Chapter 23

Undamped coupled 2^{nd}-order ODEs

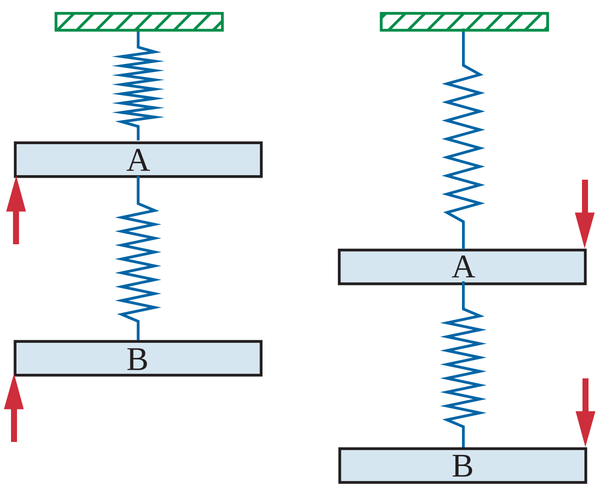
Summary (see examples in Hw 13)

Many physical phenomena are governed by **undamped**, coupled, linear, constant-coefficient, ODEs. These include certain motions of buildings, airplanes, automobiles, beams, space structures, and molecules.

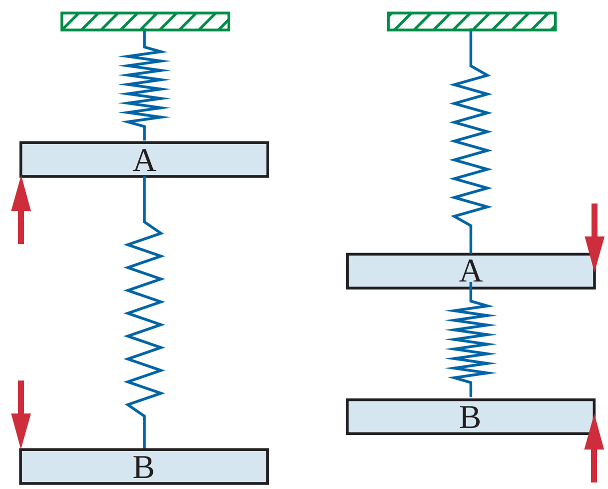
23.1 Physical insights into eigenvalues and eigenvectors of a slinky

Two long thin rulers (A and B) connected to a slinky (spring) are useful for demonstrating the physical significance of eigenvalues and eigenvectors.

Shown below is A and B moving in the ***first mode***, moving up ↑ together or down ↓ together.



Shown below is A and B moving in the ***second mode*** (moving in opposite directions).



Mode #	Eigenvalue (frequency)	Eigenvector (use + or 0 or −)	Potential Energy ($\frac{1}{2}\,k\,\delta^2$)	Kinetic Energy ($\frac{1}{2}\,m\,v^2$)	Total Energy Kinetic + Potential
1	$\boxed{\text{small}}$/large	$\begin{bmatrix} + \\ \boxed{+} \end{bmatrix}$ or $\begin{bmatrix} - \\ \boxed{-} \end{bmatrix}$	$\boxed{\text{small}}$/large $\boxed{\text{one}}$/two deformed spring $\boxed{\text{min}}$/max deformation	$\boxed{\text{small}}$/large	$\boxed{\text{small}}$/large
2	small/$\boxed{\text{large}}$	$\begin{bmatrix} + \\ \boxed{-} \end{bmatrix}$ or $\begin{bmatrix} - \\ \boxed{+} \end{bmatrix}$	small/$\boxed{\text{large}}$ one/$\boxed{\text{two}}$ deformed springs min/$\boxed{\text{max}}$ deformation	small/$\boxed{\text{large}}$	small/$\boxed{\text{large}}$

Answers at www.MotionGenesis.com ⇒ Textbooks ⇒ Resources.

23.2 Analytical solutions of *undamped, coupled,* ODEs

Any set of n coupled, linear, constant-coefficient, ***2nd-order ODEs*** may be written in the matrix form

$$\underset{(n\times n)}{M}\;\ddot{X} \;+\; \underset{(n\times n)}{B}\;\dot{X} \;+\; \underset{(n\times n)}{K}\;X \;=\; \underset{(n\times p)}{G}\;\underset{(p\times 1)}{F} \tag{1}$$

where X is a $n\times 1$ matrix of dependent variables, M, B, K are $n\times n$ matrices of constants, G is a $n\times p$ matrix of constants, and F is a $p\times 1$ matrix of functions of time $f_1(t)$, $f_2(t)$, $\ldots$, $f_p(t)$.

