

3.4 Independent, dependent, and specified variables

To classify a differential equation, four types of mathematical quantities need to be identified.

- An **independent variable** is a quantity that varies independently, i.e., it is not controlled by (or depend on) other variables.² For many dynamic systems, there is one independent variable that is either t (time) or s (a complex variable used in conjunction with the Laplace transform).
- The symbol x is a **dependent variable** if its value depends on the independent variable and its dependence is considered to be **unknown**, e.g., x is governed by a differential equation.
- The symbol x is a **specified variable** if its dependence on the independent variable can be considered to be **known**. For example, x is specified (or **prescribed**) when $x = \sin(t)$ is **known** a priori.
- The symbol c is a **constant** if its value does not change, i.e., its value does not depend on the independent variable.

3.5 Classification of algebraic and differential equations

| | | | | | |
|-----------|-----------|---------------|----------------------|------------------------|--------------|
| Uncoupled | Linear | Homogeneous | Constant-coefficient | 1 st -order | Algebraic |
| Coupled | Nonlinear | Inhomogeneous | Variable-coefficient | 2 nd -order | Differential |

In the following examples, x and y are dependent variables and t is the independent variable time.³

1. Differential and algebraic equations

In each example below, circle **differential** if the equation contains a time-derivative of a dependent variable (unknown), otherwise circle **algebraic**.

| | |
|---|--|
| $\frac{dy}{dt} = 7$ | differential/algebraic equation |
| $3 * y = 7$ | differential/algebraic equation |
| $\frac{\partial^2 y}{\partial t^2} + \frac{\partial y}{\partial s} = 0$ | differential/algebraic equation |
| $7y^4 + \cos(t) \sin(y)^3 = \tan(t) + 99$ | differential/algebraic equation |

- **Ordinary and partial differential equations**

When the dependent variable x is a function of only *one* independent variable t , the derivative of x with respect to t is called an **ordinary derivative**,⁴ and is denoted $\frac{dx}{dt}$ or \dot{x} or x' . Alternately, if x is a function of *two or more* independent variables (e.g., s and t), the derivative of x with respect to t is called the **partial derivative** of x with respect to t , and is denoted $\frac{\partial x}{\partial t}$.⁵

In each example below, circle **ordinary** if the equation is an **ODE**, i.e., contains only ordinary derivatives of dependent variables. Circle **partial** if the equation is an **PDE**, i.e., contains a partial derivative of a dependent variable.

| | |
|---|---|
| $\frac{dx}{dt} = e^t$ | ordinary/partial differential equation |
| $\frac{\partial y}{\partial t} = 0$ | ordinary/partial differential equation |
| $\frac{d^2 x}{dt^2} + \sin(t) \left(\frac{dx}{dt}\right)^2 + \tan(x) = e^t$ | ordinary/partial differential equation |
| $\frac{\partial^2 y}{\partial t^2} + \frac{\partial^2 y}{\partial s^2} = 0$ | ordinary/partial differential equation |

²To help remember t is an **independent** variable, associate t with an **independent**, uncontrollable, [teenager].

³It is helpful to make an association of **GOOD** or **BAD** with each qualifier, e.g., linear is **GOOD** and nonlinear is **BAD**.

⁴Mathematicians regard **ordinary** derivatives as **plain and boring** whereas equations with 2⁺ independent variables and **partial** derivatives are **“hot and spicy”**.

⁵This textbook studies **ODEs** (ordinary differential equations), i.e., with **one** independent variable – not **PDEs**.

