	Change in $\lvert x \rvert$ based on intuition	Change in $\lvert x \rvert$ based on expression for U_2
Increasing L	**decreases/increases**	**decreases/increases**
Increasing m_A	**decreases/increases**	**decreases/increases**
Increasing m_B	**decreases/increases**	**decreases/increases**

Note: It is difficult to intuitively discern how increasing m_B impacts $\lvert x \rvert$.

(j) Assemble the solution for $X(t)$ when $g = 9.81 \ \frac{\text{m}}{\text{s}^2}, \ m_A = 10 \text{ kg}, \ m_B = 1 \text{ kg}, \ L = 0.5 \text{ m}.$

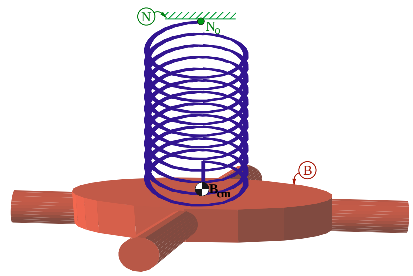
Result:

$$X(t) = c_1 \begin{bmatrix} 1 \\ \boxed{} \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ \boxed{} \end{bmatrix} \boxed{} + c_3 \begin{bmatrix} \boxed{} \\ 1 \end{bmatrix} e^{\boxed{}\, t} + c_4 \begin{bmatrix} \boxed{} \\ 1 \end{bmatrix} e^{\boxed{}\, t}$$

where the constants $c_1, \ c_2, \ c_3, \ c_4$ are usually determined from the initial values of $\boxed{}$.

Note: you will re-analyze this problem using **state-space** techniques in Homework 14.2.

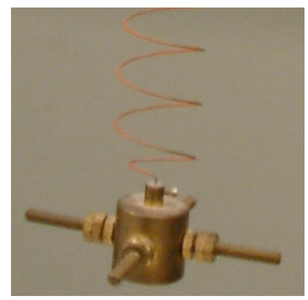
13.4 ♣ Coupled motions of WilberForce pendulum



A spring attaches B_{cm} (the center of mass of a rigid body B) to a point N_o fixed in a Newtonian reference frame N.

$$\left.\begin{aligned} m\,\ddot{x} \ + \ k_x\,x \ + \ k_c\,\theta &= 0 \\ I\,\ddot{\theta} \ + \ k_c\,x \ + \ k_\theta\,\theta &= 0 \end{aligned}\right\} \quad \text{Equations that govern this system's motion}$$

Quantity	Symbol	Type
B's mass	m	Positive constant
B's central moment of inertia about vertical axis	I	Positive constant
Extensional linear spring constant modeling	k_x	Positive constant
Torsional linear spring constant modeling	k_θ	Positive constant
Coupled linear spring constant modeling	k_c	Positive constant
Translational stretch of spring from equilibrium	x	Variable
Rotational stretch of spring from equilibrium	θ	Variable



Purchase Wilburforce pendulum at PASCO Scientific or B&B Co. (Super Spinnerama). More WilberForce information at "WilberForce pendulum oscillations and normal modes", Berg and Marshall, Am. J. Physics Vol. 59, No. 1, January 1991.

(a) Write the ODEs in the matrix form $M\,\ddot{X} + B\,\dot{X} + K\,X = [0]$. Next, substitute the assumed solution $\boxed{X(t) = U\,e^{p\,t}}$ where p is a constant (to-be-determined) and U is a ***non-zero*** 2×1 matrix of constants (to-be-determined). Show p and U are governed by a ***generalized eigenvalue problem***. Next, define $\lambda \triangleq -p^2$ and find a scalar equation that governs λ.