To solve equation (1), write $X(t) = X_h(t) + X_p(t)$ (the sum of a homogeneous and particular solution). The homogeneous solution $X_h(t)$ can be found by assuming a solution of the form shown in equation (2), where p is a yet-to-be-determined constant and U is a yet-to-be-determined **non-zero** $n\times 1$ matrix of constants.[a]

$$X_h(t) \;=\; \boxed{U\,e^{p\,t}} \tag{2}$$

$$U \;\triangleq\; \begin{bmatrix} u_1 \\ u_2 \\ \cdots \\ u_n \end{bmatrix}$$

[a] It is reasonable to guess $X_h(t) = U\,e^{p\,t}$ because it worked for **uncoupled** linear ODEs. The matrix U is **non-zero** because $U = [0]$ produces the ***trivial solution*** $X_h(t) = [0]$, which is not a solution of interest (not what we are looking for).

Substituting/differentiating equation (2) into equation (1), factoring on $U e^{p\,t}$, and simplifying, gives

Note: This proof mimics Section 21.1. Answers are at www.MotionGenesis.com ⇒ Textbooks ⇒ Resources.

$$M\left(\underset{(2)}{p^2\,U\,e^{p\,t}}\right) \;+\; B\left(\underset{(2)}{p\,U\,e^{p\,t}}\right) \;+\; K\left(\underset{(2)}{U\,e^{p\,t}}\right) \;\underset{(1)}{=}\; [\,0\,]$$

$$\left(p^2\,M \;+\; p\,B \;+\; K\right)U\,e^{p\,t} \;=\; [\,0\,] \qquad \text{Section 7.1 shows: } e^{p\,t} \neq 0$$

$$\left(p^2\,M \;+\; p\,B \;+\; K\right)U \;=\; [\,0\,] \qquad \text{Sometimes } B \approx [\,0\,] \text{ (see below)}$$

$$\left(p^2\,M \;+\; K\right)U \;=\; [\,0\,] \qquad \text{Define: } \boxed{\lambda \triangleq -p^2} \;\; \boxed{p = \pm\sqrt{-\lambda}}$$

$$\boxed{\textit{Generalized eigenvalue equation} \qquad (-\lambda\,M \;+\; K)\,U \;=\; [\,0\,] \quad \Rightarrow \quad \det\left(-\lambda\,M \;+\; K\right) \;=\; 0} \tag{3}$$

Equation (3a) is a **coupled** nonlinear algebraic equation. Its two unknowns are λ and U.

Equation (3b) is an **uncoupled** nonlinear algebraic equation. Its only unknown is λ.

(3a) ⇒ (3b) changes $\boldsymbol{n}$ equations with $\boldsymbol{n+1}$ unknowns into **1** equation with **1** unknown.

Equation (1) ⇒ (3) changes $\boldsymbol{n}$ coupled ODEs with $\boldsymbol{n}$ unknowns into a ***generalized eigenvalue equation***.

Note: It is reasonable to set $B = [\,0\,]$ if damping is small. Other circumstances in which B is set to $[\,0\,]$:

- It is difficult to analytically or experimentally determine the elements of the damping matrix.
- It makes the mathematics easier.
- In structural vibrations, damping is approximated with ***modal damping***.

For equation (3a) to produce a **non-zero** U (and **non-zero** X_h), the inverse of $(-\lambda M + K)$ must not exist. To see this, suppose $(-\lambda M + K)^{-1}$ does exist and multiply both sides of equation (3a) by $(-\lambda M + K)^{-1}$ which produces $U = [\,0\,]$. To get a **non-zero** U from equation (3a), one must find values of λ so $(-\lambda M + K)^{-1}$ does not exist. These special values of λ are denoted λ_1, λ_2, $\ldots$, λ_n and are called ***eigenvalues***.

For each λ_i, there is a corresponding non-zero U_i called the ***eigenvector*** corresponding to λ_i.[1]

The following are equivalent statements about equation (3a) and finding the ***eigenvalues*** λ:

- Find the values of λ which result in $U \neq [\,0\,]$.
- Find the values of λ so the matrix $[-\lambda\,M + K]$ is singular, i.e., $[-\lambda\,M + K]^{-1}$ does not exist.
- Find the values of λ so the determinant of $[-\lambda\,M + K]$ is zero, i.e., $\boxed{\det\left[-\lambda\,M + K\right] \;=\; 0}$ (4)

[1] The eigenvalue problem is a special type of nonlinear algebraic equation because the number of solutions is known apriori.