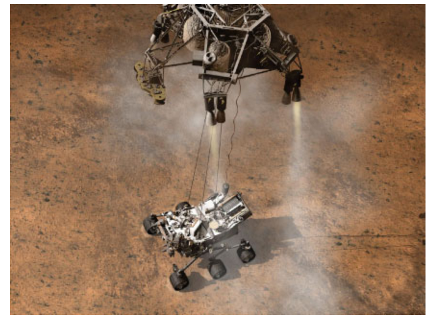
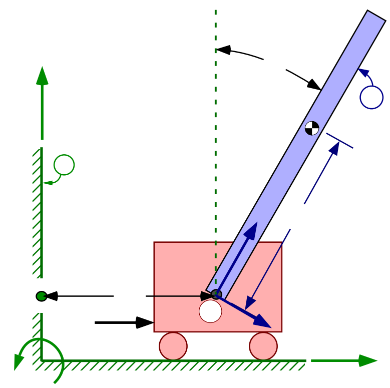


Chapter 27

Example: Inverted pendulum on a cart

The figure to the right shows a rigid inverted pendulum B attached by an ideal frictionless pin (revolute) joint to a cart A (modeled as a particle). The cart A slides on a straight horizontal frictionless track. The track is fixed in a Newtonian reference frame N.



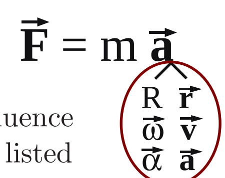
Right-handed orthogonal unit vectors $\widehat{\mathbf{n}}_x$, $\widehat{\mathbf{n}}_y$, $\widehat{\mathbf{n}}_z$ and $\widehat{\mathbf{b}}_x$, $\widehat{\mathbf{b}}_y$, $\widehat{\mathbf{b}}_z$ are fixed in N and B respectively, with:

- $\widehat{\mathbf{n}}_x$ horizontally right and $\widehat{\mathbf{n}}_y$ vertically upward
- $\widehat{\mathbf{n}}_z = \widehat{\mathbf{b}}_z$ parallel to B's axis of rotation in N
- $\widehat{\mathbf{b}}_y$ directed from A to the distal end of B

Complete the figure to the right by adding the identifiers N, A, B, B_{cm}, L, F_c, x, θ, $\widehat{\mathbf{n}}_x$, $\widehat{\mathbf{n}}_y$, $\widehat{\mathbf{n}}_z$, $\widehat{\mathbf{b}}_x$, $\widehat{\mathbf{b}}_y$, $\widehat{\mathbf{b}}_z$.

Quantity	Symbol	Value
Mass of A	m_A	10.0 kg
Mass of B	m_B	1.0 kg
Distance between A and B_{cm} (B's center of mass)	L	0.5 m
B's moment of inertia about B_{cm} for $\widehat{\mathbf{b}}_z$	I_{zz}	0.08333 kg$*$m^2
Earth's gravitational constant	g	9.8 m/s^2
$\widehat{\mathbf{n}}_x$ measure of feedback-control force applied to A	F_c	**Specified**
$\widehat{\mathbf{n}}_x$ measure of A's position vector from N_o (a point fixed in N)	x	Variable
Angle from $\widehat{\mathbf{n}}_y$ to $\widehat{\mathbf{b}}_y$ with $-\widehat{\mathbf{n}}_z$ sense	θ	Variable

27.1 Kinematics (space and time)

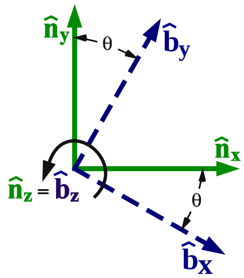


Kinematics is the study of the relationship between space and time, independent of the influence of mass or forces. The kinematic quantities normally needed for motion analysis are listed below. In many circumstances, it is efficient to form rotation matrices, angular velocities, and angular accelerations before position vectors, velocities, and accelerations.

