Chapter 14

Complex numbers

$$i \triangleq \sqrt{-1}$$

$$e^{i\,\theta} = \cos(\theta) + i\,\sin(\theta)$$

Motivating questions (see examples in Hw 8)

- The following proof involves the imaginary number i defined as $i \triangleq \sqrt{\text{-}1}$ and shows the surprising conclusion that $1 = \text{-}1$. Circle the incorrect step in the proof and explain your reasoning.

$$1 \;=\; \sqrt{1} \;=\; \boxed{\sqrt{(\text{-}1)^2} \;=\; \sqrt{\text{-}1} * \sqrt{\text{-}1}} \;=\; i * i \;=\; i^2 \;=\; \text{-}1$$

When x is negative, $\left(x^2\right)^{(1/2)} \neq \left(x^{\frac{1}{2}}\right)^2$. In general, $\left(x^a\right)^b \neq \left(x^b\right)^a$.

- Find all real and/or complex numbers that can appear on the right-hand side of the equal signs.[1]

$$1^4 = \left(1\,e^{2n\pi\,i}\right)^4_{\;n=0,\,1,\,2,\,\ldots} = e^{8n\pi\,i} = \cos(8n\pi) + i\,\sin(8n\pi) = \boxed{1}$$

$$1^{1/4} = \left(1\,e^{2n\pi\,i}\right)^{1/4}_{\;n=0,\,1,\,2,\,\ldots} = e^{\frac{n\,\pi}{2}\,i} = \cos(\tfrac{n\,\pi}{2}) + i\,\sin(\tfrac{n\,\pi}{2}) = \boxed{\pm 1,\ \pm i}$$

$$1^{1/3} = \left(1\,e^{2n\pi\,i}\right)^{1/3}_{\;n=0,\,1,\,2,\,\ldots} = e^{\frac{2n\pi}{3}\,i} = \cos(\tfrac{2n\pi}{3}) + i\,\sin(\tfrac{2n\pi}{3}) = \boxed{1,\ \text{-}0.5 \pm 0.866\,i}$$

Summary

Complex numbers are useful in many aspects of dynamic systems, including:

- Solving 2^{nd}-order, 3^{rd}-order, and higher-order ODEs
- Root locus
- Control system design
- Frequency response
- Eigen-analysis

This chapter provides basic mathematical tools for complex numbers, namely addition, multiplication, division, and exponentiation.[2]

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[1] Answers at www.MotionGenesis.com ⇒ Textbooks ⇒ Resources. Alternate: $1^{1/4} = \sqrt{\sqrt{1}} = \sqrt{\pm 1} = \pm 1$ or $\pm i$.

[2] Complex numbers are said to be "***closed***" under addition, subtraction, negation, multiplication, division, and exponentiation because when these operations are performed on complex numbers, only complex numbers result. Complex numbers are said to be "***algebraically closed***" because polynomial equations with complex number coefficients can only produce complex numbers. Real numbers are not closed under exponentiation. For example, $\text{-}4^{0.5} = \sqrt{\text{-}4} = \pm 2\,i$ produces a complex (not real) number. Real numbers are not algebraically closed. For example, although the polynomial equation $x^2 + 2\,x + 5 = 0$. has real coefficients, its roots are the complex numbers $x = \text{-}1 \pm 2\,i$.