# Chapter 1

# Math review


**Courtesy NASA**

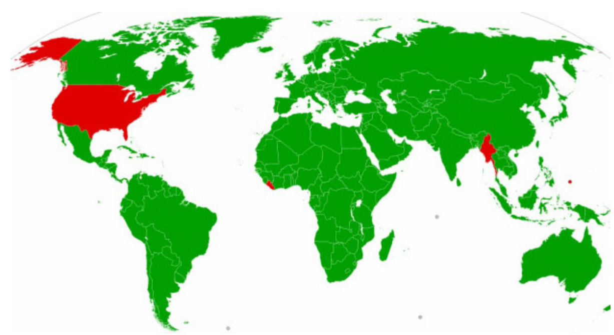
Math is a foundation for science, medicine, engineering, construction, and business. Math provides ***concepts*** (pictures, words, ideas), ***calculations*** (mathematical operations, symbols, equations, definitions), and ***context*** (situations in which the concepts and calculations are relevant and useful). More generally, math is a language and set of rules that helps us count, quantify, calculate, manipulate, relate, define, extrapolate, and abstract "stuff".[1] Advances in math depend on pictures[2] words, symbols, equations, and precise ***definitions***. For example, consider the following ***definition*** of $\pi$.

| Object | Example | Approximate age of human comprehension |
|---|---|---|
| **Picture** |  | Toddlers |
| **Spoken word** | "circle" | Pre-school |
| **Written word** | "circle", "diameter", "circumference" | Elementary school |
| **Symbol** | $d$ for diameter, $c$ for circumference | Middle school |
| **Equation** | $c = \pi\, d$ | Middle/high school |
| **Definition** | $\pi \triangleq \dfrac{c}{d}$ ($\triangleq$ means "***defined as***") | University |

## 1.1 Unit systems - SI and U.S.

Units quantify the measurement of "stuff". The ***SI*** system was first adopted by France on December 10, 1799 and is now used in all countries other than Liberia, Myanmar, and the United States.

The ***SI*** (metric) system uses a base-10 number system and decimals (not fractions) and has measures for length, mass, force, temperature, time, etc.



**Countries using SI units (green) vs. U.S. units (red).**

***NIST*** (National Institute of Standards & Technology) defines physical constants and conversion factors.

| | | | |
|---|---|---|---|
| Length | 1 inch $\triangleq$ 2.54 cm | | |
| Mass | 1 lbm $\approx$ 0.45359237 kg | 1 slug $\approx g_{\text{US}}$ lbm | $g_{\text{US}} \approx 32.174048556$ |
| Force | 1 Newton $\triangleq 1 \frac{\text{kg m}}{\text{s}^2}$ | 1 lbf $\triangleq 1 \frac{\text{slug ft}}{\text{s}^2}$ | 1 lbf $\triangleq g_{\text{US}} \frac{\text{lbm ft}}{\text{s}^2}$ |

Inaccurate unit conversions have caused ***many*** failures. In 1999, NASA lost a $125,000,000 Mars orbiter because one engineering team used SI units while another used U.S. units. In 1983, an Air Canada Boeing 767 ran out of fuel mid-flight because of a kg to lbm unit conversion.[3]

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[1]For example, the "idea" of ***value*** (answering **"how much something is worth"**) is quantified through money.

[2]***Art*** is ***not*** reserved for the sophisticated elite with knowledge and historical context for art. Appreciation for shapes, colors, and emotional expression in art is available to humans on a basic (primitive/subconscious) level.

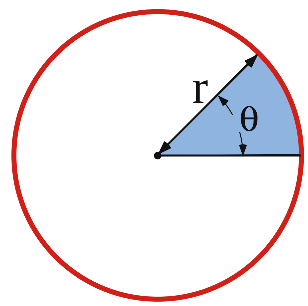
[3]Ironically, Thomas Jefferson helped United States become the first country (in 1792) to use a monetary system with decimals and a base-10 number system. The historical origin of U.S. units trace to 2575 B.C. and through ancient Egypt, Greece, and Rome. The ***inch*** approximates the width of a man's thumb. The ***foot*** approximates a foot with shoe and was somewhat standardized in England to King Henry I. The ***mile*** "mille passus" is 1000 paces (2 steps) of a Roman soldier. An

## 1.2 Geometry: Ancient Euclid and modern vectors

Geometry is the study of figures (e.g., lines, curves, surfaces, solids) and their properties (e.g., distance, area, and volume). Geometry plays a central role in construction, farming, engineering, medicine, science, etc.