Kinematic Quantity	Quantities needed for analyzing the inverted pendulum on a cart
Rotation matrix	${}^{b}R^{n}$, the rotation matrix relating $\widehat{\mathbf{b}}_x$, $\widehat{\mathbf{b}}_y$, $\widehat{\mathbf{b}}_z$ and $\widehat{\mathbf{n}}_x$, $\widehat{\mathbf{n}}_y$, $\widehat{\mathbf{n}}_z$
Angular velocity	${}^{N}\vec{\boldsymbol{\omega}}^{B}$, B's angular velocity in N
Angular acceleration	${}^{N}\vec{\boldsymbol{\alpha}}^{B}$, B's angular acceleration in N
Position vectors	$\vec{\mathbf{r}}^{A/N_o}$ and $\vec{\mathbf{r}}^{B_{cm}/A}$, the position vector of A from N_o and of B_{cm} from A
Velocity	${}^{N}\vec{\mathbf{v}}^{A}$ and ${}^{N}\vec{\mathbf{v}}^{B_{cm}}$, A's velocity in N and B_{cm}'s velocity in N
Acceleration	${}^{N}\vec{\mathbf{a}}^{A}$ and ${}^{N}\vec{\mathbf{a}}^{B_{cm}}$, A's acceleration in N and B_{cm}'s acceleration in N

27.2 Rotation matrix

The relative orientation of two sets of right-handed orthogonal unit vectors $\widehat{\mathbf{b}}_x$, $\widehat{\mathbf{b}}_y$, $\widehat{\mathbf{b}}_z$ and $\widehat{\mathbf{n}}_x$, $\widehat{\mathbf{n}}_y$, $\widehat{\mathbf{n}}_z$ is frequently stored in a 3×3 ***rotation matrix*** denoted ${}^{b}R^{n}$ whose elements are $\widehat{\mathbf{b}}_i \cdot \widehat{\mathbf{n}}_j$ $(i, j = x, y, z)$. All rotation matrices are ***orthogonal***, which means its inverse is equal to its transpose, and which means the matrix can be written as a table read horizontally or vertically. To relate $\widehat{\mathbf{b}}_x$, $\widehat{\mathbf{b}}_y$, $\widehat{\mathbf{b}}_z$ and $\widehat{\mathbf{n}}_x$, $\widehat{\mathbf{n}}_y$, $\widehat{\mathbf{n}}_z$, **redraw** these unit vectors in the geometrically-suggestive way shown below. To determine the 1^{st} row of the ${}^{b}R^{n}$ rotation matrix, $\widehat{\mathbf{b}}_x$ is expressed in terms of $\widehat{\mathbf{n}}_x$, $\widehat{\mathbf{n}}_y$, $\widehat{\mathbf{n}}_z$ as shown below. Similarly, the 2^{nd} and 3^{rd} rows of ${}^{b}R^{n}$ are found by expressing $\widehat{\mathbf{b}}_y$ and $\widehat{\mathbf{b}}_z$ in terms of $\widehat{\mathbf{n}}_x$, $\widehat{\mathbf{n}}_y$, $\widehat{\mathbf{n}}_z$.



$$\begin{aligned}
\widehat{\mathbf{b}}_x &= \cos(\theta)\,\widehat{\mathbf{n}}_x \;-\; \sin(\theta)\,\widehat{\mathbf{n}}_y \\
\widehat{\mathbf{b}}_y &= \sin(\theta)\;\widehat{\mathbf{n}}_x \;+\; \cos(\theta)\;\widehat{\mathbf{n}}_y \\
\widehat{\mathbf{b}}_z &= \widehat{\mathbf{n}}_z
\end{aligned}$$

${}^{b}R^{n}$	$\widehat{\mathbf{n}}_x$	$\widehat{\mathbf{n}}_y$	$\widehat{\mathbf{n}}_z$
$\widehat{\mathbf{b}}_x$	$\cos(\theta)$	$-\sin(\theta)$	0
$\widehat{\mathbf{b}}_y$	$\sin(\theta)$	$\cos(\theta)$	0
$\widehat{\mathbf{b}}_z$	0	0	1

27.3 Angular velocity (special 2D case)

Since 3D angular velocity is complicated, many textbooks only define ***simple angular velocity***, which is useful for ***two-dimensional analysis***.[1]

When a vector $\vec{\boldsymbol{\lambda}}$ is **fixed** in both a reference frame B and a reference frame N, B has a ***simple angular velocity*** in N that can be calculated via equation (1).

