

# Contents

<b>1</b>	<b>What is a dynamic system?</b>	<b>1</b>
1.1	Important concepts in dynamic systems . . . . .	1
1.2	High-level objectives of this textbook . . . . .	2
1.3	Skills taught in this textbook . . . . .	2
1.4	MIPSI: A procedure for studying dynamic systems . . . . .	2
<b>2</b>	<b>Math review</b>	<b>3</b>
2.1	Why math is important . . . . .	3
2.2	Mathematical operations . . . . .	4
2.2.1	Order of operations: Established mathematical convention? . . . . .	5
2.2.2	Patterns in mathematics . . . . .	5
2.3	Unit systems - SI and U.S. . . . .	6
2.4	Geometry: Ancient Euclid and modern vectors . . . . .	6
2.5	Circles and their properties . . . . .	6
2.6	Triangles and ratios of their sides (sine, cosine, tangent) . . . . .	7
2.6.1	Properties of sine and cosine and useful trigonometric formulas . . . . .	7
2.6.2	Sine and cosine as functions (Euler, circa 1730) . . . . .	8
2.6.3	The amplitude-phase formulas for sine and cosine . . . . .	9
2.6.4	The function $\text{atan2}(y, x)$ . . . . .	9
2.6.5	Optional: The sinc function . . . . .	9
2.7	Types of scalars: Variable, Specified, Constant . . . . .	9
2.8	Differentiation . . . . .	10
2.8.1	Definition of an ordinary derivative of a scalar function . . . . .	10
2.8.2	Definition of a partial derivative of a scalar function . . . . .	10
2.8.3	Definition of the differential of an independent variable and scalar function . . . . .	10
2.8.4	Definition of the total derivative of a scalar function . . . . .	11
2.8.5	Short table of derivatives frequently encountered in engineering . . . . .	11
2.8.6	Example: Partial and ordinary differentiation . . . . .	11
2.8.7	Good product rule for differentiation (for scalars, vectors, matrices, . . .) . . . . .	12
2.8.8	Quotient rule for derivatives: Use exponents and the product rule . . . . .	12
2.8.9	Chain rule for derivatives . . . . .	12
2.8.10	Implicit differentiation: A useful tool for calculating derivatives . . . . .	12
2.9	Integration and a short table of integrals . . . . .	12
2.10	Solutions of <i>polynomial</i> equations (roots) . . . . .	13
2.11	Optional: Continuous solutions of <i>nonlinear</i> algebraic equations . . . . .	14
2.12	Optional: Proofs . . . . .	14
2.12.1	Proof of the addition formula for the sine function . . . . .	14
2.12.2	Geometrical proof of eqn (21), the amplitude-phase trigonometric identity . . . . .	14
2.12.3	Trigonometric proof of eqn (21), the amplitude-phase trigonometric identity . . . . .	15

<b>3</b>	<b>Introduction to differential equations</b>	<b>17</b>
3.1	Motivating example: Chaotic plate pendulum (see examples in Hw 2)	17
3.2	Why we care about differential equations	18
3.3	Classification of differential equations	18
3.4	Independent, dependent, and specified variables	18
3.5	Classification of algebraic and differential equations	19
3.6	Example of classification of differential equations	21
3.7	What is a vibration?	22
3.8	Solution techniques for ordinary differential equations (ODEs)	22
3.9	Optional: Numerical solution of ODEs - concepts	22
	Summary of ODE methods	23
<b>4</b>	<b>Separation of variables &amp; 1<sup>st</sup>-order ODEs</b>	<b>25</b>
4.1	Separation of variables and integration	25
4.1.1	Example: Separation of variables and integration to solve a 1 <sup>st</sup> -order ODE	25
