$, it is denoted $f(t_1, \ldots, t_n)$.[10]
There are n quantities $\frac{\partial f}{\partial t_i}$ called "first ***partial derivatives*** of f with respect to t_i", defined as

$$\frac{\partial f}{\partial t_i} \;\triangleq\; \lim_{h \to 0} \frac{f(t_1, \ldots,\;\; t_i + h,\;\; \ldots, t_n) \;-\; f(t_1, \ldots,\;\; t_i,\;\; \ldots, t_n)}{h} \qquad (i = 1, \ldots, n) \qquad (25)$$

The definition of the ***partial derivative*** of f with respect to t in equation (25) reduces to the ***ordinary derivative*** of f with respect to t when f is a function of ***one*** independent variable,[11] i.e., $\frac{df}{dt} = \frac{\partial f}{\partial t}$.

Since $\frac{\partial f}{\partial t_i}$ is defined as a limit and is not a ratio of differentials, one cannot cancel the ∂t_i in the denominator by multiplying through by ∂t_i. In other words ∂t_i is not an entity in its own right.

2.8.3 Definition of the differential of an independent variable and scalar function

The ***differentials*** of the independent scalar variables $t_1, \ldots, t_n$ are denoted $dt_1, \ldots, dt_n$ and defined as arbitrary **non-zero** scalar quantities having the same dimension (units) as $t_1, \ldots, t_n$. When a scalar variable f is regarded as a function of n independent scalar variables $t_1, \ldots, t_n$, one may define the:

differential of the function f
$$\boxed{df \;\triangleq\; \frac{\partial f}{\partial t_1} * dt_1 \;+\; \frac{\partial f}{\partial t_2} * dt_2 \;+\; \ldots \; \frac{\partial f}{\partial t_n} * dt_n} \qquad (26)$$

Leibniz regarded differentials as "infinitesimal" whereas Cauchy (tutored by Laplace and Lagrange, inventor of limits) did not.

When f is regarded as a function of ***one*** scalar variable t, equation (26) simplifies as shown below-left.

Since the ***differential*** dt is defined as a **non-zero** scalar quantity, divide by dt to produce the **ratio** of df to dt, i.e.,

$$df \underset{(26)}{=} \frac{\partial f}{\partial t} * dt \qquad \Rightarrow \qquad \frac{df}{dt} \;=\; \frac{\partial f}{\partial t} \qquad (27)$$

[10]Euler invented the function notation, e.g., $f(t)$, $f(x, y)$, circa 1730.
[11]Synonyms for ordinary (as in ordinary derivative) are "plain" and "boring" because f is a function of only ***one*** variable, whereas a "hot and spicy" partial derivative is a function of ***two or more variables***.

Hence, when f is a function of ***one*** independent variable t, the symbol $\frac{df}{dt}$ can means both a **ratio** of the differential df to the differential dt and as a **limit** (or ***ordinary derivative***) in the sense of equation (24). Although "overloading" the symbol $\frac{df}{dt}$ may be confusing, it is useful - particular for integration.

2.8.4 Definition of the total derivative of a scalar function

At times, a function f can be regarded as either depending on **1** scalar quantity t, or regarded as a function of $\mathbf{n+1}$ scalar quantities $x_1, \ldots, x_n$ and t, where $x_1, \ldots, x_n$ are themselves functions of t. When f is regarded as a function of $x_1, \ldots, x_n$ and t, f is denoted $f(x_1(t), \ldots, x_n(t), t)$, and the ordinary derivative of f with respect to t is called the ***total derivative*** of f with respect to t and can be calculated as

$$\boxed{\begin{aligned}\frac{df}{dt} &= \frac{\partial f}{\partial x_1} * \frac{dx_1}{dt} + \frac{\partial f}{\partial x_2} * \frac{dx_2}{dt} + \ldots \frac{\partial f}{\partial x_n} * \frac{dx_n}{dt} + \frac{\partial f}{\partial t} \\ &= \frac{\partial f}{\partial x_1} \dot{x}_1 + \frac{\partial f}{\partial x_2} \dot{x}_2 + \ldots \frac{\partial f}{\partial x_n} \dot{x}_n + \frac{\partial f}{\partial t}\end{aligned}} \tag{28}$$