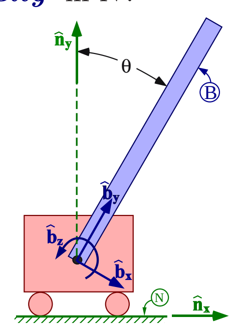
$$\boxed{{}^{N}\vec{\boldsymbol{\omega}}^{B} \underset{(\textbf{simple})}{=} \pm\,\dot{\theta}\,\vec{\boldsymbol{\lambda}}} \tag{1}$$

The sign of $\dot{\theta}\,\vec{\boldsymbol{\lambda}}$ is determined by the right-hand rule. If increasing θ causes a right-hand rotation of B in N about $+\vec{\boldsymbol{\lambda}}$, the sign is positive, otherwise it is negative.

27.3.1 Simple angular velocity example: Step-by-step process to calculate ${}^{N}\vec{\boldsymbol{\omega}}^{B}$

Due to the pin joint, $\widehat{\mathbf{b}}_z$ is **fixed**[a] in both B and N, so B has a ***simple angular velocity*** in N.

- $\widehat{\mathbf{b}}_z$ is a unit vector **fixed** in both N and B (parallel to the pin joint)
- $\widehat{\mathbf{n}}_y$ is fixed in N and perpendicular to $\widehat{\mathbf{b}}_z$
- $\widehat{\mathbf{b}}_y$ is fixed in B and perpendicular to $\widehat{\mathbf{b}}_z$
- θ is the angle between $\widehat{\mathbf{n}}_y$ and $\widehat{\mathbf{b}}_y$, and $\dot{\theta}$ is its time-derivative
- After pointing the four fingers of your **right** hand in the direction of $\widehat{\mathbf{n}}_y$ and curling them in the direction of $\widehat{\mathbf{b}}_y$, your thumb points in the $-\widehat{\mathbf{b}}_z$ direction.



Since the right-hand rule produces a sign of $\widehat{\mathbf{b}}_z$ that is negative: $\boxed{{}^{N}\vec{\boldsymbol{\omega}}^{B} = -\dot{\theta}\,\widehat{\mathbf{b}}_z}$

[a] A vector is said to be **fixed** in reference frame B if its magnitude is constant and its direction does not change in B.

27.3.2 Angular velocity and vector differentiation

For ***any*** vector $\vec{\mathbf{v}}$, the ***golden rule for vector differentiation*** relates $\frac{{}^{N}d\vec{\mathbf{v}}}{dt}$ (the ordinary time-derivative of $\vec{\mathbf{v}}$ in a reference frame N) to:

- ${}^{N}\vec{\boldsymbol{\omega}}^{B}$, B's angular velocity in N
- $\frac{{}^{B}d\vec{\mathbf{v}}}{dt}$, the ordinary time-derivative of $\vec{\mathbf{v}}$ in B

$$\boxed{\frac{{}^{N}d\vec{\mathbf{v}}}{dt} = \frac{{}^{B}d\vec{\mathbf{v}}}{dt} + {}^{N}\vec{\boldsymbol{\omega}}^{B} \times \vec{\mathbf{v}}} \tag{2}$$

[1] One of the major obstacles in ***three-dimensional*** kinematics is properly calculating angular velocity.
Note: See 3D dynamics textbooks for undergraduates/professionals at www.MotionGenesis.com.

Equation (2) is one of the ***most important formulas in kinematics*** because $\vec{\mathbf{v}}$ can be ***any*** vector, e.g., a unit vector, a position vector, a velocity vector, a linear/angular acceleration vector, a linear/angular momentum vector, or a force or torque vector.

27.4 Angular acceleration

Equation (3) defines the angular acceleration of a reference frame B in a reference frame N.

${}^{N}\vec{\boldsymbol{\alpha}}^{B}$ also ***happens*** to be equal to the time-derivative in $\boldsymbol{B}$ of ${}^{N}\vec{\boldsymbol{\omega}}^{B}$.