4.1.2	Finding a value for the undetermined constant $C$	26
4.1.3	Example: Olympic swimming via separation of variables	26
4.2	Time constant	27
4.3	Assumed solution $y(t) = C e^{pt}$ for homogeneous, 1 <sup>st</sup> -order, ODEs	28
4.3.1	Example: Olympic swimming via assumed solution	29
4.3.2	Example: The biology of life	29
4.4	Optional: The origins of $e$	30
4.4.1	Optional: An amazing property of the exponential function $e^t$	30
4.4.2	Optional: Mathematical connections with $e$	31
4.4.3	Optional: $e$ is for Euler, $e$ is for engineer, $e$ is for excellent	31
<b>5</b>	<b>Second-order, homogeneous, ODEs</b>	<b>33</b>
5.1	Analytical techniques for solving ODEs (see examples in Hw 3)	33
5.2	Analytical solutions of homogeneous, second-order, ODEs	33
5.3	Euler's formula for $e^{i\theta}$ - memorize!	34
5.4	Example: <i>Undamped</i> , homogeneous, second-order, ODE	35
5.5	Mathematical and physical significance of $\zeta$ , $\omega_n$ , and $\omega_d$	35
5.6	Example: Bungee jumping	36
5.7	Solution process for linear, 2 <sup>nd</sup> -order, ODEs	37
5.7.1	Solution process for linear, $n^{\text{th}}$ -order, constant-coefficient ODEs	37
5.7.2	Derivation of solution for <i>undamped</i> , second-order, ODE	37
5.7.3	Optional: Derivation of solution for <i>critically-damped</i> ODE	38
<b>6</b>	<b>Time specifications for 2<sup>nd</sup>-order ODEs</b>	<b>39</b>
6.1	Period of vibration	39
6.2	Step response and time-specifications	40
6.3	Decay ratio and logarithmic decrement	41
6.4	Rise time	41
6.5	Peak time and maximum overshoot	42
6.6	Settling ratio and settling time	42
6.7	Example: Laboratory data and time specifications	43
<b>7</b>	<b>Roots, root locus, and <math>e^{pt}</math></b>	<b>45</b>
7.1	Properties of $e^{pt}$ ( $p$ is a constant and $t$ is time)	45
7.2	Roots of polynomials and the Fundamental Theorem of Algebra	47
7.3	Roots of polynomials, independent functions, and stability of ODEs	47
7.4	Necessary condition for a polynomial to have negative roots	47

7.5	What is a root locus? . . . . .	49
7.6	Why use a root locus? . . . . .	49
7.7	Root locus for $2^{nd}$ -order ODE (for $m$ , $b$ , or $k$ ) . . . . .	49
7.8	$2^{nd}$ -order ODEs: $\omega_n$ , $\zeta$ in terms of poles $p_1$ , $p_2$ . . . . .	52
7.9	Roots of a $5^{th}$ -order polynomial with MATLAB <sup>®</sup> or MotionGenesis . . . . .	53
7.10	Optional: Drawing a root locus with MATLAB <sup>®</sup> . . . . .	53
<b>8</b>	<b>Newton/Euler dynamics</b> . . . . .	<b>55</b>
8.1	Newton's law of motion for a particle . . . . .	55
8.2	Newton's equation for a system . . . . .	55
8.3	Euler's equation for a rigid body with simple angular velocity . . . . .	55
	Summary of dynamics equations . . . . .	56
<b>9</b>	<b>Power/energy-rate principle</b> . . . . .	<b>57</b>
9.1	Power of a force, set of forces, or torque on a rigid body . . . . .	58
9.2	Forces that do not contribute to power (workless forces) . . . . .	59
9.3	Power/energy-rate for a commercial spring scale . . . . .	59
9.4	Optional: Kinetic energy . . . . .	60
9.4.1	Kinetic energy of a particle . . . . .	60
9.4.2	Kinetic energy of a rigid body with a simple angular velocity . . . . .	60
<b>10</b>	<b>MIPSI: Classic particle pendulum</b> . . . . .	<b>61</b>
10.1	Modeling the classic particle pendulum . . . . .	61
