Chapter 22

Eigenvalues and eigenvectors

$$\boldsymbol{A\,U = \lambda\,U}$$
$$\boldsymbol{A\,U = \lambda\,B\,U}$$

What is a matrix eigenvalue problem? (see examples in Hw 12)

An ***eigenvalue*** is a "special value" of λ that allows equation (1) to produce ***non-zero*** U.[1]

- λ is an unknown ***scalar*** (called an ***eigenvalue***).
- $U \neq [0]$ is a unknown $n\times 1$ column matrix (called an ***eigenvector***).
- $\text{Matrix}(\lambda)$ is an $n\times n$ matrix that depends on λ.

$$\text{Matrix}(\lambda) * U = [0] \qquad (1)$$

$[0]$ is the $n\times 1$ zero matrix.

Eigenvalue problem	Equation form	Alternate form	Solution for λ
Standard eigenvalue	$[-\lambda\,I + A]\,U = [0]$	$A\,U = \lambda\,U$	$\det[-\lambda\,I + A] = 0$
Generalized eigenvalue	$[-\lambda\,B + A]\,U = [0]$	$A\,U = \lambda\,B\,U$	$\det[-\lambda\,B + A] = 0$
Quadratic eigenvalue	$[M\,\lambda^2 + B\,\lambda + K]\,U = [0]$	Not applicable	$\det[M\,\lambda^2 + B\,\lambda + K] = 0$
Nonlinear eigenvalue	$\text{Matrix}(\lambda) * U = [0]$	Not applicable	$\det[\text{Matrix}(\lambda)] = 0$

22.1 Recognize and remember: Solving an eigenvalue problem

There are similarities between the familiar ***quadratic equation*** and an ***eigenvalue problem***. Both are algebraic equations that are ***nonlinear*** in their unknowns, and both have known solutions. It is important to ***recognize*** these equations and ***remember*** their solutions.

	Quadratic equation	**Standard eigenvalue**	**Generalized eigenvalue**
Equation form Alternate form	$a\,x^2 + b\,x + c = 0$ $a\,x^2 + b\,x = -c$	$[-\lambda\,I + A]\,U = [0]$ $A\,U = \lambda\,U$	$[-\lambda\,B + A]\,U = [0]$ $A\,U = \lambda\,B\,U$
Unknowns Equation type	x **Nonlinear**	$\lambda,\ U$ **Nonlinear**	$\lambda,\ U$ **Nonlinear**
Solution	$x = \dfrac{-b \pm \sqrt{b^2 - 4\,a\,c}}{2\,a}$	$\det[-\lambda\,I + A] = 0$	$\det[-\lambda\,B + A] = 0$

22.2 Motivating questions for eigenvalues and eigenvectors (Hw 12.9)

Question 1: Solve the following equation for the two unknowns λ and U (with the condition $U \neq 0$). Since this equation has the special form $\text{Matrix}(\lambda)\,U = 0$, it is ***recognized*** as an ***eigenvalue problem***.

$$[-\lambda + 3\,] * U = 0 \qquad \text{(or alternately } 3\,U = \lambda\,U\,)$$

The solution to this equation is a ***"special value"*** of λ and an associated ***non-zero*** U.

Eigenvalue: $\lambda = 3$ ***Eigenvector:*** $U =$ any number

[1] U is a "right" eigenvector for $\text{Matrix}(\lambda) * U = [0]$, whereas U is a "left" eigenvector for $U * \text{Matrix}(\lambda) = [0]$.

Question 2: Find a ***non-zero*** solution $y(t)$ to the ODE shown below-right.

Start by substituting the assumed solution $y(t) = U\,e^{p\,t}$ into the ODE where p is a constant (to-be-determined) and U is a ***non-zero*** constant.[a] Subsequently, rearrange and simplify using $e^{p\,t} \neq 0$.

The equation for p is ***recognized*** as an ***eigenvalue problem***.

The ***"special value"*** of p and associated ***non-zero*** U are[b]

ODE:	$\dot{y} - 3\,y$	$= 0$
Eigen-problem:	$(p-3)\,U$	$= 0$
Solution:	$y(t)$	$= U\,e^{3\,t}$

Eigenvalue: $p = 3$ ***Eigenvector:*** $U =$ any constant

[a] Note: $U = 0$ produces the trivial (degenerate) solution $y(t) = 0$, which is not what we are looking for. Hence $U \neq 0$.

[b] Note: In ODEs, this ***"special value"*** of p is called a ***pole*** whereas in matrix algebra p is called an ***eigenvalue***.

Question 3: Eigenvalue and eigenvector concepts. (Answers: www.MotionGenesis.com ⇒ Textbooks ⇒ Resources)

Consider the following set of algebraic equations governing the unknowns u_1, u_2, and λ.

$$\begin{aligned} \lambda\,u_1 + u_2 &= 0 \\ 4\,u_1 + \lambda\,u_2 &= 0 \end{aligned} \quad \overset{}{\Longleftrightarrow}\ \text{or} \quad \begin{bmatrix} \lambda & 1 \\ 4 & \lambda \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Find "special values" of λ (called ***eigenvalues***) that allow for $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Result: $\lambda_1 = 2$ $\lambda_2 = -2$

For each special value of λ determine a corresponding "special ratio" of u_2 to u_1.

Result: (These "special ratios" are called ***eigenvectors*** and c_1 and c_2 are arbitrary constants.)

For λ_1: $U_1 \triangleq \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ For λ_2: $U_2 \triangleq \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Question 4: Eigenvalues for an unusual (nonlinear) eigenvalue problem.

Consider the following set of algebraic equations governing the unknowns u_1, u_2, and λ.

$$\begin{aligned} \lambda^2\,u_1 + 5\,u_2 &= 0 \\ \big(\cos(\lambda) - 0.9\big)\,u_1 + \lambda\,u_2 &= 0 \end{aligned}$$

whose matrix form is:

$$\begin{bmatrix} \lambda^2 & 5 \\ \cos(\lambda) - 0.9 & \lambda \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Find an equation, which when solved produces "special values" of λ that allow for $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Result: (These "special values" of λ are called ***eigenvalues***.) $\lambda^3 - 5\cos(\lambda) + 4.5 = 0$

Note: It is questionable whether this eigen-problem can be cast as a standard or generalized eigenvalue problem.
Three eigenvalues that satisfy this equation are: $\lambda_1 = -1.7574$, $\lambda_2 = -0.5078$, $\lambda_3 = +0.4166$.

Question 5: Solve the following set of linear algebraic equations for x and y (for given values of d).

$$\begin{aligned} x - y &= 0 \\ x + d\,y &= 0 \end{aligned} \quad \Longleftrightarrow\ \text{or} \quad \begin{bmatrix} 1 & -1 \\ 1 & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$d = 0$	$x = 0$	$y = 0$
$d = 1$	$x = 0$	$y = 0$
$d = 2$	$x = 0$	$y = 0$
$d = 3$	$x = 0$	$y = 0$
$d = -1$	$x =$ any number	$y = x$

Answers at www.MotionGenesis.com ⇒ Textbooks ⇒ Resources.

Note: The ***special value*** $d = -1$ is the **only** value of d that produces a **non-zero** solution for x and y.
Note: One way to solve for this ***special value*** of d is by setting the determinant of the 2×2 matrix equal to 0.
Note: This is recognized as an eigenvalue problem **if** the question is: Find the special value of d that allows for $\begin{bmatrix} x \\ y \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.