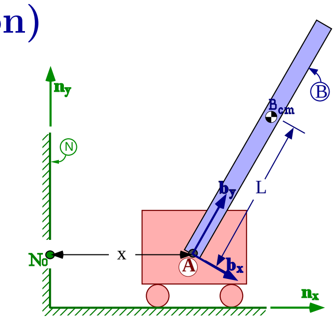
$$\boxed{{}^{N}\vec{\boldsymbol{\alpha}}^{B} \triangleq \frac{{}^{N}d\,{}^{N}\vec{\boldsymbol{\omega}}^{B}}{dt} = \frac{{}^{B}d\,{}^{N}\vec{\boldsymbol{\omega}}^{B}}{dt}} \qquad (3)$$

Note: Calculate with $\frac{{}^{B}d\,{}^{N}\vec{\boldsymbol{\omega}}^{B}}{dt}$ if it is easier to compute than $\frac{{}^{N}d\,{}^{N}\vec{\boldsymbol{\omega}}^{B}}{dt}$.

B's angular acceleration in N is most easily calculated with its alternate definition, i.e.,

$$ {}^{N}\vec{\boldsymbol{\alpha}}^{B} = \frac{{}^{B}d\,{}^{N}\vec{\boldsymbol{\omega}}^{B}}{dt} = \frac{{}^{B}d\,(\text{-}\dot{\theta}\,\widehat{\mathbf{b}}_\text{z})}{dt} = \boxed{\text{-}\ddot{\theta}\,\widehat{\mathbf{b}}_\text{z}}$$

27.5 Position vectors (inspection and vector addition)



- Inspection of the figure: $\boxed{\vec{\mathbf{r}}^{\,A/N_o} = x\,\widehat{\mathbf{n}}_\text{x}}$ (A's position vector from N)
- Inspection of the figure: $\boxed{\vec{\mathbf{r}}^{\,B_\text{cm}/A} = L\,\widehat{\mathbf{b}}_\text{y}}$ (B_cm's position vector from A)
- Vector addition: $\vec{\mathbf{r}}^{\,B_\text{cm}/N_o} = \vec{\mathbf{r}}^{\,B_\text{cm}/A} + \vec{\mathbf{r}}^{\,A/N_o} = \boxed{L\,\widehat{\mathbf{b}}_\text{y} + x\,\widehat{\mathbf{n}}_\text{x}}$
  (B_cm's position vector from N_o)

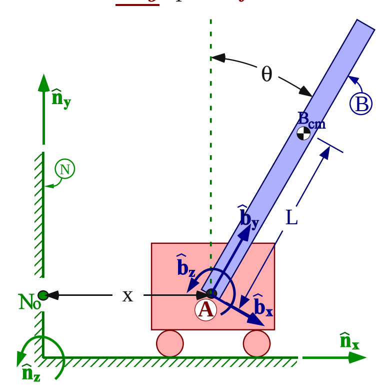
27.6 Velocity and acceleration

${}^{N}\vec{\mathbf{v}}^{B_\text{cm}}$ (the velocity of a point B_cm in a reference frame N) is defined as the time-derivative in N of $\vec{\mathbf{r}}^{\,B_\text{cm}/N_o}$ (B_cm's position vector from N_o).

$$\boxed{{}^{N}\vec{\mathbf{v}}^{B_\text{cm}} \triangleq \frac{{}^{N}d\,\vec{\mathbf{r}}^{\,B_\text{cm}/N_o}}{dt}} \qquad (4)$$