2.8.5 Short table of derivatives frequently encountered in engineering

Function and its derivative			Function and its derivative		
$F(t) = \sin(t)$	$\frac{\partial F}{\partial t} = \boxed{\cos(t)}$		$F(t) = \cos(t)$	$\frac{\partial F}{\partial t} = \boxed{-\sin(t)}$	
$F(t) = t^n$	$\frac{\partial F}{\partial t} = \boxed{n * t^{n-1}}$	n = constant	$F(t) = \tan(t)$	$\frac{\partial F}{\partial t} = \frac{1}{\cos^2(t)}$	
$F(t) = \ln(t)$	$\frac{\partial F}{\partial t} = \boxed{t^{-1} = \frac{1}{t}}$		$F(t) = e^t$	$\frac{\partial F}{\partial t} = \boxed{e^t}$	important for ODEs $e = 2.71828\ldots$
$F(t) = \int_{x=t_0}^{t} f(x)\, dx$	$\frac{\partial F}{\partial t} = f(t)$	***Fundamental Theorem of Calculus***			
$F(t) = \int_{s=g(t)}^{h(t)} f(s,t)\, ds$	$\frac{\partial F}{\partial t} = \int_{s=g(t)}^{h(t)} \frac{\partial f(s,t)}{\partial t}\, ds - f[s=g(t),\, t]\, \frac{d\,[g(t)]}{dt} + f[s=h(t),\, t]\, \frac{d\,[h(t)]}{dt}$				

2.8.6 Example: Partial and ordinary differentiation

Example A: Consider a function f that only depends on **1** independent variable t (time), but which is expressed in terms of dependent variables x and y (both x and y depend on t). The function f can also be ***regarded*** as a function of **3** independent scalar quantities (x, y, t).

$$f(x, y, t) = \sin(x)\, y^2 + e^{3\,t}$$

Partial derivatives of $f(x, y, t)$ with respect to x, y, or t and the ordinary (total) derivative of f are

$$\frac{\partial f}{\partial x} = \cos(x)\, y^2 \qquad \frac{\partial f}{\partial y} = 2\, \sin(x)\, y \qquad \frac{\partial f}{\partial t} = 3\, e^{3\,t} \qquad \frac{df}{dt} = \cos(x)\, \dot{x}\, y^2 + 2\, \sin(x)\, y\, \dot{y} + 3\, e^{3\,t}$$

Example B: Consider a function g that depends on **1** independent variable t (time), but which is expressed in terms of a dependent variable x and its ordinary time-derivative $\dot{x}$. The function g can also be ***regarded*** as a function of **3** independent scalars $(x, \dot{x}, t)$ as

$$g(x, \dot{x}, t) = \sin(x)\, \dot{x}^2 + e^{3\,t}$$

Partial derivatives of $g(x, \dot{x}, t)$ with respect to x, $\dot{x}$, or t and the ordinary (total) derivative of g are

$$\frac{\partial g}{\partial x} = \cos(x)\, \dot{x}^2 \qquad \frac{\partial g}{\partial \dot{x}} = 2\, \sin(x)\, \dot{x} \qquad \frac{\partial g}{\partial t} = 3\, e^{3\,t} \qquad \frac{dg}{dt} = \cos(x)\, \dot{x}^3 + 2\, \sin(x)\, \dot{x}\, \ddot{x} + 3\, e^{3\,t}$$

2.8.7 Good product rule for differentiation (for scalars, $\vec{\text{v}}$ectors, matrices, ...)

Good product rule:
$$\boxed{\frac{\partial(u * v * w)}{\partial t} \;=\; \frac{\partial u}{\partial t} * v * w \;+\; u * \frac{\partial v}{\partial t} * w \;+\; u * v * \frac{\partial w}{\partial t}} \tag{29}$$

Example:
$$\frac{\partial\left[\, t^2 * \sin(t) * e^t \,\right]}{\partial t} \;=\; 2\,t\,\sin(t)\,e^t \;+\; t^2\,\cos(t)\,e^t \;+\; t^2\,\sin(t)\,e^t$$

Unfortunately, many calculus books use the "**bad**" ***product rule for differentiation*** $\frac{d\,(u*v)}{dt} = \text{u} * \frac{dv}{dt} + \text{v} * \frac{du}{dt}$, which fails if u and v are $\vec{\text{v}}$ectors or matrices, and is inefficient for differentiating 3^+ scalars (e.g., $u * v * w$).

