Chapter 25

Undamped coupled 2^{nd}-order ODEs

Summary (see examples in Hw 13)

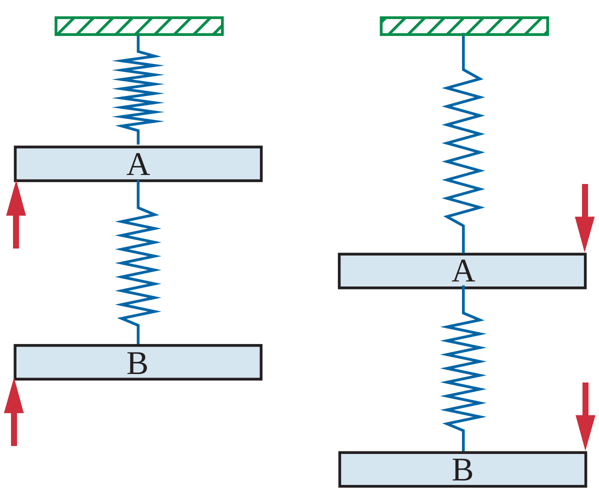
Many physical phenomena are governed by a set of n **undamped**, coupled, linear, constant-coefficient, homogeneous ODEs. These include free vibrations of buildings, airplanes, automobiles, space structures, and molecules.

$$\boxed{M\,\ddot{X} \;+\; K\,X \;=\; [0]} \qquad X = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \cdots \\ x_n(t) \end{bmatrix}$$

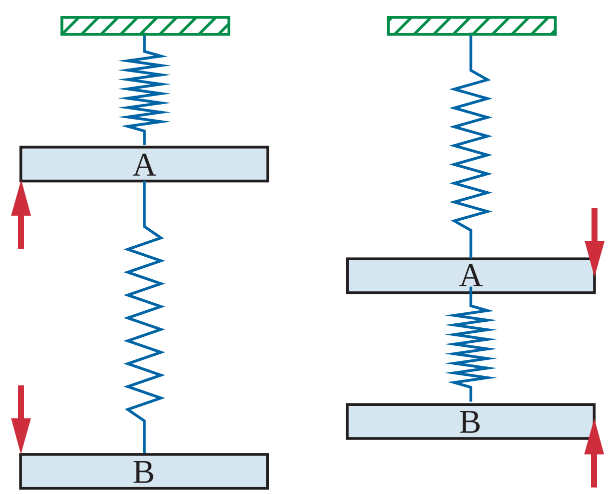
25.1 Physical insights into eigenvalues and eigenvectors of a slinky

Two long thin rulers (A and B) connected to a slinky (spring) are useful for demonstrating the physical significance of eigenvalues and eigenvectors.

Shown below is A and B moving in the ***first mode***, moving up ↑ together or down ↓ together.



Shown below is A and B moving in the ***second mode*** (moving in opposite directions).



Mode #	Eigenvalue (frequency)	Eigenvector (use + or 0 or −)	Potential Energy ($\frac{1}{2}\,k\,\delta^2$)	Kinetic Energy ($\frac{1}{2}\,m\,v^2$)	Total Energy Kinetic + Potential
1	$\boxed{\text{small}}$/large	$\begin{bmatrix}+\\+\end{bmatrix}$ or $\begin{bmatrix}-\\-\end{bmatrix}$	$\boxed{\text{small}}$/large $\boxed{\text{one}}$/two deformed spring $\boxed{\text{min}}$/max deformation	$\boxed{\text{small}}$/large	$\boxed{\text{small}}$/large
2	small/$\boxed{\text{large}}$	$\begin{bmatrix}+\\-\end{bmatrix}$ or $\begin{bmatrix}-\\+\end{bmatrix}$	small/$\boxed{\text{large}}$ one/$\boxed{\text{two}}$ deformed springs min/$\boxed{\text{max}}$ deformation	small/$\boxed{\text{large}}$	small/$\boxed{\text{large}}$

Answers at www.MotionGenesis.com ⇒ Textbooks ⇒ Resources.

25.2 Analytical solutions of *undamped, coupled,* ODEs

X	$n \times 1$ matrix of dependent variables
M, B, K	$n \times n$ matrices of constants
G	$n \times p$ matrix of constants
F	$p \times 1$ matrix of known functions of time $f_1(t)$, $f_2(t)$, $\ldots$, $f_p(t)$.