Point N_o is ***any*** point ***fixed*** in N

$$\begin{aligned}
{}^{N}\vec{\mathbf{v}}^{B_\text{cm}} &\triangleq \frac{{}^{N}d\,\vec{\mathbf{r}}^{\,B_\text{cm}/N_o}}{dt} = \frac{{}^{N}d\,(x\,\widehat{\mathbf{n}}_\text{x} + L\,\widehat{\mathbf{b}}_\text{y})}{dt} \\
&= \frac{{}^{N}d\,(x\,\widehat{\mathbf{n}}_\text{x})}{dt} + \frac{{}^{N}d\,(L\,\widehat{\mathbf{b}}_\text{y})}{dt} \\
&= \dot{x}\,\widehat{\mathbf{n}}_\text{x} + \frac{{}^{B}d\,(L\,\widehat{\mathbf{b}}_\text{y})}{dt} + {}^{N}\vec{\boldsymbol{\omega}}^{B} \times L\,\widehat{\mathbf{b}}_\text{y} \\
&= \dot{x}\,\widehat{\mathbf{n}}_\text{x} + \vec{\mathbf{0}} + \text{-}\dot{\theta}\,\widehat{\mathbf{b}}_\text{z} \times L\,\widehat{\mathbf{b}}_\text{y} \\
&= \boxed{\dot{x}\,\widehat{\mathbf{n}}_\text{x} + \dot{\theta}\,L\,\widehat{\mathbf{b}}_\text{x}}
\end{aligned}$$



${}^{N}\vec{\mathbf{a}}^{B_\text{cm}}$ (the acceleration of point B_cm in reference frame N) is defined as the time-derivative in N of ${}^{N}\vec{\mathbf{v}}^{B_\text{cm}}$ (B_cm's velocity in N).

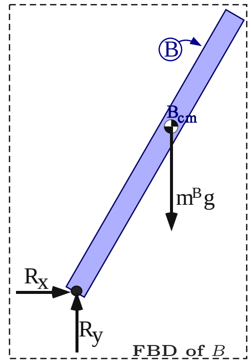
$$\boxed{{}^{N}\vec{\mathbf{a}}^{B_\text{cm}} \triangleq \frac{{}^{N}d\,{}^{N}\vec{\mathbf{v}}^{B_\text{cm}}}{dt}} \qquad (5)$$

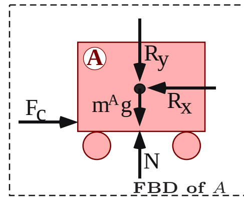
$$\begin{aligned}
{}^{N}\vec{\mathbf{a}}^{B_\text{cm}} &\triangleq \frac{{}^{N}d\,{}^{N}\vec{\mathbf{v}}^{B_\text{cm}}}{dt} = \frac{{}^{N}d\,(\dot{x}\,\widehat{\mathbf{n}}_\text{x} + \dot{\theta}\,L\,\widehat{\mathbf{b}}_\text{x})}{dt} = \frac{{}^{N}d\,(\dot{x}\,\widehat{\mathbf{n}}_\text{x})}{dt} + \frac{{}^{N}d\,(\dot{\theta}\,L\,\widehat{\mathbf{b}}_\text{x})}{dt} \\
&= \ddot{x}\,\widehat{\mathbf{n}}_\text{x} + \frac{{}^{B}d\,(\dot{\theta}\,L\,\widehat{\mathbf{b}}_\text{x})}{dt} + {}^{N}\vec{\boldsymbol{\omega}}^{B} \times (\dot{\theta}\,L\,\widehat{\mathbf{b}}_\text{x}) = \boxed{\ddot{x}\,\widehat{\mathbf{n}}_\text{x} + \ddot{\theta}\,L\,\widehat{\mathbf{b}}_\text{x} + \text{-}\dot{\theta}^2\,L\,\widehat{\mathbf{b}}_\text{y}}
\end{aligned}$$

27.7 Forces, moments, and free-body diagrams (2D)

$\vec{\mathbf{F}} = m\,\vec{\mathbf{a}}$

To draw a ***free-body diagram (FBD)***, isolate a single body (or system S of A and B) and draw all the external contact and distance forces that act on it. Shown right are FBDs with all the external forces on the cart A and pendulum B.[a]





