5.7 Solution process for linear, 2^{nd}-order, ODEs

To solve equation (2) $\frac{d^2y}{dt^2} + 2\,\zeta\,\omega_n \frac{dy}{dt} + \omega_n^2\,y = 0$, begin by assuming a solution of the form (assume a solution of the form $C\,e^{p\,t}$ because it worked for 1^{st}-order ODEs in Section 4.3 and it is the best guess we have).

$$\boxed{y(t) \;=\; C\,e^{p\,t}} \qquad \text{whose } 1^{st} \text{ and } 2^{nd} \text{ derivatives are} \qquad \frac{dy}{dt} = \boxed{p\,C\,e^{p\,t}} \qquad \frac{d^2y}{dt^2} = \boxed{p^2\,C\,e^{p\,t}} \tag{12}$$

where C and p are **constants** to be determined [$C \neq 0$ since $C = 0$ gives the trivial (boring) solution $y(t) = 0$].

Substitute $y(t)$, $\dot{y}(t)$, $\ddot{y}(t)$ from eqn (12) into (2). $\quad \boxed{p^2\,C\,e^{p\,t}} + 2\,\zeta\,\omega_n \left(\boxed{p\,C\,e^{p\,t}}\right) + \omega_n^2 \left(\boxed{C\,e^{p\,t}}\right) \underset{(2)}{=} 0$

Factor out $C\,e^{p\,t}$ to the right. $\quad \left[\boxed{p^2 + 2\,\zeta\,\omega_n\,p + \omega_n^2}\right] * C * e^{p\,t} = 0$

Section 7.1 and equation (7.1) show $e^{p\,t} \neq 0$. $\quad \left[\boxed{p^2 + 2\,\zeta\,\omega_n\,p + \omega_n^2}\right] * C = 0$

Use $C \neq 0$ to get "***characteristic equation***": $\quad \left[\boxed{p^2 + 2\,\zeta\,\omega_n\,p + \omega_n^2}\right] = 0$

Solve for p (the values of p are called "***poles***"). $\quad p_{1,2} = -\zeta\,\omega_n \pm \omega_n\,\sqrt{\zeta^2 - 1} \quad (13)$

Assemble the solution. $\quad \boxed{y(t) \;=\; C_1\,e^{p_1\,t} + C_2\,e^{p_2\,t}}$

Completed answer for blanks at www.MotionGenesis.com ⇒ Textbooks ⇒ Resources.

5.7.1 Solution process for linear, n^{th}-order, constant-coefficient ODEs

The ***characteristic equation*** in the previous section was a ***quadratic equation*** for p and had **two** roots, p_1 and p_2. More generally, the solution process in the previous section works for **any order** linear, constant-coefficient, ODE and results in a ***characteristic polynomial equation*** in p.

1^{st}-order	$a_1\,p + a_0 = 0$	Solve linear equation as $p = -a_0/a_1$.
2^{nd}-order	$a_2\,p^2 + a_1\,p + a_0 = 0$	Solve with ***quadratic equation***.
3^{rd}-order	$a_3\,p^3 + a_2\,p^2 + a_1\,p + a_0 = 0$	Solve with calculator or computer.
n^{th}-order	$a_n\,p^n \ldots + a_3\,p^3 + a_2\,p^2 + a_1\,p + a_0 = 0$	Solve with calculator or computer (eigenvalue routine).

5.7.2 Derivation of solution for *undamped*, second-order, ODE

When $\zeta = 0$ undamped, equation (2) simplifies to $\frac{d^2y}{dt^2} + \omega_n^2\,y \underset{(2)}{=} 0$ and the roots p_1 and p_2 simplify to

$$p_1 \underset{(13)}{=} +i\,\omega_n \qquad\qquad p_2 \underset{(13)}{=} -i\,\omega_n \tag{14}$$

With these two roots and Euler's formula [equation (5.8)], there are two solutions of the ODE.

$$y_1(t) \underset{(12)}{=} C_1\,e^{p_1\,t} \underset{(14)}{=} C_1\,e^{i\,\omega_n\,t} \underset{(5.8)}{=} C_1\,[\cos(\omega_n\,t) + i\,\sin(\omega_n\,t)] \tag{15}$$

$$y_2(t) \underset{(12)}{=} C_2\,e^{p_2\,t} \underset{(14)}{=} C_2\,e^{-i\,\omega_n\,t} \underset{(5.8)}{=} C_2\,[\cos(\omega_n\,t) - i\,\sin(\omega_n\,t)] \tag{16}$$

$$y(t) = y_1(t) + y_2(t) \underset{(15\,16)}{=} (\boxed{C_1 + C_2})\cos(\omega_n t) + (\boxed{C_1 - C_2})\,i\,\sin(\omega_n t) \tag{17}$$

Since C_1 and C_2 are yet-to-be-determined constants, they can be replaced by two new other undetermined constants A and B (defined below) so $y(t)$ can be written as:

$$A \triangleq C_1 + C_2 \qquad B \triangleq (C_1 - C_2)\,i \qquad \Rightarrow \qquad y(t) \underset{(17)}{=} A\,\cos(\omega_n\,t) + B\,\sin(\omega_n\,t)$$

Alternately, $y(t)$ can be rewritten using the ***"amplitude-phase" formula*** [equation (2.21)] as

$$y(t) = \bar{C}\,\cos(\omega_n\,t + \phi) \qquad \text{where} \qquad \bar{C} = \sqrt{A^2 + B^2} \qquad \phi = \texttt{atan2}(-A,\; B)$$