Many students spend $2^+$ years learning ancient ($\approx$300 BC) 2D Euclidean geometry and trigonometry (trigonometry translates to "triangle measurement"). The invention of ***vectors*** (Gibbs $\approx$ 1900 AD) and its easy-to-use vector addition, dot-products, and cross-products have **greatly simplified** 2D and 3D geometry. Unfortunately, few instructors teach geometry or trigonometry with vectors.

## 1.3 Circles and their properties




The ratio of ***any*** circle's ***circumference*** to its ***diameter*** is the number[a]

$$\pi \triangleq \frac{\text{circumference}}{\text{diameter}} \approx 3.14159265358979323846264338\ldots$$

$\boldsymbol{\pi}$ is called an "***irrational number***" because it is not a whole number or fraction, nor does it terminate or repeat. It is chaotic, disorderly, and has no discernible pattern ($\pi$ has been memorized to $67,890^+$ digits).

The ***arc-length*** of a portion of the circle's periphery and the ***area*** of a wedge of the circle can be calculated in terms of the circle's ***radius*** $r$ and the ***angle*** $\theta$ as shown right.[6]

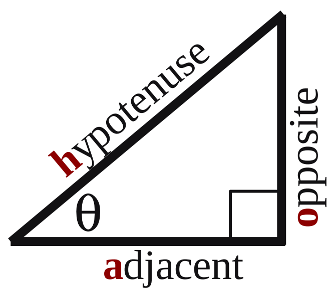
| | |
|---|---|
| Arc-length $= \theta\, r$ | Area of wedge $= \frac{\theta}{2}\, r^2$ |
| Circumference $= 2\,\pi\, r$ | Area of circle $= \pi\, r^2$ |

[a] The symbol $\boldsymbol{\pi}$ was popularized by Euler circa 1750, but the value $\pi \approx 3.14$ was known in Egypt circa 3000 BC.[4]

## 1.4 Triangles and ratios of their sides (sine, cosine, tangent)

A triangle ("three angles") is a 3-sided planar geometric shape widely used in construction, engineering, and science.

SohCahToa is a ***mnemonic*** for memorizing the definitions of ***Sine***, ***Cosine***, and ***Tangent*** (ratios of various sides of a right triangle).




$$\sin(\theta) \triangleq \frac{\text{opposite}}{\text{hypotenuse}} \qquad \cos(\theta) \triangleq \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \tan(\theta) \triangleq \frac{\textit{opposite}}{\textit{adjacent}} = \frac{\sin(\theta)}{\cos(\theta)} \tag{1}$$

The ***Pythagorean theorem*** in equation (2) relates lengths of sides of a right triangle. Combining the definitions of $\sin(\theta)$ and $\cos(\theta)$ with the Pythagorean theorem gives the second relationship to the right.

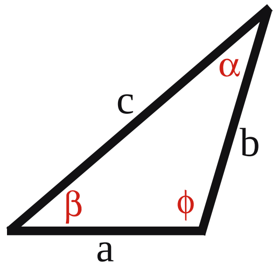
$$\text{hypotenuse}^2 = \text{adjacent}^2 + \text{opposite}^2 \qquad \sin^2(\theta) + \cos^2(\theta) \underset{(1)}{=} 1 \tag{2}$$

**Note: Numbers under = refer to equation numbers, e.g., $\underset{(1)}{=}$ means "refers to equation (1)".**

Australian study found that switching from British units to metric units freed $\frac{1}{2}$-year in science education. U.S. lawmakers have consistently failed to legislate changes in federal systems, e.g., road signs, NASA, DOD, and NSF.

[4] An ***angle*** involves two lines (or vectors) and is measured in radians or degrees. A radian is the ratio of the arc-length of part of a circle's perimeter to its radius. A degree is an archaic unit of angle measurement based on the ancient Babylonian year which had 360 days (12 months $*$ 30 days). Each degree represents one day of Earth's travel about the sun and the degree symbol's circular appearance $^\circ$ is a reminder that $360^\circ$ measures the Earth's quasi-circular travel around the sun.