10.2	Identifiers for the classic particle pendulum . . . . .	62
10.3	Physics: Equations of motion of the classic particle pendulum . . . . .	62
10.3.1	$\vec{F} = m\vec{a}$ for the classic particle pendulum . . . . .	63
10.3.2	Angular momentum principle for the classic particle pendulum . . . . .	64
10.3.3	Euler's rigid body equation for the classic particle pendulum . . . . .	64
10.3.4	Kinetic energy for the classic particle pendulum . . . . .	64
10.3.5	Power/energy-rate principle for the classic particle pendulum . . . . .	65
10.3.6	Conservation of mechanical energy for the classic particle pendulum . . . . .	65
10.4	Solution of the classic particle pendulum ODE . . . . .	65
10.4.1	Numerical solution of pendulum ODE via MotionGenesis and/or MATLAB <sup>®</sup> . . . . .	66
10.4.2	Analytical (closed-form) solution of the classic particle pendulum ODE . . . . .	66
10.4.3	Simplification and analytical solution of the classic particle pendulum ODE . . . . .	66
10.5	Interpretation of results for the classic particle pendulum . . . . .	66
<b>11</b>	<b>Example: Inverted pendulum on cart</b> . . . . .	<b>69</b>
11.1	Kinematics (space and time) . . . . .	69
11.2	Rotation matrix . . . . .	70
11.3	Angular velocity (special 2D case) . . . . .	70
11.3.1	Simple angular velocity example: Step-by-step process to calculate ${}^N\vec{\omega}^B$ . . . . .	70
11.3.2	Angular velocity and vector differentiation . . . . .	70
11.4	Angular acceleration . . . . .	71
11.5	Position vectors (inspection and vector addition) . . . . .	71
11.6	Velocity and acceleration . . . . .	71
11.7	Forces, moments, and free-body diagrams (2D) . . . . .	72
11.8	Mass, center of mass, inertia (required by dynamics) . . . . .	72
11.9	Newton/Euler laws of motion for $A$ and $B$ separately (inefficient) . . . . .	72
11.9.1	Dynamics for a rigid body with a simple angular velocity (special 2D case) . . . . .	73
11.10	Equations of motion via MG road-maps/D'Alembert (efficient) . . . . .	73
11.11	Matrix form of nonlinear and linearized equations of motion . . . . .	73

<b>12 Inhomogeneous ODEs</b>	<b>75</b>
12.1 Analytical solutions of an <i>inhomogeneous</i> , 1 <sup>st</sup> -order ODE . . . . .	76
12.2 Two techniques for finding a particular solution $y_p(t)$ . . . . .	76
12.3 Integral technique for particular solution to 1 <sup>st</sup> -order ODE . . . . .	76
12.4 Table of assumed particular solutions (inhomogeneous ODEs) . . . . .	77
12.5 Steady-state and transient parts of $y(t)$ . . . . .	77
12.6 Example: Speed of a parachutist . . . . .	78
12.6.1 Homogeneous solution (from Chapter 4) . . . . .	78
12.6.2 Integral technique for particular solution . . . . .	78
12.6.3 Assumed particular solution technique (use table in Section 12.4) . . . . .	78
12.6.4 Steady-state and transient parts of $v(t)$ . . . . .	79
12.7 Analytical solutions of an <i>inhomogeneous</i> , 2 <sup>nd</sup> -order, ODE . . . . .	79
12.8 Integral technique for particular solution to 2 <sup>nd</sup> -order ODE . . . . .	80
12.9 Example: Undamped, <i>inhomogeneous</i> , 2 <sup>nd</sup> -order, ODE . . . . .	80
12.9.1 Integral solution to <i>inhomogeneous</i> , 2 <sup>nd</sup> -order, ODE from equation (13) . . . . .	80
12.9.2 Assumed solution to <i>inhomogeneous</i> , 2 <sup>nd</sup> -order, ODE . . . . .	80
