

5.7 Solution process for linear, 2nd-order, ODEs

To solve equation (2) $\frac{d^2y}{dt^2} + 2\zeta\omega_n \frac{dy}{dt} + \omega_n^2 y = 0$, begin by assuming a solution of the form (assume a solution of the form $C e^{pt}$ because it worked for 1st-order ODEs in Section 4.3 and it is the best guess we have).

$$y(t) = C e^{pt} \quad \text{whose 1}^{st} \text{ and 2}^{nd} \text{ derivatives are} \quad \frac{dy}{dt} = \quad \frac{d^2y}{dt^2} = \quad (12)$$

where C and p are **constants** to be determined [$C \neq 0$ since $C = 0$ gives the trivial (boring) solution $y(t) = 0$].

Substitute $y(t)$, $\dot{y}(t)$, $\ddot{y}(t)$ from eqn (12) into (2). $\quad + 2\zeta\omega_n (\quad) + \omega_n^2 (\quad) = 0$ (2)

Factor out $C e^{pt}$ to the right. $[\quad] * C * e^{pt} = 0$

Section 7.1 and equation (7.1) show $e^{pt} \neq 0$. $[\quad] * C = 0$

Use $C \neq 0$ to get “**characteristic equation**”: $[\quad] = 0$

Solve for p (the values of p are called “**poles**”). $p_{1,2} = -\zeta\omega_n \pm \omega_n \sqrt{\zeta^2 - 1}$ (13)

Assemble the solution. $y(t) = C_1 e^{p_1 t} + C_2 e^{p_2 t}$

Completed answer for blanks at www.MotionGenesis.com \Rightarrow [Textbooks](#) \Rightarrow [Resources](#).

5.7.1 Solution process for linear, n^{th} -order, constant-coefficient ODEs

The **characteristic equation** in the previous section was a **quadratic equation** for p and had **two** roots, p_1 and p_2 . More generally, the solution process in the previous section works for **any order** linear, constant-coefficient, ODE and results in a **characteristic polynomial equation** in p .

| | | |
|------------------------|---|---|
| 1 st -order | $a_1 p + a_0 = 0$ | Solve linear equation as $p = -a_0/a_1$. |
| 2 nd -order | $a_2 p^2 + a_1 p + a_0 = 0$ | Solve with quadratic equation . |
| 3 rd -order | $a_3 p^3 + a_2 p^2 + a_1 p + a_0 = 0$ | Solve with calculator or computer. |
| n^{th} -order | $a_n p^n + \dots + a_3 p^3 + a_2 p^2 + a_1 p + a_0 = 0$ | Solve with calculator or computer (eigenvalue routine). |

5.7.2 Derivation of solution for **undamped**, second-order, ODE

When $\zeta = 0$ undamped, equation (2) simplifies to $\frac{d^2y}{dt^2} + \omega_n^2 y = 0$ and the roots p_1 and p_2 simplify to

$$p_1 \stackrel{(13)}{=} +i\omega_n \quad p_2 \stackrel{(13)}{=} -i\omega_n \quad (14)$$

With these two roots and Euler’s formula [equation (5.8)], there are two solutions of the ODE.

$$y_1(t) \stackrel{(12)}{=} C_1 e^{p_1 t} \stackrel{(14)}{=} C_1 e^{i\omega_n t} \stackrel{(5.8)}{=} C_1 [\cos(\omega_n t) + i \sin(\omega_n t)] \quad (15)$$

$$y_2(t) \stackrel{(12)}{=} C_2 e^{p_2 t} \stackrel{(14)}{=} C_2 e^{-i\omega_n t} \stackrel{(5.8)}{=} C_2 [\cos(\omega_n t) - i \sin(\omega_n t)] \quad (16)$$

$$y(t) = y_1(t) + y_2(t) \stackrel{(15,16)}{=} (C_1 + C_2) \cos(\omega_n t) + (C_1 - C_2) i \sin(\omega_n t) \quad (17)$$

Since C_1 and C_2 are yet-to-be-determined constants, they can be replaced by two new other undetermined constants A and B (defined below) so $y(t)$ can be written as:

$$A \triangleq C_1 + C_2 \quad B \triangleq (C_1 - C_2) i \quad \Rightarrow \quad y(t) \stackrel{(17)}{=} A \cos(\omega_n t) + B \sin(\omega_n t)$$

Alternately, $y(t)$ can be rewritten using the “**amplitude-phase**” formula [equation (2.21)] as

$$y(t) = \bar{C} \cos(\omega_n t + \phi) \quad \text{where} \quad \bar{C} = \sqrt{A^2 + B^2} \quad \phi = \text{atan2}(-A, B)$$