### 1.4.1 Formulas involving sine and cosine




| | |
|---|---|
| ***Law of cosines*** | Euclid of Alexandria Egypt, 300 BC |
| ***Law of sines*** | Ptolemy of Alexandria Egypt, 100 AD |
| ***Addition formula for sine*** | Ptolemy of Alexandria Egypt, 100 AD |

$$c^2 \;=\; a^2 \;+\; b^2 \;-\; 2\,a\,b\,\cos(\phi) \qquad \textbf{\textit{Law of cosines}} \qquad (3)$$

$$\frac{\sin(\alpha)}{a} \;=\; \frac{\sin(\beta)}{b} \;=\; \frac{\sin(\phi)}{c} \qquad \textbf{\textit{Law of sines}} \qquad (4)$$

$$\sin(-\alpha) \;=\; -\sin(\alpha) \qquad (5)$$

$$\cos(-\alpha) \;=\; \cos(\alpha) \qquad (6)$$

$$\sin(\alpha+\beta) \;=\; \sin(\alpha)\cos(\beta) \;+\; \sin(\beta)\cos(\alpha) \qquad \textbf{\textit{Addition formula for sine}} \qquad (7)$$

$$\cos(\alpha+\beta) \;\underset{(7)}{=}\; \cos(\alpha)\cos(\beta) \;-\; \sin(\alpha)\sin(\beta) \qquad \textbf{\textit{Addition formula for cosine}} \qquad (8)$$

### 1.4.2 Sine and cosine as functions (Euler, circa 1730)

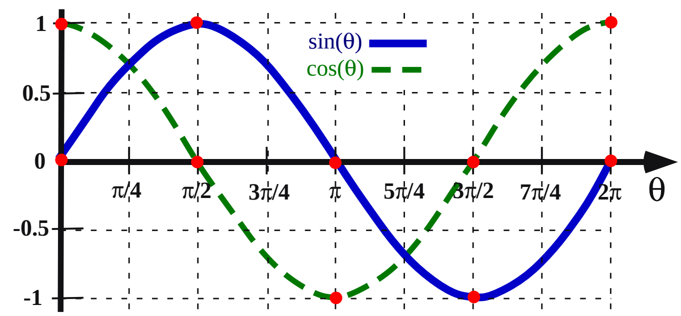
Euler's interpretation of ***cosine*** and ***sine*** as ***functions*** (not just ratios of sides of a triangle) was a major breakthrough for trigonometry and functions.[5]

**Triangle:** $\cos(\theta) \triangleq \dfrac{\textbf{a}\text{djacent}}{\textbf{h}\text{ypotenuse}}$

whereas the **cosine function** is shown right

**Triangle:** $\sin(\theta) \triangleq \dfrac{\textbf{o}\text{pposite}}{\textbf{h}\text{ypotenuse}}$

whereas the **sine function** is shown right




## 1.5 Types of scalars: Variable, specified, constant

- An ***independent variable*** is a quantity that varies independently, i.e., it does not depend on other variables. Many dynamic systems have one independent variable, namely ***time*** $t$.
- A ***dependent variable*** is a quantity whose value depends on the independent variable and its dependence is considered to be **unknown**, e.g., governed by an algebraic or differential equation.
- A ***specified variable*** is a quantity that varies in a **known** way, e.g., it is ***prescribed*** as a function of constants, time, and other variables, such as $x = \sin(t)$.
- A ***constant*** is a quantity whose value does not change (a constant may be **known** or **unknown**).

[5] The Babylonians used right triangle formulas for thousands of years before their proofs by Pythagoras of Samos [≈500 BC]. The definitions of ***sine***, ***cosine***, and ***tangent*** as ratios of sides of a right triangle predate 140 BC when the Greek Hipparchus made sine, cosine, and tangent tables. Euler's interpretation of sine, cosine, and tangent as ***functions*** was a breakthrough for math. Gibb's invention of vectors (≈1900 AD) significantly simplified 3D geometry and trigonometry and proofs of ***law of cosines***, ***law of sines***, and ***sine addition formula***, from which other trigonometric formulas are derived [***cosine addition formula***, ***half or double-angle formulas***, etc.].