12.9.3 Steady-state and transient solutions of an undamped ODE . . . . .	81
12.10 BIBO stability of homogeneous and inhomogeneous ODEs . . . . .	81
12.11 Optional: Derivation for an <i>inhomogeneous</i> , 1 <sup>st</sup> -order, ODE . . . . .	82
<b>13 Harmonic forcing, steady-state response</b>	<b>83</b>
13.1 Steady-state response to harmonic forcing $\bar{A} \sin(\Omega t + \theta)$ . . . . .	83
13.2 Steady-state response to harmonic forcing $A\Omega^2 \sin(\Omega t + \theta)$ ( $\bar{A} = A\Omega^2$ ) . . . . .	84
13.3 Undamped ( $\zeta = 0$ ) vibration and harmonic-forcing $\bar{A} \sin(\Omega t + \theta)$ . . . . .	85
13.3.1 Resonance ( $\Omega = \omega_n$ ) and undamped harmonically-forced vibrations . . . . .	85
13.3.2 The beat phenomenon ( $\Omega \approx \omega_n$ ) for undamped vibrations . . . . .	85
13.4 Optional: Underdamped vibration to harmonic-forcing $\bar{A} \sin(\Omega t + \theta)$ . . . . .	86
13.5 Optional: Undamped ( $\zeta = 0$ ) vibration and harmonic-forcing . . . . .	86
<b>14 Motors, sensors, and electrical circuits</b>	<b>87</b>
14.1 DC (direct current) permanent magnet motors . . . . .	89
14.2 Electrical current $i$ . . . . .	89
14.3 Voltage $v$ and its units . . . . .	89
14.4 Definitions of series and parallel electrical elements . . . . .	90
14.5 Example: Motor circuit . . . . .	90
<b>15 Complex numbers</b>	<b>95</b>
15.1 Definition of a complex number and equivalent complex numbers . . . . .	96
15.2 Complex conjugate . . . . .	97
15.3 Complex number addition and subtraction . . . . .	97
15.4 Complex number multiplied (or divided) by a real number . . . . .	97
15.5 Complex number multiplied by a complex number . . . . .	98
15.6 Exponentiation and complex numbers . . . . .	98
15.7 Complex number divided by a complex number . . . . .	99
15.8 Optional: Complex numbers and other mathematical functions . . . . .	99
15.9 Optional: Complex numbers and 2D (planar) vectors . . . . .	100
<b>16 Laplace transforms</b>	<b>101</b>
16.1 What is time $t$ and the Laplace variable $s$ ? . . . . .	101
16.2 What is a Laplace transform? . . . . .	101
16.3 Why use a Laplace transform – connections to <u>linear</u> ODEs . . . . .	102
16.4 Complex functions and transfer functions . . . . .	103

16.5	Poles and zeros of a transfer function . . . . .	103
16.6	Bode plots: Visualizing frequency response . . . . .	103
16.7	Sinusoidal transfer function and harmonic forcing . . . . .	104
16.8	Thinking in $s$ and $t$ (also see Homework 8.7) . . . . .	105
16.9	Optional: Final value theorem . . . . .	105
16.10	Example: An RC circuit as a filter (output voltage) . . . . .	106
<b>17</b>	<b>PID control of dynamic systems</b>	<b>109</b>
17.1	What is a control system? (see examples in Hw 8, 9) . . . . .	109
17.2	Bio-dynamic control systems in the human body . . . . .	109
17.3	Conceptual overview of control system design . . . . .	109
17.4	Control system design for a linear ODE . . . . .	110
17.5	Block diagrams and control system components . . . . .	110
17.6	What is PID control? . . . . .	110
17.7	Cruise control for a simple car model . . . . .	110
17.7.1	Proportional (P) feedback control for a simple car model . . . . .	111
17.7.2	Proportional-Integral (PI) cruise control for a simple car model . . . . .	112
