



Chapter 23

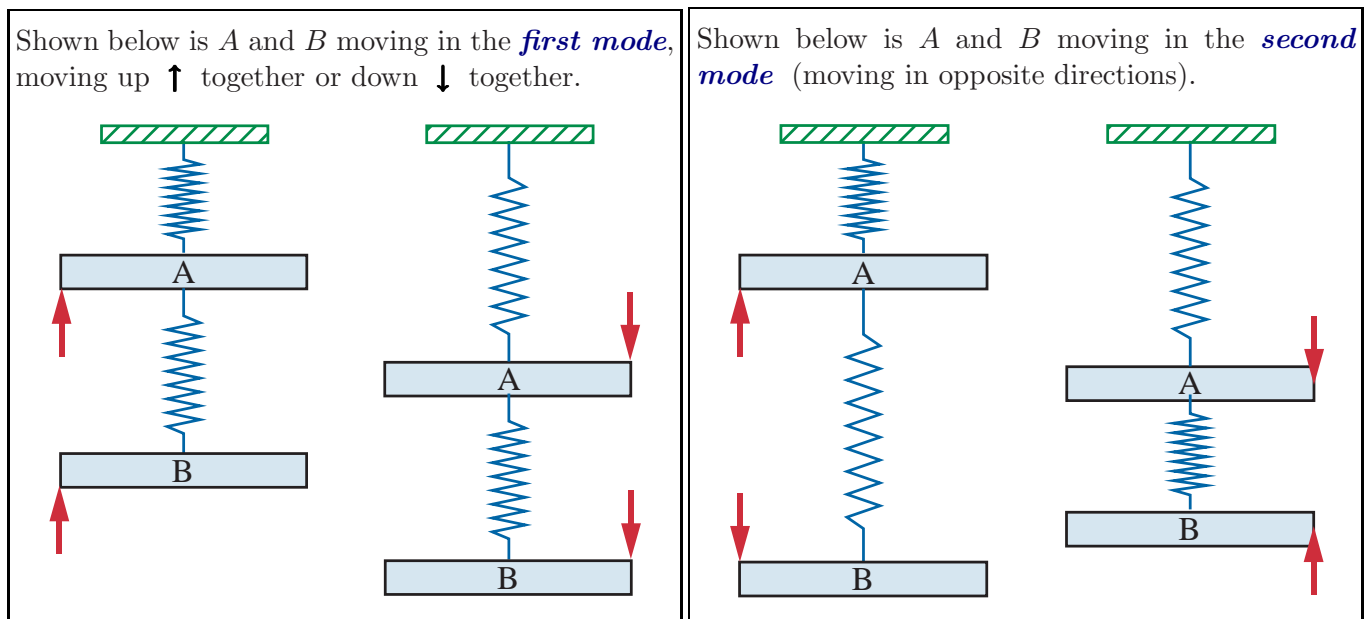
Undamped coupled 2nd-order ODEs

Summary (see examples in Hw 12)

Many physical phenomena are governed by **undamped**, coupled, linear, constant-coefficient, ODEs. These include certain motions of buildings, airplanes, automobiles, beams, space structures, and molecules.

23.1 Physical insights into eigenvalues and eigenvectors of a slinky

Two long thin rulers (*A* and *B*) connected to a slinky (spring) are useful for demonstrating the physical significance of eigenvalues and eigenvectors.



Mode #	Eigenvalue (frequency)	Eigenvector (use + or 0 or -)	Potential Energy ($\frac{1}{2} k \delta^2$)	Kinetic Energy ($\frac{1}{2} m v^2$)	Total Energy Kinetic + Potential
1	small /large	$\begin{bmatrix} + \\ + \end{bmatrix}$ or $\begin{bmatrix} - \\ - \end{bmatrix}$	small /large one/two deformed spring min /max deformation	small /large	small /large
2	small/ large	$\begin{bmatrix} + \\ - \end{bmatrix}$ or $\begin{bmatrix} - \\ + \end{bmatrix}$	small/ large one/two deformed springs min/ max deformation	small/ large	small/ large

Answers at www.MotionGenesis.com \Rightarrow [Textbooks](#) \Rightarrow [Resources](#).

