Chapter 15

Complex numbers

$$\boxed{\begin{aligned} i &\triangleq \sqrt{-1} \\ e^{i\theta} &= \cos(\theta) + i\,\sin(\theta) \end{aligned}}$$

Summary: Tools and use for complex numbers (see examples in Hw 8, 9)

This chapter provides algebraic tools for complex numbers ($+$, $*$, $/$, $\sqrt{\ }$, and powers).[1]
Complex numbers are useful in many aspects of dynamic systems, including:

Solving 2^{nd}, 3^{rd}, and higher-order ODEs	Root locus	
Control system design	Frequency response	Eigen-analysis

Motivating questions

- The following proof involves the imaginary number i defined as $i \triangleq \sqrt{-1}$ and shows the surprising conclusion that $1 = -1$. Circle the incorrect step in the proof and explain your reasoning.

$$1 \;=\; \sqrt{1} \;=\; \boxed{\sqrt{(-1)^2} \;=\; \sqrt{-1} * \sqrt{-1}} \;=\; i * i \;=\; i^2 \;=\; -1$$

When x is negative, $\left(x^2\right)^{(1/2)} \neq \left(x^{\frac{1}{2}}\right)^2$. In general, $\left(x^a\right)^b \neq \left(x^b\right)^a$.

- Find all real and/or complex numbers that can appear on the right-hand side of the equal signs.

$$1^4 = \left(1\,e^{2n\pi\,i}\right)^4 {}_{n=0,\,1,\,2,\,...} = e^{8n\pi\,i} = \cos(8n\pi) + i\,\sin(8n\pi) = \boxed{1}$$

$$1^{1/4} = \left(1\,e^{2n\pi\,i}\right)^{1/4} {}_{n=0,\,1,\,2,\,...} = e^{\frac{n\,\pi}{2}\,i} = \cos(\tfrac{n\,\pi}{2}) + i\,\sin(\tfrac{n\,\pi}{2}) = \boxed{\pm 1,\ \pm i}$$

$$1^{1/3} = \left(1\,e^{2n\pi\,i}\right)^{1/3} {}_{n=0,\,1,\,2,\,...} = e^{\frac{2n\pi}{3}\,i} = \cos(\tfrac{2n\pi}{3}) + i\,\sin(\tfrac{2n\pi}{3}) = \boxed{1,\ -0.5 \pm 0.866\,i}$$

Answers at www.MotionGenesis.com ⇒ Textbooks ⇒ Resources. Alternate: $1^{1/4} = \sqrt{\sqrt{1}} = \sqrt{\pm 1} = \pm 1$ or $\pm i$.

[1] Complex numbers are said to be "***closed***" under addition, subtraction, negation, multiplication, division, and exponentiation because when these operations are performed on complex numbers, only complex numbers result. Complex numbers are said to be "***algebraically closed***" because polynomial equations with complex number coefficients can only produce complex numbers. Real numbers are not closed under exponentiation. For example, $-4^{0.5} = \sqrt{-4} = \pm 2\,i$ produces a complex (not real) number. Real numbers are not algebraically closed. For example, although the polynomial equation $x^2 + 2\,x + 5 = 0$. has real coefficients, its roots are the complex numbers $x = -1 \pm 2\,i$.