17.8	Cruise control for an improved car model . . . . .	112
<b>18</b>	<b>Linearization and small approximations</b>	<b>113</b>
18.1	Approximating products, powers, functions (examples in Hw 11) . . . . .	113
18.2	Example: Linearization of a particle sliding on a beam . . . . .	113
18.3	Why linearize? . . . . .	114
18.4	Simplification and solution of the classic particle pendulum . . . . .	114
18.5	Simplification and solution of the <i>inverted</i> particle pendulum . . . . .	115
18.6	Optional: Further investigation of small angle approximations . . . . .	116
<b>19</b>	<b>Optional: Linearization and Taylor series</b>	<b>117</b>
19.1	Taylor series expansion in a single variable . . . . .	117
19.2	Multi-variable Taylor series expansion . . . . .	118
19.3	Linearization of a function . . . . .	118
19.4	Linearization of ordinary differential equations . . . . .	119
19.5	Example: Linearization and stability of particle pendulum ODE . . . . .	119
19.6	Example: Linearization of ODEs for a swinging spring . . . . .	121
19.7	Optional: Motivating/proving the Taylor series . . . . .	123
<b>20</b>	<b>Optional: Fitting data to functions</b>	<b>125</b>
20.1	Fitting periodic data with Fourier Transforms . . . . .	125
<b>21</b>	<b>Matrices and <math>AX = B</math></b>	<b>129</b>
21.1	Motivating question: Solving <u>one</u> linear algebraic equation . . . . .	129
21.2	Why use matrices – getting organized . . . . .	129
21.3	What is a matrix? . . . . .	129
21.4	Row and column matrices . . . . .	130
21.5	Addition of matrices . . . . .	130
21.6	Multiplication of a matrix with a scalar . . . . .	130
21.7	Multiplication of matrices . . . . .	130
21.8	The zero matrix and the identity matrix . . . . .	130
21.9	Transpose of a matrix . . . . .	131
21.10	Submatrices of a matrix . . . . .	131
21.11	Determinant of a matrix, minors and cofactors . . . . .	131
21.12	Adjugate matrix (or classical adjoint matrix) . . . . .	132

21.13	Inverse of a matrix . . . . .	132
21.14	Changing linear algebraic equations into matrix form $AX = B$ . . . . .	133
21.15	Solving the matrix linear algebraic equation $AX = B$ . . . . .	133
21.16	Optional: Pseudo-inverse solutions to $AX = B$ . . . . .	133
21.17	Partial and ordinary derivative of a matrix . . . . .	134
21.18	Optional: Other matrices and their properties . . . . .	134
21.19	Optional: A history of matrices and determinants . . . . .	134
<b>22</b>	<b>Eigenvalues and eigenvectors</b>	<b>135</b>
22.1	Recognize and remember: Solving an eigenvalue problem . . . . .	135
22.2	Motivating questions for eigenvalues and eigenvectors (Hw 12.9) . . . . .	135
22.3	Solving the standard algebraic eigenvalue problem . . . . .	137
22.3.1	Special eigenvalues for the $n \times n$ matrix $A$ . . . . .	137
22.3.2	Eigenvalues and the roots of a polynomial . . . . .	137
22.4	Situations when $\text{Matrix}(\lambda)$ is nonlinear in $\lambda$ . . . . .	138
22.5	Generalized eigenvalue problem . . . . .	139
<b>23</b>	<b>Coupled, linear, 1<sup>st</sup>-order ODEs</b>	<b>141</b>
23.1	Solutions of <i>coupled</i> , linear, constant-coefficient 1 <sup>st</sup> -order ODEs . . . . .	141
23.2	Molecular dynamics: RNA folding and <i>coupled</i> 1 <sup>st</sup> -order ODEs . . . . .	142
