3.4 Independent, dependent, and specified variables

To classify a differential equation, four types of mathematical quantities need to be identified.

- An ***independent variable*** is a quantity that varies independently, i.e., it is not controlled by (or depend on) other variables.[1] For many dynamic systems, there is one independent variable that is either t (time) or s (a complex variable used in conjunction with the Laplace transform).
- The symbol x is a ***dependent variable*** if its value depends on the independent variable and its dependence is considered to be **unknown**, e.g., x is governed by a differential equation.
- The symbol x is a ***specified variable*** if its dependence on the independent variable can be considered to be **known**. For example, x is specified (or ***prescribed***) when $x = sin(t)$ is **known** apriori.
- The symbol c is a ***constant*** if its value does not change, i.e., its value does not depend on the independent variable.

3.5 Classification of algebraic and differential equations

Uncoupled	**Linear**	**Homogeneous**	**Constant-coefficient**	**1^{st}-order**	**Algebraic**
Coupled	**Nonlinear**	**Inhomogeneous**	**Variable-coefficient**	**2^{nd}-order**	**Differential**

In the following examples, x and y are dependent variables and t is the independent variable time.[2]

1. **Differential and algebraic equations**
   In each example below, circle ***differential*** if the equation contains a time-derivative of a dependent variable (unknown), otherwise circle ***algebraic***.

Note: Answers to these interactive questions are at www.MotionGenesis.com ⇒ Textbooks ⇒ Resources.

$\frac{dy}{dt} = 7$	**[differential]/algebraic** equation
$3 * y = 7$	**differential/[algebraic]** equation
$\frac{\partial^2 y}{\partial t^2} + \frac{\partial y}{\partial s} = 0$	**[differential]/algebraic** equation
$7y^4 + \cos(t)\sin(y)^3 = \tan(t) + 99$	**differential/[algebraic]** equation

- **Ordinary and partial differential equations**
  When the dependent variable x is a function of only *one* independent variable t, the derivative of x with respect to t is called an ***ordinary derivative***,[3] and is denoted $\frac{dx}{dt}$ or $\dot{x}$ or x'. Alternately, if x is a function of *two or more* independent variables (e.g., s and t), the derivative of x with respect to t is called the ***partial derivative*** of x with respect to t, and is denoted $\frac{\partial x}{\partial t}$.[4]
  In each example below, circle ***ordinary*** if the equation is an ***ODE***, i.e., contains only ordinary derivatives of dependent variables. Circle ***partial*** is the equation is an ***PDE***, i.e., contains a partial derivative of a dependent variable.

$\frac{dx}{dt} = e^t$	**[ordinary]/partial** differential equation
$\frac{\partial y}{\partial t} = 0$	**ordinary/[partial]** differential equation
$\frac{d^2x}{dt^2} + \sin(t)\left(\frac{dx}{dt}\right)^2 + \tan(x) = e^t$	**[ordinary]/partial** differential equation
$\frac{\partial^2 y}{\partial t^2} + \frac{\partial^2 y}{\partial s^2} = 0$	**ordinary/[partial]** differential equation

[1] To help remember t is an ***independent*** variable, associate t with an ***independent***, uncontrollable, []teenager.
[2] It is helpful to make an association of **GOOD** or **BAD** with each qualifier, e.g., linear is **GOOD** and nonlinear is **BAD**.
[3] Mathematicians regard ***ordinary*** derivatives as ***plain and boring*** whereas equations with 2^+ independent variables and ***partial*** derivatives are "***hot and spicy***" .
[4] This textbook studies ***ODEs*** (ordinary differential equations), i.e., with one independent variable – not ***PDEs***.

- **First-order, second-order, third-order, . . . , differential equations**
  The order of a differential equation is the order of the highest derivative of a dependent variable in the equation.[5] Complete each blank below with 1^{st}**-order**, 2^{nd}**-order**, or 3^{rd}**-order**.