2.8.8 Quotient rule for derivatives: Use exponents and the product rule

Since the quotient $\frac{u}{v}$ is equivalent to $u\,v^{-1}$, the derivative of $\frac{u}{v}$ with respect to t can be implemented with the ***product rule*** and exponents (without memorizing special ***quotient-rule*** formulas).

$$\boxed{\frac{\partial}{\partial t}\left(\frac{u}{v}\right) \;=\; \frac{\partial u}{\partial t}\,v^{-1} \;-\; u\,v^{-2}\,\frac{\partial v}{\partial t}} \tag{30}$$

2.8.9 Chain rule for derivatives

When the variable x depends on the variable t, the derivative of the function $f(x)$ with respect to t can be written via the ***chain rule for differentiation*** as shown in equation (31).

$$\boxed{\frac{\partial\, f(x)}{\partial t} \;=\; \frac{\partial\, f(x)}{\partial x}\,\frac{\partial x}{\partial t}} \tag{31}$$

2.8.10 Implicit differentiation: A useful tool for calculating derivatives

Example: In general, it is difficult to solve the nonlinear equation below to find y explicitly in terms of t. However, ***implicit differentiation*** calculates $\frac{dy}{dt}$ **without** first solving for y, e.g.,

$$y^2 \;+\; \sin(y) \;=\; \cos(t) \qquad\Rightarrow\qquad 2\,y\,\frac{dy}{dt} \;+\; \cos(y)\,\frac{dy}{dt} \;=\; \text{-}\sin(t) \qquad\Rightarrow\qquad \frac{dy}{dt} \;=\; \frac{\text{-}\sin(t)}{2\,y + \cos(y)}$$

Example: The use of implicit differentiation in conjunction with ***natural logarithms*** is useful for calculating the ordinary time-derivative of $y = c^t$ (c is a constant and t is time), as shown below.

$$y \;=\; c^t \qquad\Rightarrow\qquad \ln(y) \;=\; t\,\ln(c) \qquad\Rightarrow\qquad d\,[\ln(y)] \;=\; \ln(c)\,dt \qquad\Rightarrow\qquad \frac{1}{y}\,dy \;=\; \ln(c)\,dt$$

$$\frac{dy}{dt} \;=\; \ln(c)\,y \;=\; \ln(c)\,c^t$$

Note: When $c = e = 2.718281828$, $\frac{dy}{dt} = y$.
This plays a **central role** in solving ordinary differential equations.

2.9 Integration and a short table of integrals

An ***integral*** can be regarded as either an ***anti-derivative*** or as a ***sum*** (e.g., **area under a curve**).

Function	Integral of $F(t)$	
$F(t) = t^n$	$\int F(t)\,dt = \frac{t^{n+1}}{n+1} + C$	(n is a number other than -1)
$F(t) = t^{-1}$	$\int F(t)\,dt = \ln(t) + C$	
$F(t) = e^t$	$\int F(t)\,dt = e^t + C$	
$F(t) = \sin(t)$	$\int F(t)\,dt = \text{-}\cos(t) + C$	
$F(t) = \cos(t)$	$\int F(t)\,dt = \sin(t) + C$	

The website www.WolframResearch.com is a valuable resource for calculating integrals.

History: In 1675, Leibniz invented the integral notation $\int$ (Latin abbreviation for summa - sum) and its natural extension to double and triple integrals. Newton's integral notation was so defective, it was never popular – even in England. Euler was the first to use a symbol for an integral's limits, and its modern notation, e.g., $\int_a^b x\,dx$, was invented by Fourier in 1820.

Optional: Short history of differentiation

The modern differential notation $\frac{df}{dt}$ was introduced by Gottfried Leibniz in **1675** and relates to the ratio of differentials df and dt. The dot-notation $\dot{f}$ was introduced by Newton in his "***method of fluxions***" around **1675** and relates to his idea of flux (time-rates of change) of "***fluents***" (now called ***variables***). The prime notation f' was introduced by Lagrange in 1797 in his *Théorie des fonctiones analytiques*. Lagrange called $f'(t)$ the "derived function" of $f(t)$, from which the modern term ***derivative*** comes [8, pgs. 95-97]. An important concept introduced by Euler and Lagrange was that the derivative was a ***function*** which itself could be differentiated. The limit notation was introduced by Cauchy in **1823** and refined in the 1840s and 1850s by Cauchy, Seidel, Stokes, and Weierstrauss.

Although ***Newton and Leibniz*** share the discovery of ***calculus***, their relationship was contentious - with Newton and Leibniz and their respective supporters alleging plagiarism and undermining each other's credibility. As President of the Royal Society, Newton appointed an "impartial" committee to decide whether he or Leibniz invented calculus. He wrote the committee's official published report (although not under his name) and then wrote a review (again anonymously) which appeared in the Philosophical Transactions of the Royal Society. Ironically, the introverted Newton died at 80-years old a national hero of England with a state funeral of the highest honors whereas the more sociable Leibniz's died at 70-years old, almost completely forgotten, with a funeral attended by only his secretary. Newton's daunting reputation intimidated British mathematicians. England did not produce another first-rate mathematician for over a century. Undaunted and unintimidated by their English neighbors, the rest of Europe, lead by the Bernoulli family, Leonard Euler, D'Alembert, Lagrange, Laplace, Fourier, and many others, quickly expanded analytical analysis through differential equations, the calculus of variations, etc.