## 1.6 Differentiation

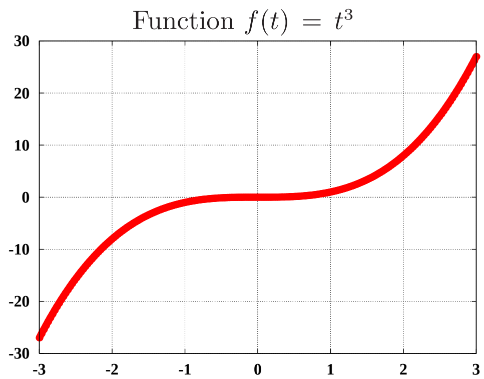
### 1.6.1 Definition of an ordinary derivative of a scalar function

When a function $f$ is regarded to depend on **1** scalar variable $t$, it is denoted $f(t)$.

The ordinary ***1st-derivative*** of $f$ with respect to $t$" is denoted in various ways as shown in equation (9).[a]

$$f' = \dot{f} = \frac{df}{dt} = \lim_{h \to 0} \frac{f(t+h) - f(t)}{h} \tag{9}$$

[a] The notation using a ratio (fraction) of differentials $\frac{df}{dt}$ was invented by Leibniz in 1675, the dot-notation $\dot{f}$ by Newton $\approx$ 1675, the prime notation $f'$ by Lagrange in 1797, and the limit notation by Cauchy and Weierstrauss in 1850.




The derivative of the derivative with respect to $t$ is called the "***2nd-derivative*** of $f(t)$ with respect to $t$", and is denoted in various ways as shown below.

$$f'' = \ddot{f} = \frac{d^2 f}{dt^2} \triangleq \frac{d}{dt}\left(\frac{df}{dt}\right)$$

From a geometric (Newton's) perspective, the $1^{st}$-derivative is ***slope*** and the $2^{nd}$-derivative is ***curvature***. From a function (Euler's) perspective, the derivative of a function is a function.

### 1.6.2 Definition of a partial derivative of a scalar function

When a function $f$ depends on $n$ independent scalar variables $t_1, \ldots, t_n$, it is denoted $f(t_1, \ldots, t_n)$.[6] There are $n$ quantities $\frac{\partial f}{\partial t_i}$ called "first ***partial derivatives*** of $f$ with respect to $t_i$", defined as

$$\frac{\partial f}{\partial t_i} \triangleq \lim_{h \to 0} \frac{f(t_1, \ldots, \; t_i + h, \; \ldots, t_n) - f(t_1, \ldots, \; t_i, \; \ldots, t_n)}{h} \qquad (i = 1, \ldots, n) \tag{10}$$

The definition of the ***partial derivative*** of $f$ with respect to $t$ in equation (10) reduces to the ***ordinary derivative*** of $f$ with respect to $t$ when $f$ is a function of ***one*** independent variable,[7] i.e., $\frac{df}{dt} = \frac{\partial f}{\partial t}$.

Since $\frac{\partial f}{\partial t_i}$ is defined as a limit and is not a ratio of differentials, one cannot cancel the $\partial t_i$ in the denominator by multiplying through by $\partial t_i$. In other words $\partial t_i$ is not an entity in its own right.

### 1.6.3 Definition of the total derivative of a scalar function

At times, a function $f$ can be regarded as either depending on **1** scalar quantity $t$, or regarded as a function of $\mathbf{n+1}$ scalar quantities $x_1, \ldots, x_n$ and $t$, where $x_1, \ldots, x_n$ are themselves functions of $t$. When $f$ is regarded as a function of $x_1, \ldots, x_n$ and $t$, $f$ is denoted $f(x_1(t), \ldots, x_n(t), t)$, and the ordinary derivative of $f$ with respect to $t$ is called the ***total derivative*** of $f$ with respect to $t$ and can be calculated as

$$\begin{aligned} \frac{df}{dt} &= \frac{\partial f}{\partial x_1} * \frac{dx_1}{dt} + \frac{\partial f}{\partial x_2} * \frac{dx_2}{dt} + \ldots \frac{\partial f}{\partial x_n} * \frac{dx_n}{dt} + \frac{\partial f}{\partial t} \\ &= \frac{\partial f}{\partial x_1} \; \dot{x}_1 + \frac{\partial f}{\partial x_2} \; \dot{x}_2 + \ldots \frac{\partial f}{\partial x_n} \; \dot{x}_n + \frac{\partial f}{\partial t} \end{aligned} \tag{11}$$

[6] Euler invented the function notation, e.g., $f(t)$, $f(x, y)$, circa 1730.

[7] Synonyms for ordinary (as in ordinary derivative) are "plain" and "boring" because $f$ is a function of only ***one*** variable, whereas a "hot and spicy" partial derivative is a function of ***two or more variables***.