$\sin(t)\left(\frac{dx}{dt}\right)^3 + \tan(x) = 0$	**1^{st}** -order ODE
$\frac{d^2x}{dt^2} + \left(\frac{dx}{dt}\right)^3 + \tan(x) = 0$	**2^{nd}** -order ODE
$\frac{\partial^3 y}{\partial t^3} + \frac{\partial y}{\partial s} = 0$	**3^{rd}** -order PDE

- **Constant, periodic, and variable coefficient differential equations**
  A differential equation is classified by the coefficients of factors containing dependent variables. Coefficients are either constant or depend exclusively on ***independent variables***, e.g.,[6]

Constant-coefficient	***All*** the coefficients are constants.
Periodic-coefficient	***All*** the coefficients are constants or periodic.
Variable-coefficient	***One*** or more coefficients are neither constant nor periodic.

  Complete each blank below with **constant**, **periodic**, or **variable**.

$\ddot{x} + 3 * x - \sin(t) = 0$	**constant** -coefficient, 2^{nd}-order, ODE
$\ddot{x} + t * x = 0$	**variable** -coefficient, 2^{nd}-order, ODE
$3\ddot{x} + x * \dot{x} + 7 * \sin(x) * x = 0$	**constant** -coefficient, 2^{nd}-order, ODE
$\sin(t) * \dot{x} + x = 0$	**periodic** -coefficient, 1^{st}-order, ODE

2. **Homogeneous and inhomogeneous (differential or algebraic equations)**
   A differential equation is classified by whether it not all its terms have a dependent variable.

Homogeneous	***All*** terms contain one or more dependent variables.
Inhomogeneous	***One*** term is constant or depends exclusively on ***independent variables***

   Complete each example below by circling **homogeneous** or **inhomogeneous**.

$x + y = 0$	**[homogeneous]/inhomogeneous**	algebraic equation
$x + y = t$	**homogeneous/[inhomogeneous]**	algebraic equation
$t^2\,\ddot{x} + \sin(t)\,\dot{x}^2 + x = 0$	**[homogeneous]/inhomogeneous**	ODE
$\ddot{x} + 3 = 0$	**homogeneous/[inhomogeneous]**	ODE

   Note: Equations are frequently written in a "standard form" with all the **unknowns** (e.g., terms containing x, $\dot{x}$, $\ddot{x}$, y, $\dot{y}$, $\ddot{y}$) on the left-hand side of the equal sign and all the remaining terms on the right-hand side. The equation's right-hand side will either be 0, a constant, or exclusively a function of independent variables. If the right-hand side is 0, the equation is ***homogeneous*** otherwise it is ***inhomogeneous***.[7]

3. **Linear and nonlinear (differential or algebraic equations)**
   To determine if an expression is ***linear*** or ***nonlinear***, one must first ask *"Linear in what?"*. In the context of differential equations, a differential equation is ***linear*** if it is linear in its ***dependent variables*** and their derivatives (e.g., linear in x, $\dot{x}$, $\ddot{x}$, y, $\dot{y}$, $\ddot{y}$).

   For example, a ***linear*** 2^{nd}-order ODE with one dependent variable x can be expressed as

$$c_0 + c_1 * x + c_2 * \dot{x} + c_3 * \ddot{x} = 0$$

   where c_i (i=0,1,2,3) are **not** functions of x, $\dot{x}$, $\ddot{x}$, (but may be functions of t).

[5] A physical system that is governed by a n^{th}-order differential equation is called an n^{th}-order system.
[6] Periodic-coefficient ODEs are a special subset of variable-coefficient ODEs whose solution involves Floquet theory.
[7] Separating terms with a dependent variable from those without is analogous to a farmer separating sheep from goats. If the farmer only has sheep, the herd is ***homogeneous***. If, after the farmer has put all dependent variables into the barn, there are any terms still out in the fields, the equation is ***inhomogeneous***.

A ***linear*** 1^{st}-order ODE with dependent variables x, y, z can be expressed as

$$c_0 \;+\; c_1 * x \;+\; c_2 * \dot{x} \;+\; c_3 * y \;+\; c_4 * \dot{y} \;+\; c_5 * z \;+\; c_6 * \dot{z} \;=\; 0$$

where c_i (i = 0, ..., 6) are **not** functions of x, $\dot{x}$, y, $\dot{y}$, z, $\dot{z}$, (but may be functions of t).