The order in which certain mathematics are taught can be non-intuitive as evidenced by history. For example, derivatives are usually taught starting with limits whereas the derivative was first ***used*** (Fermat and Descartes, 1637), then ***discovered*** (Newton and Leibniz, 1669-1684), then ***explored*** and ***developed*** (Taylor, Euler, Maclaurin, Lagrange, 1755-1797) and finally ***defined*** (Cauchy and Wiestrass, 1823-1861) [5]. Similarly, the ***Pythagorean theorem*** was used for thousands of years before its proof by Pythagoras $\approx$ 500 BC.

2.10 Solutions of *polynomial* equations (roots)

Polynomial equations are a special class of nonlinear algebraic equations. A special polynomial equation is the ***quadratic equation***, which is a polynomial equation of degree **2**. Shown below is a quadratic equation in x and its **2** ***roots*** (solutions).

Quadratic equation

$$\boxed{a\,x^2 + b\,x + c = 0}$$

Solution to quadratic equation

$$\boxed{x = \frac{-b + \sqrt{b^2 - 4\,a\,c}}{2\,a} \quad \text{and} \quad x = \frac{-b - \sqrt{b^2 - 4\,a\,c}}{2\,a}}$$

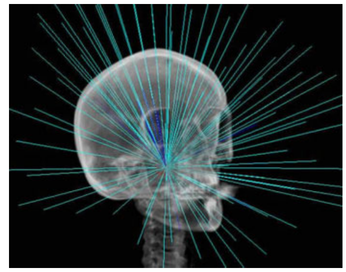
Two other polynomial equations with "closed-form solutions" are the ***cubic*** and ***quartic*** equations

$$x^3 + c_2\,x^2 + c_1\,x + c_0 = 0 \qquad \text{and} \qquad x^4 + c_3\,x^3 + c_2\,x^2 + c_1\,x + c_0 = 0$$

The ***Fundamental Theorem of Algebra***, states that any polynomial of degree n with complex coefficients has n complex roots.[12] In 1824, Abel proved that no general closed-form solution for 5^{th}-order (or higher) polynomials exist. Numerical methods are useful for calculating roots of polynomials of any order.

[12]The proof of the ***Fundamental Theorem of Algebra*** is difficult and was presented with various rigor between 1608 and 1981 by great mathematicians including, Rothe(1608) Girard (1629), Leibniz (1702), Bernoulli (1742), d'Alembert (1746), Euler (1749), Lagrange (1772), Laplace (1795), Gauss (1799), Argand (**1806**), Gauss (again in 1816 and 1849), Cauchy (1821), Weierstrauss (1891), Hellmuth Kneser (1940), and his son Martin Kneser (**1981**).

2.11 Optional: Continuous solutions of *nonlinear* algebraic equations


Courtesy Accuray Inc.

One way to find a continuous solution for x in the range $0 \le t \le 8$ for

$$x^2 \;-\; \cos^2(x) \;=\; 0.3\,\sin(t)$$

is to differentiate this ***nonlinear*** equation with respect to t and then solve the derivative equation that is ***linear*** in $\dot{x}$ as

$$2\,x\,\dot{x} + 2\,\cos(x)\,\sin(x)\,\dot{x} \;=\; 0.3\,\sin(t) \qquad \Rightarrow \qquad \dot{x} \;=\; \frac{0.3\,\cos(t)}{2\,x \;+\; 2\,\cos(x)\,\sin(x)}$$

Solving the nonlinear equation **once** at $t = 0$ gives $x(t = 0) \;\approx\; 0.74$. With this initial value for x and continuous formula for $\dot{x}$, ODE techniques can numerically integrate $\dot{x}(t)$ to solve for $x(t)$.

2.12 Optional: Proofs

Trigonometric formulas can be tedious or difficult to prove. For example, a proof of the law of cosines can be found in the book Engineering Mechanics OnLine, by Thomas R. Kane and David A. Levinson, 1999. Several trigonometric proofs are shown below.