- **First-order, second-order, third-order, . . . , differential equations**

The order of a differential equation is the order of the highest derivative of a dependent variable in the equation.⁶ Complete each blank below with 1st-order, 2nd-order, or 3rd-order.

$$\sin(t) \left(\frac{dx}{dt}\right)^3 + \tan(x) = 0 \quad \text{[]-order ODE}$$

$$\frac{d^2x}{dt^2} + \left(\frac{dx}{dt}\right)^3 + \tan(x) = 0 \quad \text{[]-order ODE}$$

$$\frac{\partial^3y}{\partial t^3} + \frac{\partial y}{\partial s} = 0 \quad \text{[]-order PDE}$$

- **Constant, periodic, and variable coefficient differential equations**

A differential equation is classified by the coefficients of factors containing dependent variables. Coefficients are either constant or depend exclusively on *independent variables*, e.g.,⁷

| | |
|-----------------------------|--|
| Constant-coefficient | All the coefficients are constants. |
| Periodic-coefficient | All the coefficients are constants or periodic. |
| Variable-coefficient | One or more coefficients are neither constant nor periodic. |

Complete each blank below with **constant**, **periodic**, or **variable**.

$$\ddot{x} + 3 * x - \sin(t) = 0 \quad \text{[]-coefficient, 2nd-order, ODE}$$

$$\ddot{x} + t * x = 0 \quad \text{[]-coefficient, 2nd-order, ODE}$$

$$3\ddot{x} + x * \dot{x} + 7 * \sin(x) * x = 0 \quad \text{[]-coefficient, 2nd-order, ODE}$$

$$\sin(t) * \dot{x} + x = 0 \quad \text{[]-coefficient, 1st-order, ODE}$$

2. Homogeneous and inhomogeneous (differential or algebraic equations)

A differential equation is classified by whether it not all its terms have a dependent variable.

| | |
|----------------------|--|
| Homogeneous | All terms contain one or more dependent variables. |
| Inhomogeneous | One term is constant or depends exclusively on <i>independent variables</i> |

Complete each example below by circling **homogeneous** or **inhomogeneous**.

$$x + y = 0 \quad \text{homogeneous/inhomogeneous algebraic equation}$$

$$x + y = t \quad \text{homogeneous/inhomogeneous algebraic equation}$$

$$t^2 \ddot{x} + \sin(t) \dot{x}^2 + x = 0 \quad \text{homogeneous/inhomogeneous ODE}$$

$$\ddot{x} + 3 = 0 \quad \text{homogeneous/inhomogeneous ODE}$$

Note: Equations are frequently written in a “standard form” with all the **unknowns** (e.g., terms containing x , \dot{x} , \ddot{x} , y , \dot{y} , \ddot{y}) on the left-hand side of the equal sign and all the remaining terms on the right-hand side. The equation’s right-hand side will either be 0, a constant, or exclusively a function of independent variables. If the right-hand side is 0, the equation is *homogeneous* otherwise it is *inhomogeneous*.⁸

3. Linear and nonlinear (differential or algebraic equations)

To determine if an expression is *linear* or *nonlinear*, one must first ask “*Linear in what?*”. In the context of differential equations, a differential equation is *linear* if it is linear in its *dependent variables* and their derivatives (e.g., linear in x , \dot{x} , \ddot{x} , y , \dot{y} , \ddot{y}).

For example, a *linear* 2nd-order ODE with one dependent variable x can be expressed as

$$c_0 + c_1 * x + c_2 * \dot{x} + c_3 * \ddot{x} = 0$$

where c_i ($i=0,1,2,3$) are **not** functions of x , \dot{x} , \ddot{x} , (but may be functions of t).

⁶A physical system that is governed by a n^{th} -order differential equation is called an n^{th} -order system.

⁷Periodic-coefficient ODEs are a special subset of variable-coefficient ODEs whose solution involves Floquet theory.

⁸Separating terms with a dependent variable from those without is analogous to a farmer separating sheep from goats. If the farmer only has sheep, the herd is *homogeneous*. If, after the farmer has put all dependent variables into the barn, there are any terms still out in the fields, the equation is *inhomogeneous*.