Quantity	Description	Type
F_c	$\widehat{\mathbf{n}}_x$ measure of control force applied to A	Contact
N	$\widehat{\mathbf{n}}_y$ measure of the resultant normal force on A from N	Contact
R_x	$\widehat{\mathbf{n}}_x$ measure of the force on B from A across the revolute joint	Contact
R_y	$\widehat{\mathbf{n}}_y$ measure of the force on B from A across the revolute joint	Contact
$m_A\,g$	$-\widehat{\mathbf{n}}_y$ measure of Earth's gravitational force on A	Distance
$m_B\,g$	$-\widehat{\mathbf{n}}_y$ measure of Earth's gravitational force on B	Distance

$$\begin{aligned}
\text{Resultant force on } A\text{:}\quad \vec{\mathbf{F}}^{A} &= (F_c - R_x)\ \widehat{\mathbf{n}}_x \ + \ (N - m_A\,g - R_y)\ \widehat{\mathbf{n}}_y \\
\text{Resultant force on } B\text{:}\quad \vec{\mathbf{F}}^{B} &= R_x\ \widehat{\mathbf{n}}_x \ + \ (R_y - m_B\,g)\ \widehat{\mathbf{n}}_y \\
\text{Resultant force on } S\text{:}\quad \vec{\mathbf{F}}^{S} &= F_c\ \widehat{\mathbf{n}}_x \ + \ [N \ - \ (m_A + m_B)\,g]\ \widehat{\mathbf{n}}_y
\end{aligned}$$

[a] Alternately, to use the efficient ***road-map/D'Alembert method***) to eliminate "constraint forces" R_x and R_y, **draw** a FBD of the system S consisting of A and B (no need to draw A alone). Since the revolute joint between A and B is ideal, action/reaction is used to minimize the number of unknowns.

The $\widehat{\mathbf{b}}_z$ component of the moment of all forces on B about B_{cm} is[a]

$$\begin{aligned}
\vec{\mathbf{M}}_z^{B/B_{\text{cm}}} &= \vec{\mathbf{r}}^{A/B_{\text{cm}}} \times (R_x\,\widehat{\mathbf{n}}_x + R_y\,\widehat{\mathbf{n}}_y) \ + \ \overbrace{\cancel{\vec{\mathbf{r}}^{B_{\text{cm}}/B_{\text{cm}}}}}^{\vec{0}} \times (-m_B\,g\,\widehat{\mathbf{n}}_y) \\
&= -L\,\widehat{\mathbf{b}}_y \times (R_x\,\widehat{\mathbf{n}}_x + R_y\,\widehat{\mathbf{n}}_y) \ = \ [L\cos(\theta) R_x \ - \ L\sin(\theta) R_y]\ \widehat{\mathbf{b}}_z
\end{aligned}$$

[a] Note: The rotation table to useful for calculating the cross-products $(\widehat{\mathbf{b}}_y \times \widehat{\mathbf{n}}_x)$ and $(\widehat{\mathbf{b}}_y \times \widehat{\mathbf{n}}_y)$.

27.8 Dynamics requires: Mass, center of mass, inertia

$\vec{\mathbf{F}} = m\,\vec{\mathbf{a}}$

- Mass of each particle and body, e.g., m_A (mass of particle A) and m_B (mass of body B).
- Location of each particle and body center of mass, e.g., $\vec{\mathbf{r}}^{A/N_o}$ and $\vec{\mathbf{r}}^{B_{\text{cm}}/A}$.
- Inertia dyadic of each rigid body about a point fixed on the body. Since B's angular velocity in N is **simple**, I_{zz} (B's moment of inertia about B_{cm} for $\widehat{\mathbf{b}}_z$) suffices for this analyses.

27.9 Newton/Euler laws of motion for A and B separately (inefficient)

An inefficient way to form this system's equations of motion is with separate analyses of A and B. Using $\vec{\mathbf{F}} = m\,\vec{\mathbf{a}}$ for particle A and body B [in conjunction with the previous ***free-body diagrams (FBDs)***] yields,

$$\vec{\mathbf{F}}^{A} = m_A * {}^{N}\vec{\mathbf{a}}^{A} \quad \Rightarrow \quad (F_c - R_x)\,\widehat{\mathbf{n}}_x \ + \ (N - m_A g - R_y)\,\widehat{\mathbf{n}}_y \ = \ m_A\,\ddot{x}\,\widehat{\mathbf{n}}_x$$