2.12.1 Proof of the addition formula for the sine function

On way to prove equation (7) and equation (8) is to note that Euler's formula provides

$$e^{i\,(\alpha+\beta)} \;=\; \cos(\alpha+\beta) \;+\; i\,\sin(\alpha+\beta) \tag{32}$$

A second way to write $e^{i\,(\alpha+\beta)}$ is to use the addition property of exponents, namely

$$\begin{aligned} e^{i\,(\alpha+\beta)} \;&=\; e^{i\,\alpha} * e^{i\,\beta} \;=\; (\cos\alpha + i\sin\alpha) * (\cos\beta + i\sin\beta) \\ &=\; \cos(\alpha)\,\cos(\beta) - \sin(\alpha)\,\sin(\beta) \;+\; i\,[\sin(\alpha)\,\cos(\beta) + \sin(\beta)\,\cos(\alpha)] \end{aligned} \tag{33}$$

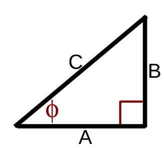
Equating the real parts of the right hand-sides of equations (32) and (33) leads directly to equation (8). Similarly, equating the imaginary parts of the equations (32) and (33) leads directly to equation (7).

2.12.2 Geometrical proof of eqn (21), the amplitude-phase trigonometric identity

Multiplying and dividing the left-hand side of equation (21) by ${}^{+}\!\sqrt{A^2+B^2}$ leads to

$$A\sin(x) \;+\; B\cos(x) \;=\; {}^{+}\!\sqrt{A^2+B^2}\left[\frac{A}{{}^{+}\!\sqrt{A^2+B^2}}\,\sin(x) \;+\; \frac{B}{{}^{+}\!\sqrt{A^2+B^2}}\,\cos(x)\right] \tag{34}$$

The geometry of a right triangle is used to form expressions for C, $\cos(\phi)$, and $\sin(\phi)$.



$$C \;=\; {}^{+}\!\sqrt{A^2 \;+\; B^2} \qquad\qquad \cos(\phi) \;=\; \frac{A}{{}^{+}\!\sqrt{A^2+B^2}} \qquad\qquad \sin(\phi) \;=\; \frac{B}{{}^{+}\!\sqrt{A^2+B^2}}$$

With the expressions for C, $\cos(\phi)$, and $\sin(\phi)$ in hand, equation (34) can be rewritten as

$$A\sin(x) \;+\; B\cos(x) \underset{(34)}{=} C\,[\cos(\phi)\,\sin(x) \;+\; \sin(\phi)\,\cos(x)] \tag{35}$$

In view of equation (7), it is possible to rewrite the right-hand side of equation (35) as

$$A\sin(x) \;+\; B\cos(x) \underset{(7,35)}{=} C\,\sin(x+\phi) \tag{36}$$

2.12.3 Trigonometric proof of eqn (21), the amplitude-phase trigonometric identity

An alternate mathematical proof of equation (21) begins with

$$C\cos(x+\phi) \overset{?}{=} A\sin(x) + B\cos(x) \tag{37}$$

$$\underset{(8)}{C[\cos(x)\cos(\phi) - \sin(x)\sin(\phi)]} \underset{(37)}{=} A\sin(x) + B\cos(x) \tag{38}$$

$$[C\cos(\phi) - B]\cos(x) - [C\sin(\phi) + A]\sin(x) \underset{(38)}{=} 0 \tag{39}$$

Since x may be assigned any value and $\cos(x)$ and $\sin(x)$ are linearly independent functions[13], the coefficients of both $\cos(x)$ and $\sin(x)$ must be zero. Thus,

$$C\cos(\phi) - B \underset{(39)}{=} 0 \quad \Rightarrow \quad C\cos(\phi) = B \tag{40}$$

$$C\sin(\phi) + A \underset{(39)}{=} 0 \quad \Rightarrow \quad C\sin(\phi) = \text{-}A \tag{41}$$

One way to solve ***nonlinear*** algebraic equations (40) and (41) for the two unknowns C and ϕ is:

$$C^2\left[\sin^2(\phi) + \cos^2(\phi)\right] \underset{(40,\,41)}{=} A^2 + B^2 \quad \Rightarrow \quad C \underset{(2)}{=} {}^{+}\sqrt{A^2 + B^2}$$

$$\cos(\phi) \underset{(40)}{=} \frac{B}{C} \quad \text{and} \quad \sin(\phi) \underset{(41)}{=} \frac{\text{-}A}{C} \quad \Rightarrow \quad \phi \underset{(23)}{=} \texttt{atan2}(\text{-}A, B)$$

Geometry and mathematics are heavily used in structural design

[13]In other words, it is *not* possible to write $\cos(x) = a * \sin(x) + b$ when a and b are not functions of x. Another way to see that the coefficients must be zero is to recognize that x may be assigned *any* value. Choosing $x = 0$ results in $\cos(x) = 1$ and $\sin(x) = 0$, while choosing $x = \pi/2$ results in $\cos(x) = 0$ and $\sin(x) = 1$.