Complete each example below by circling **linear** or **nonlinear**.

$[t^2 + \sin(t)]\, x \;+\; \tan(t) \;=\; 4$	**[linear]**/**nonlinear**	inhomogeneous algebraic equation
$x^2 \;+\; \sin(x) \;=\; 4$	**linear**/**[nonlinear]**	inhomogeneous algebraic equation
$t^2\, \ddot{x} \;+\; \sin(t)\, \dot{x} \;+\; x \;=\; 0$	**[linear]**/**nonlinear**	homogeneous ODE
$\ddot{x} \;+\; \sin(x) \;=\; 0$	**linear**/**[nonlinear]**	homogeneous ODE

Note: Solving ***nonlinear equations*** is usually significantly ***more difficult*** than solving ***linear equations***. Even determining the ***number*** of solutions to nonlinear equations can be difficult, e.g., a nonlinear equation may have no solution, 1 solution, 2 solutions, 17 solutions, or infinite solutions. For example, the solutions to $x^2 + \sin(x) = 4$ are $x \approx 1.736$ and $x \approx 2.194$, and can be obtained by a variety of methods, e.g., Newton-Rhapson or graphing.

4. **Uncoupled and coupled (differential or algebraic equations)**
To determine if a set of equations is ***coupled***, one must first ask *"Coupled in what?"*. A *set* of equations is ***coupled*** when the same unknown appears in more than one equation. A set of ODEs is coupled if a ***dependent variable*** or its derivative (e.g., x, $\dot{x}$, $\ddot{x}$) appear in more than one equation.
Complete each blank below with ***coupled*** or ***uncoupled***.

$$t^2 \cos(x^3) \;+\; \sin(t)\, x^2 \;+\; x \;=\; 0 \qquad y \;+\; t^3 \sin(y) \;-\; 9 \;=\; 0$$

uncoupled nonlinear inhomogeneous algebraic

$$x \;+\; y \;=\; 0 \qquad x \;=\; 7$$

coupled linear inhomogeneous algebraic

$$t^2 \ddot{x} \;+\; \sin(t)\dot{x}^2 \;+\; x \;=\; 0 \qquad \ddot{y} \;+\; t^3 \sin(y) \;-\; \tan(t) \;=\; 0$$

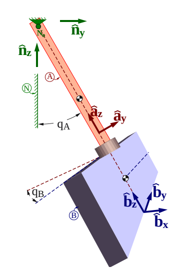
uncoupled nonlinear inhomogeneous ODE

$$\dot{x} \;+\; \dot{y} \;=\; 0 \qquad \ddot{x} \;=\; 0$$

coupled linear homogeneous ODE

3.6 Example of classification of differential equations

The equations governing the motion of the system described in Section 3.1 are



$$\ddot{q}_A \;=\; \frac{2\,[508.89 \sin(q_A) \;-\; \sin(q_B) \cos(q_B)\, \dot{q}_A\, \dot{q}_B]}{-21.556 + \sin^2(q_B)}$$

$$\ddot{q}_B \;=\; -\sin(q_B) \cos(q_B)\, \dot{q}_A^2$$

This set of equations is classified as (circle the appropriate qualifiers)

Uncoupled	Linear	**[Homogeneous]**	**[Constant-coefficient]**	1^{st}-order	Algebraic
[Coupled]	**[Nonlinear]**	Inhomogeneous	Variable-coefficient	**[2^{nd}-order]**	**[Differential]**

It is worth noting that in general, an algebraic equation is **good** if it is linear. A differential equation is **good** if it is linear, homogeneous, constant-coefficient, first-order, and ordinary.[8]

[8] A **good** equation means that it is easy to manipulate and obtain a solution. A **bad** equation means that instead of enjoying time with your friends, you are struggling with difficult mathematics.