Dot-multiply with $\widehat{\mathbf{n}}_x$: $\boxed{F_c \ - \ R_x \ = \ m_A\,\ddot{x}}$ Dot-multiply with $\widehat{\mathbf{n}}_y$: $\boxed{N \ - \ m_A\,g \ - \ R_y \ = \ 0}$

$$\vec{\mathbf{F}}^{B} = m_B * {}^{N}\vec{\mathbf{a}}^{B_{\text{cm}}} \quad \Rightarrow \quad R_x\,\widehat{\mathbf{n}}_x \ + \ (R_y - m_B g)\,\widehat{\mathbf{n}}_y \ = \ m_B\,(\ddot{x}\,\widehat{\mathbf{n}}_x \ + \ \ddot{\theta}\,L\,\widehat{\mathbf{b}}_x \ - \ \dot{\theta}^2\,L\,\widehat{\mathbf{b}}_y)$$

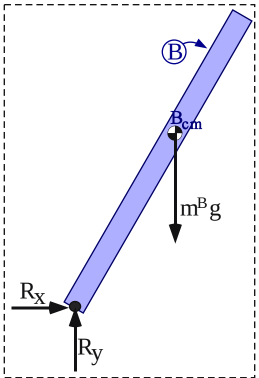
Dot-multiplication with $\widehat{\mathbf{n}}_x$ and $\widehat{\mathbf{n}}_y$ (use the rotation table to calculate dot-products) gives

$$R_x \ = \ m_B\,[\ddot{x} \ + \ \ddot{\theta}\,L\,(\widehat{\mathbf{b}}_x \cdot \widehat{\mathbf{n}}_x) \ - \ \dot{\theta}^2\,L\,(\widehat{\mathbf{b}}_y \cdot \widehat{\mathbf{n}}_x)] \qquad (R_y - m_B g) \ = \ m_B\,[\ddot{\theta}\,L\,(\widehat{\mathbf{b}}_x \cdot \widehat{\mathbf{n}}_y) \ - \ \dot{\theta}^2\,L\,(\widehat{\mathbf{b}}_y \cdot \widehat{\mathbf{n}}_y)]$$

$$\boxed{R_x \ = \ m_B\,[\ddot{x} \ + \ \ddot{\theta}\,L\,\cos(\theta) \ - \ \dot{\theta}^2\,L\,\sin(\theta)]} \qquad \boxed{(R_y - m_B g) \ = \ m_B\,[-\ddot{\theta}\,L\,\sin(\theta) \ - \ \dot{\theta}^2\,L\,\cos(\theta)]}$$

Note: Separate analyses of A and B is less efficient than the ***road-map/D'Alembert method*** and Homework ??.12.

27.9.1 Dynamics for a rigid body with a simple angular velocity (special 2D case)



Euler's equation for a rigid body B with a ***simple angular velocity*** in a Newtonian reference frame N is:

$$\boxed{\vec{\mathbf{M}}_z^{B/B_{cm}} \underset{(8.3)}{=} \mathrm{I}_{zz} \; {}^N\vec{\boldsymbol{\alpha}}^B}$$

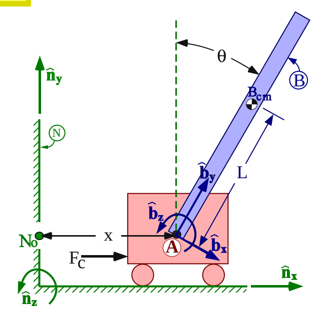
- $\vec{\mathbf{M}}_z^{B/B_{cm}}$ is the $\widehat{\mathbf{b}}_z = \widehat{\mathbf{n}}_z$ component of the moment of all forces on B about B_{cm}.
- I_{zz} is B's moment of inertia about the line passing through B_{cm} and parallel to $\widehat{\mathbf{b}}_z$.
- ${}^N\vec{\boldsymbol{\alpha}}^B$ is B's angular acceleration in N.