23.2 Analytical solutions of *undamped, coupled, ODEs*

Any set of n coupled, linear, constant-coefficient, 2^{nd} -order *ODEs* may be written in the matrix form

$$\underset{(n \times n)}{M} \ddot{X} + \underset{(n \times n)}{B} \dot{X} + \underset{(n \times n)}{K} X = \underset{(n \times p)}{G} \underset{(p \times 1)}{F} \quad (1)$$

where X is a $n \times 1$ matrix of dependent variables, M, B, K are $n \times n$ matrices of constants, G is a $n \times p$ matrix of constants, and F is a $p \times 1$ matrix of functions of time $f_1(t), f_2(t), \dots, f_p(t)$.

To solve equation (1), write $X(t) = X_h(t) + X_p(t)$ (the sum of a homogeneous and particular solution). The homogeneous solution $X_h(t)$ can be found by assuming a solution of the form shown in equation (2), where p is a yet-to-be-determined constant and U is a yet-to-be-determined **non-zero** $n \times 1$ matrix of constants.^a

$$X_h(t) = U e^{pt} \quad (2)$$

$$U \triangleq \begin{bmatrix} u_1 \\ u_2 \\ \dots \\ u_n \end{bmatrix}$$

^aIt is reasonable to guess $X_h(t) = U e^{pt}$ because it worked for **uncoupled** linear ODEs. The matrix U is **non-zero** because $U = [0]$ produces the **trivial solution** $X_h(t) = [0]$, which is not a solution of interest (not what we are looking for).

Substituting/differentiating equation (2) into equation (1), factoring on $U e^{pt}$, and simplifying, gives¹

$$\begin{aligned} M \left(\underset{(2)}{p^2 U e^{pt}} \right) + B \left(\underset{(2)}{p U e^{pt}} \right) + K \left(\underset{(1)}{U e^{pt}} \right) &= [0] \\ \left(\underset{(2)}{p^2 M + p B + K} \right) U e^{pt} &= [0] && \text{Section 5.4 shows: } e^{pt} \neq 0 \\ \left(\underset{(2)}{p^2 M + p B + K} \right) U &= [0] && \text{Sometimes } B \approx [0] \text{ (see below)} \\ \left(\underset{(2)}{p^2 M + K} \right) U &= [0] && \text{Define: } \lambda \triangleq -p^2 \quad p = \pm \sqrt{-\lambda} \end{aligned}$$

$$\boxed{\text{Generalized eigenvalue equation}} \quad (-\lambda M + K) U = [0] \Rightarrow \det(-\lambda M + K) = 0 \quad (3)$$

Equation (3a) is a **coupled** nonlinear algebraic equation. Its two unknowns are λ and U .

Equation (3b) is an **uncoupled** nonlinear algebraic equation. Its only unknown is λ .

(3a) \Rightarrow (3b) changes n equations with $n+1$ unknowns into 1 equation with 1 unknown.

Equation (1) \Rightarrow (3) changes n coupled ODEs with n unknowns into a **generalized eigenvalue equation**.

Note: It is reasonable to set $B = [0]$ if damping is small. Other circumstances in which B is set to $[0]$:

- It is difficult to analytically or experimentally determine the elements of the damping matrix.
- It makes the mathematics easier.
- In structural vibrations, damping is approximated with **modal damping**.

For equation (3a) to produce a **non-zero** U (and **non-zero** X_h), the inverse of $(-\lambda M + K)$ must not exist. To see this, suppose $(-\lambda M + K)^{-1}$ does exist and multiply both sides of equation (3a) by $(-\lambda M + K)^{-1}$ which produces $U = [0]$. To get a **non-zero** U from equation (3a), one must find values of λ so $(-\lambda M + K)^{-1}$ does not exist. These special values of λ are denoted $\lambda_1, \lambda_2, \dots, \lambda_n$ and are called **eigenvalues**. For each λ_i , there is a corresponding non-zero U_i called the **eigenvector** corresponding to λ_i .²

The following are equivalent statements about equation (3a) and finding the **eigenvalues** λ :

- Find the values of λ which result in $U \neq [0]$.
- Find the values of λ so the matrix $[-\lambda M + K]$ is singular, i.e., $[-\lambda M + K]^{-1}$ does not exist.
- Find the values of λ so the determinant of $[-\lambda M + K]$ is zero, i.e., $\det[-\lambda M + K] = 0$ (4)

¹Note: This proof mimics Section 21.1. Answers are at www.MotionGenesis.com \Rightarrow [Textbooks](#) \Rightarrow [Resources](#).

²The eigenvalue problem is a special type of nonlinear algebraic equation because the number of solutions is known a priori.