An undamped set of n coupled, linear, constant-coefficient, ***2^{nd}-order ODEs*** can be written in the matrix form

$$\boxed{\underset{(n\times n)}{M}\ \ddot{X} \ +\ \underset{(n\times n)}{B}\ \dot{X} \ +\ \underset{(n\times n)}{K}\ X \ =\ \underset{(n\times p)}{G}\ \underset{(p\times 1)}{F}} \qquad (1)$$

To solve equation (1), write $X(t) = X_h(t) + X_p(t)$ (the sum of a homogeneous and particular solution). The homogeneous solution $X_h(t)$ can be found by assuming a solution of the form shown in equation (2), where p is a yet-to-be-determined constant and U is a yet-to-be-determined **non-zero** $n \times 1$ matrix of constants.[a]

$$X_h(t) \ =\ \boxed{U\, e^{p\,t}} \qquad (2)$$

$$U \ \triangleq\ \begin{bmatrix} u_1 \\ u_2 \\ \ldots \\ u_n \end{bmatrix}$$

[a] It is reasonable to guess $X_h(t) = U\, e^{p\,t}$ because it worked for **uncoupled** linear ODEs. The matrix U is **non-zero** because $U = [0]$ produces the ***trivial solution*** $X_h(t) = [0]$, which is not a solution of interest (not what we are looking for).

Substituting/differentiating equation (2) into the homogeneous part of equation (1) $(F = [0])$, gives

Note: This proof mimics Section 23.1. Answers are at www.MotionGenesis.com ⇒ Textbooks ⇒ Resources.

$$M\left[p^2\, U\, e^{p\,t}\right] \ +\ B\left[p\, U\, e^{p\,t}\right] \ +\ K\left[U\, e^{p\,t}\right] \ \underset{(1,\,2)}{=}\ [0] \qquad \text{Factor on } U e^{p\,t}$$

$$\left[p^2\, M \ +\ p\, B \ +\ K\right]\ U\, e^{p\,t} \ =\ [0] \qquad \text{Section 7.1 shows } e^{p\,t} \neq 0$$

$$\left[p^2\, M \ +\ p\, \overset{0}{B} \ +\ K\right] U \ =\ [0] \qquad \text{Sometimes } \boxed{B \approx [0]} \text{ (see **undamped** below)}$$

$$\left[p^2\, M \ +\ K\right]\ U \ =\ [0] \qquad \text{Define: } \boxed{\lambda \ \triangleq\ -p^2} \ \boxed{p = \pm\sqrt{-\lambda}}$$

$$\boxed{\textit{Generalized eigen-equation} \quad \left[-\lambda\, M \ +\ K\right] U \ =\ [0]} \quad \Rightarrow \quad \boxed{\det\left[-\lambda\, M \ +\ K\right] \ =\ 0} \qquad (3)$$

This eigen-equation is a **coupled** nonlinear algebraic equation. It has ***n*** equations and ***n* + 1** unknowns in λ and U.
The related $\det[-\lambda\, M \ +\ K] = 0$ is **1** **uncoupled** nonlinear algebraic equation with **1** unknown λ.
Equation (1) ⇒ (3) changes ***n*** coupled ODEs with ***n*** unknowns into a ***eigen-equation***.

Undamped: It can be reasonable to set $B = [0]$ if:
- Damping is small.
- It is difficult to analytically or experimentally determine the elements of the damping matrix.
- In structural vibrations, damping is approximated with ***modal damping***.

For eigen-equation (3) to produce a **non-zero** U (and **non-zero** X_h), the inverse of $[-\lambda\, M + K]$ must not exist. To see this, suppose $[-\lambda M + K]^{-1}$ does exist and multiply both sides of the eigen-equation by $[-\lambda\, M + K]^{-1}$, which produces $U = [0]$. For $U \neq [0]$, one must find ***special*** values of λ so $[-\lambda\, M + K]^{-1}$ does not exist. These ***special*** values λ_1, λ_2, $\ldots$, λ_n are called ***eigenvalues***. For each λ_i, there is a corresponding ***special*** non-zero U_i called the ***eigenvector*** corresponding to λ_i. Note: The eigenvalue problem is a ***special*** nonlinear algebraic equation because the number of solutions is known apriori.

The following are equivalent statements about equation (4) and finding the ***eigenvalues*** λ:
- Find the values of λ which result in $U \neq [0]$.
- Find the values of λ so the matrix $[-\lambda\, M + K]$ is singular, i.e., $[-\lambda\, M + K]^{-1}$ does not exist.
- Find the values of λ so the determinant of $[-\lambda\, M + K]$ is zero, i.e., $\boxed{\det[-\lambda\, M + K] \ =\ 0}$

Computational notes: For small sets of coupled equations $(n \leq 3)$, setting the determinate to zero is an effective way to determine λ - and this method works when M or K have symbolic (non-numeric) elements. When n is large, numerical eigenvalue algorithms are used to determine λ. Robust eigenvalue and eigenvector algorithms are relatively new (circa 1960) and are available in computer programs such as MATLAB® and **M**otion**G**enesis. If both M and K are symmetric and positive definite, the eigenvalues λ_i (i=1,…n) are real positive numbers and eigenvectors U_i (i=1,…n) are real (not complex), and there are special efficient algorithms for calculating λ_i and U_i (especially important when n is large).