Result:

$$\underbrace{\begin{bmatrix} \boxed{} & \boxed{} \\ \boxed{} & \boxed{} \end{bmatrix}}_{M} \underbrace{\begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix}}_{\ddot{X}} + \underbrace{\begin{bmatrix} \boxed{} & \boxed{} \\ \boxed{} & \boxed{} \end{bmatrix}}_{B} \underbrace{\begin{bmatrix} \dot{x} \\ \dot{\theta} \end{bmatrix}}_{\dot{X}} + \underbrace{\begin{bmatrix} \boxed{} & \boxed{} \\ \boxed{} & \boxed{} \end{bmatrix}}_{K} \underbrace{\begin{bmatrix} x \\ \theta \end{bmatrix}}_{X} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left[p^2\,M + K\right]\,U\,e^{p\,t} = [0] \qquad \Rightarrow \qquad \left[p^2\,M + K\right]\,U = [0]$$

$$\boxed{\lambda \triangleq -p^2} \quad \Rightarrow \quad \left[\boxed{}\,\boxed{} + \boxed{}\right]\,U = [0] \qquad \Rightarrow \qquad \det\left[\boxed{}\right] = 0$$

(b) Show this can be recast as a ***standard eigenvalue problem*** $[-\lambda\,I + A]\,U = [0]$. Next, complete the blanks in the following polynomial equation that governs $\boxed{\lambda \triangleq -p^2}$.

Result – determine A: (note I is the 2×2 identity matrix. The blanks only involve $m, \ I, \ k_x, \ k_\theta, \ k_c$).

$$[-\lambda\,I + A]\,U = [0] \qquad \text{where} \qquad A = M^{-1}\,K = \begin{bmatrix} \boxed{} & \boxed{} \\ \boxed{} & \boxed{} \end{bmatrix}$$

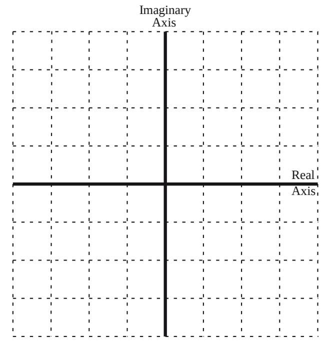
$$\text{Equation the governs } \lambda \qquad \lambda^2 + \left(\frac{-k_x}{m} + \frac{-k_\theta}{I}\right) * \lambda + \frac{\boxed{}}{\boxed{}} = 0$$

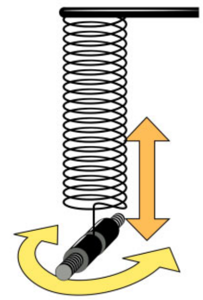
(c) For certain values of $m, \ I, \ k_x, \ k_\theta, \ k_c$, the matrix $A \triangleq M^{-1}\,K = \begin{bmatrix} 10 & 1 \\ 1 & 10 \end{bmatrix}$.

Find the ***eigenvalues*** and corresponding ***eigenvectors*** of A.

Result: $\lambda_1 = 9 \qquad \lambda_2 = \boxed{\quad}$

$$U_1 = \begin{bmatrix} \boxed{\quad} \\ 1 \end{bmatrix} \qquad U_2 = \begin{bmatrix} \boxed{\quad} \\ 1 \end{bmatrix}$$





Calculate the values of p_1, p_2, p_3, p_4.
Draw their locations in the complex plane.

$$p_{1,2} = \boxed{\qquad\qquad\qquad}$$

$$p_{3,4} = \boxed{\qquad\qquad\qquad}$$

The solution is **stable/neutrally stable/unstable**.