Assembling these terms and subsequent dot-multiplication with $\widehat{\mathbf{b}}_z$ produces

$$\boxed{L\cos(\theta)\,R_x \;-\; L\sin(\theta)\,R_y \;=\; \text{-}\mathrm{I}_{zz}\,\ddot{\theta}}$$

Summary of Newton/Euler equations of motion (inefficient)

$$\begin{array}{rcl}
F_c - R_x &=& m_A\,\ddot{x} \\
N - m_A\,g - R_y &=& 0 \\
R_x &=& m_B\,[\ddot{x} + \ddot{\theta}\,L\cos(\theta) - \dot{\theta}^2\,L\sin(\theta)] \\
(R_y - m_B g) &=& m_B\,[\text{-}\ddot{\theta}\,L\sin(\theta) - \dot{\theta}^2\,L\cos(\theta)] \\
L\cos(\theta)\,R_x - L\sin(\theta)\,R_y &=& \text{-}\mathrm{I}_{zz}\,\ddot{\theta}
\end{array}$$



There are **5** unknown variables in the previous set of equations, namely R_x, R_y, N, x, θ.
Note: Once $\theta(t)$ is known, $\dot{\theta}(t)$ and $\ddot{\theta}(t)$ are known. Similarly, once $x(t)$ is known, $\dot{x}(t)$ and $\ddot{x}(t)$ are known.

27.9.2 Equations of motion via road-maps/D'Alembert (efficient)

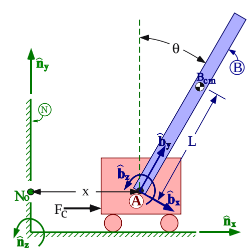
For various purposes (e.g., control system design), it is useful to eliminate the unknown "constraint forces" R_x, R_y, N. Although tedious linear-algebra can reduce the previous set of 5 equations in 5 unknowns to 2 equations in 2 unknowns ($\ddot{x}$, $\ddot{\theta}$), it is **more efficient** to use ***road maps/D'Alembert*** or the methods of Lagrange or Kane as they automatically eliminate R_x, R_y, N.

Road-map equation for x $\Rightarrow$ $\boxed{F_c \;=\; (m_A + m_B)\,\ddot{x} \;+\; m_B\,L\cos(\theta)\,\ddot{\theta} \;-\; m_B\,L\sin(\theta)\,\dot{\theta}^2}$

Road-map equation for θ $\Rightarrow$ $\boxed{m_B\,g\,L\sin(\theta) \;=\; m_B\,L\cos(\theta)\,\ddot{x} \;+\; (\mathrm{I}_{zz} + m_B\,L^2)\,\ddot{\theta}}$

27.10 Matrix form of nonlinear and linearized equations of motion

For numerical solution and various control-systems techniques, it can be useful to write this system's nonlinear equations of motion in matrix form as



$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} F_c \end{bmatrix} = \begin{bmatrix} m_A + m_B & m_B\,L\cos(\theta) \\ m_B\,L\cos(\theta) & \mathrm{I}_{zz} + m_B\,L^2 \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} \text{-}m_B\,L\sin(\theta)\,\dot{\theta}^2 \\ \text{-}m_B\,g\,L\sin(\theta) \end{bmatrix}$$

Certain stability analyses and control-system techniques depend on ***linear*** ODEs.
Using the small angle approximations $\cos(\theta) \approx 1$, $\sin(\theta) \approx \theta$, and the small rate approximation $\dot{\theta}^2 \approx 0$, this system's equations of motion can be ***linearized*** and put into various matrix forms, e.g.,

Linearized matrix form

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} F_c \end{bmatrix} = \begin{bmatrix} m_A + m_B & m_B\,L \\ m_B\,L & \mathrm{I}_{zz} + m_B\,L^2 \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \text{-}m_B\,g\,L \end{bmatrix} \begin{bmatrix} x \\ \theta \end{bmatrix}$$

State-space form

$d \;=\; \mathrm{I}_{zz}\,(m_A + m_B) \;+\; m_A\,m_B\,L^2$

$$\begin{bmatrix} x \\ \theta \\ \dot{x} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & (-g\,L^2\,m_B^2)\,/\,d & 0 & 0 \\ 0 & [g\,L\,m_B\,(m_A + m_B)]\,/\,d & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ (\mathrm{I}_{zz} + m_B\,L^2)\,/\,d \\ (\text{-}L\,m_B)\,/\,d \end{bmatrix} F_c$$