<b>24</b>	<b>Matrix form of <i>coupled</i>, 2<sup>nd</sup>-order, linear, ODEs</b>	<b>145</b>
24.1	Model, identifiers, and physics of a two rotor system . . . . .	145
24.2	Matrix form of <i>coupled</i> , linear, second-order, ODEs . . . . .	146
<b>25</b>	<b>Undamped coupled 2<sup>nd</sup>-order ODEs</b>	<b>147</b>
25.1	Physical insights into eigenvalues and eigenvectors of a slinky . . . . .	147
25.2	Analytical solutions of <i>undamped</i> , <i>coupled</i> , ODEs . . . . .	148
25.3	<i>Undamped</i> , <i>coupled</i> , ODEs for two rotor system . . . . .	149
25.4	Relationship to standard algebraic eigenvalue problem . . . . .	151
25.5	Mode dance . . . . .	152
<b>26</b>	<b>State-space and damped, coupled, ODEs</b>	<b>153</b>
26.1	Generalized eigenvalue solutions of <i>damped</i> , <i>coupled</i> , ODEs . . . . .	153
26.2	Alternate solution technique: Standard eigen-equation . . . . .	154
26.3	<i>Damped</i> , <i>coupled</i> , ODEs for two rotor system . . . . .	155
<b>27</b>	<b>State-space feedback control</b>	<b>157</b>
27.1	Motivating example: Balancing an inverted pendulum . . . . .	157
27.2	Solution for $Y(t)$ with state-space control . . . . .	158
27.3	State-space control of coupled 2 <sup>nd</sup> -order ODEs (repeated from Section 26.2) . . . . .	158
27.4	Optional: Direct control for coupled 2 <sup>nd</sup> -order ODEs . . . . .	159
<b>28</b>	<b>Feed-forward control</b>	<b>161</b>
28.1	Feed-forward control for directly controlling $Q$ or $U$ . . . . .	162
28.1.1	Example: Feed-forward control of a rocket sled's <u>velocity</u> . . . . .	162
28.1.2	Example: Feed-forward control of rocket sled's <u>position</u> . . . . .	163
28.1.3	Example: Feed-forward control with additional actuator . . . . .	164
28.2	Derivation/insights into the mathematics of feed-forward control . . . . .	165

<b>Homework</b>	<b>169</b>
Homework 1: Geometry and calculus . . . . .	169
Homework 2: Uncoupled 1 <sup>st</sup> -order ODEs . . . . .	179
Homework 3: Uncoupled 2 <sup>nd</sup> -order ODEs (vibrations) . . . . .	191
Homework 4: Time specifications for ODEs and control . . . . .	199
Homework 5: $\vec{\mathbf{F}} = m\vec{\mathbf{a}}$ : Equations of motion and ODEs for mechanical systems . . . . .	205
Homework 6: Inhomogeneous ODEs and harmonic forcing . . . . .	219
Homework 7: Root locus . . . . .	229
Homework 8: Complex numbers, circuits, Laplace transforms, frequency response . . . . .	235
Homework 9: <b>Optional:</b> Circuits, Laplace transforms, frequency response II . . . . .	243
Homework 10: PID control of electromechanical systems . . . . .	251
Homework 11: Linearizing ODEs . . . . .	261
Homework 12: Matrix algebra . . . . .	269
Homework 13: Coupled ODEs . . . . .	273
Homework 14: State-space, coupled ODEs, and control . . . . .	281
<b>Working Model simulations/labs (www.MotionGenesis.com <math>\Rightarrow</math> Textbooks <math>\Rightarrow</math> Resources)</b>	<b>1</b>
Lab 2 . . . . .	1
Lab 3 . . . . .	3
Lab 4 . . . . .	5
Lab 5 . . . . .	8
Lab 6 . . . . .	10
Lab 10 . . . . .	12
Lab 14 . . . . .	14
Example: Classic particle pendulum (www.MotionGenesis.com $\Rightarrow$ Textbooks $\Rightarrow$ Resources) . . . . .	i
Example: Inverted pendulum on cart (www.MotionGenesis.com $\Rightarrow$ Textbooks $\Rightarrow$ Resources) . . . . .	i
Index . . . . .	ii