(d) Write the solution for $X(t)$ in terms of the sine and cosine functions, and t.
Result: (also in terms of the yet-to-be-determined constants c_1, c_2, c_3, c_4)

$$\begin{bmatrix} x(t) \\ \theta(t) \end{bmatrix} = \begin{bmatrix} \boxed{\quad} \\ 1 \end{bmatrix} \{c_1 \sin(\boxed{\quad}\, t) + c_2 \cos(\boxed{\quad}\, t)\} + \begin{bmatrix} \boxed{\quad} \\ 1 \end{bmatrix} \{c_3 \sin(\boxed{\quad}\, t) + c_4 \cos(\boxed{\quad}\, t)\}$$

(e) Determine c_1, c_2, c_3, c_4 when $x(0) = 0.2$, $\theta(0) = 0$, $\dot{x}(0) = 0$, $\dot{\theta}(0) = 0$.
Write explicit solutions for $x(t)$ and $\theta(t)$ using the initial values
Result: $c_1 = \boxed{\quad} \qquad c_2 = \boxed{\quad} \qquad c_3 = \boxed{\quad} \qquad c_4 = \boxed{\quad}$

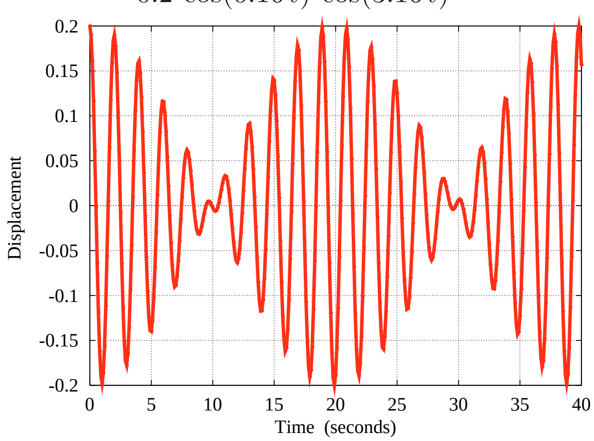
$$x(t) = \boxed{\quad} \cos(3\,t) + \boxed{\quad} \cos(\boxed{\quad}\, t)$$

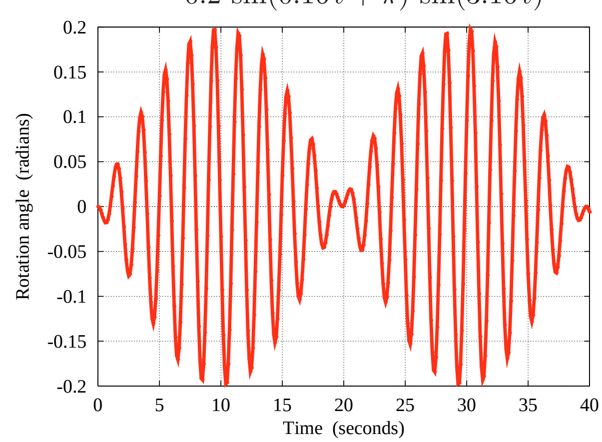
$$\theta(t) = \boxed{\quad} \cos(3\,t) + \boxed{\quad} \cos(\boxed{\quad}\, t)$$

(f) Using trigonometric identities, equation (2.10), and $\cos(a) = -\cos(a+\pi)$, it can be shown that the previous solution for $x(t)$ and $\theta(t)$ is (you do not need to show this)

$$x(t) = 0.2 \sin(-0.16\,t + \frac{\pi}{2}) \sin(3.16\,t + \frac{\pi}{2}) = 0.2 \cos(0.16\,t) \cos(3.16\,t)$$

$$\theta(t) = 0.2 \sin(-0.16\,t) \sin(3.16\,t) = 0.2 \sin(0.16\,t + \pi) \sin(3.16\,t)$$





Interpret the time-behavior of $x(t)$ and $\theta(t)$.
Both $x(t)$ and $\theta(t)$ exhibit the ***beat phenomena*** with a high-frequency of $\boxed{\quad}$ $\frac{\text{rad}}{\text{sec}}$ and a low-frequency of $\boxed{\quad}$ $\frac{\text{rad}}{\text{sec}}$.
Since $x(t)$ and $\theta(t)$ are ***coupled***, x's maximum amplitude coincides with θ's minimum amplitude (and vice-versa).