A **linear** 1st-order ODE with dependent variables x, y, z can be expressed as

$$c_0 + c_1 * x + c_2 * \dot{x} + c_3 * y + c_4 * \dot{y} + c_5 * z + c_6 * \dot{z} = 0$$

where c_i ($i = 0, \dots, 6$) are **not** functions of $x, \dot{x}, y, \dot{y}, z, \dot{z}$, (but may be functions of t).

Complete each example below by circling **linear** or **nonlinear**.

| | |
|--|--|
| $[t^2 + \sin(t)]x + \tan(t) = 4$ | linear/nonlinear inhomogeneous algebraic equation |
| $x^2 + \sin(x) = 4$ | linear/nonlinear inhomogeneous algebraic equation |
| $t^2 \ddot{x} + \sin(t) \dot{x} + x = 0$ | linear/nonlinear homogeneous ODE |
| $\ddot{x} + \sin(x) = 0$ | linear/nonlinear homogeneous ODE |

Note: Solving **nonlinear equations** is usually significantly **more difficult** than solving **linear equations**. Even determining the **number** of solutions to nonlinear equations can be difficult, e.g., a nonlinear equation may have no solution, 1 solution, 2 solutions, 17 solutions, or infinite solutions. For example, the solutions to $x^2 + \sin(x) = 4$ are $x \approx 1.736$ and $x \approx 2.194$, and can be obtained by a variety of methods, e.g., Newton-Rhapson or graphing.

4. Uncoupled and coupled (differential or algebraic equations)

To determine if a set of equations is **coupled**, one must first ask “*Coupled in what?*”. A set of equations is **coupled** when the same unknown appears in more than one equation. A set of ODEs is coupled if a **dependent variable** or its derivative (e.g., x, \dot{x}, \ddot{x}) appear in more than one equation.

Complete each blank below with **coupled** or **uncoupled**.

| | | |
|--------------------------------------|----------------------|-----------------------------------|
| $t^2 \cos(x^3) + \sin(t)x^2 + x = 0$ | <input type="text"/> | nonlinear inhomogeneous algebraic |
| $y + t^3 \sin(y) - 9 = 0$ | | |

| | | |
|-------------|----------------------|--------------------------------|
| $x + y = 0$ | <input type="text"/> | linear inhomogeneous algebraic |
| $x = 7$ | | |

| | | |
|--|----------------------|-----------------------------|
| $t^2 \ddot{x} + \sin(t) \dot{x}^2 + x = 0$ | <input type="text"/> | nonlinear inhomogeneous ODE |
| $\ddot{y} + t^3 \sin(y) - \tan(t) = 0$ | | |

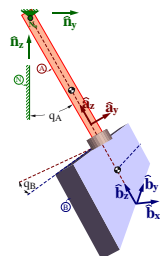
| | | |
|-------------------------|----------------------|------------------------|
| $\dot{x} + \dot{y} = 0$ | <input type="text"/> | linear homogeneous ODE |
| $\ddot{x} = 0$ | | |

3.6 Example of classification of differential equations

The equations governing the motion of the system described in Section 3.1 are

$$\ddot{q}_A = \frac{2 [508.89 \sin(q_A) - \sin(q_B) \cos(q_B) \dot{q}_A \dot{q}_B]}{-21.556 + \sin^2(q_B)}$$

$$\ddot{q}_B = -\sin(q_B) \cos(q_B) \dot{q}_A^2$$



This set of equations is classified as (circle the appropriate qualifiers)

| | | | | | |
|-----------|-----------|---------------|----------------------|------------------------|--------------|
| Uncoupled | Linear | Homogeneous | Constant-coefficient | 1 st -order | Algebraic |
| Coupled | Nonlinear | Inhomogeneous | Variable-coefficient | 2 nd -order | Differential |

It is worth noting that in general, an algebraic equation is **good** if it is linear. A differential equation is **good** if it is linear, homogeneous, constant-coefficient, first-order, and ordinary.⁹

⁹A **good** equation means that it is easy to manipulate and obtain a solution. A **bad** equation means that instead of enjoying time with your friends, you are struggling with difficult mathematics.