### 1.6.4 Short table of derivatives frequently encountered in engineering

| Function and its derivative | | | Function and its derivative | | |
|---|---|---|---|---|---|
| $F(t) = \sin(t)$ | $\frac{\partial F}{\partial t} = \cos(t)$ | | $F(t) = \cos(t)$ | $\frac{\partial F}{\partial t} = -\sin(t)$ | |
| $F(t) = t^n$ | $\frac{\partial F}{\partial t} = n * t^{n-1}$ | n = constant | $F(t) = \tan(t)$ | $\frac{\partial F}{\partial t} = \frac{1}{\cos^2(t)}$ | |
| $F(t) = \ln(t)$ | $\frac{\partial F}{\partial t} = t^{-1} = \frac{1}{t}$ | | $F(t) = e^t$ | $\frac{\partial F}{\partial t} = e^t$ | important for ODEs $e = 2.71828\ldots$ |

### 1.6.5 Example: Partial and ordinary differentiation

**Example A:** Consider a function $f$ that only depends on **1** independent variable $t$ (time), but which is expressed in terms of dependent variables $x$ and $y$ (both $x$ and $y$ depend on $t$). The function $f$ can also be ***regarded*** as a function of **3** independent scalar quantities $(x, y, t)$.

$$f(x, y, t) \;=\; \sin(x)\, y^2 \;+\; e^{3\,t}$$

Partial derivatives of $f(x, y, t)$ with respect to $x$, $y$, or $t$ and the ordinary (total) derivative of $f$ are

$$\frac{\partial f}{\partial x} = \cos(x)\, y^2 \qquad \frac{\partial f}{\partial y} = 2\,\sin(x)\, y \qquad \frac{\partial f}{\partial t} = 3\, e^{3\,t} \qquad \frac{df}{dt} = \cos(x)\, \dot{x}\, y^2 \;+\; 2\,\sin(x)\, y\, \dot{y} \;+\; 3\, e^{3\,t}$$

**Example B:** Consider a function $g$ that depends on **1** independent variable $t$ (time), but which is expressed in terms of a dependent variable $x$ and its ordinary time-derivative $\dot{x}$. The function $g$ can also be ***regarded*** as a function of **3** independent scalars $(x, \dot{x}, t)$ as

$$g(x, \dot{x}, t) \;=\; \sin(x)\, \dot{x}^2 \;+\; e^{3\,t}$$

Partial derivatives of $g(x, \dot{x}, t)$ with respect to $x$, $\dot{x}$, or $t$ and the ordinary (total) derivative of $g$ are

$$\frac{\partial g}{\partial x} = \cos(x)\, \dot{x}^2 \qquad \frac{\partial g}{\partial \dot{x}} = 2\,\sin(x)\, \dot{x} \qquad \frac{\partial g}{\partial t} = 3\, e^{3\,t} \qquad \frac{dg}{dt} = \cos(x)\, \dot{x}^3 \;+\; 2\,\sin(x)\, \dot{x}\, \ddot{x} \;+\; 3\, e^{3\,t}$$

### 1.6.6 Good product rule for differentiation (for scalars, $\vec{\text{v}}$ectors, matrices, ...)

**Good product rule:**

$$\frac{\partial(u * v * w)}{\partial t} \;=\; \frac{\partial u}{\partial t} * v * w \;+\; u * \frac{\partial v}{\partial t} * w \;+\; u * v * \frac{\partial w}{\partial t} \qquad (12)$$

**Example:**

$$\frac{\partial\big[\, t^2 * \sin(t) * e^t \,\big]}{\partial t} \;=\; 2\,t\,\sin(t)\, e^t \;+\; t^2\,\cos(t)\, e^t \;+\; t^2\,\sin(t)\, e^t$$

Unfortunately, many calculus books use the "**bad**" ***product rule for differentiation*** $\frac{d\,(u * v)}{dt} = u * \frac{dv}{dt} + v * \frac{du}{dt}$, which fails if $u$ and $v$ are $\vec{\text{v}}$ectors or matrices, and is inefficient for differentiating $3^+$ scalars (e.g., $u * v * w$).

### 1.6.7 Quotient rule for derivatives: Use exponents and the product rule

Since the quotient $\frac{u}{v}$ is equivalent to $u\, v^{-1}$, the derivative of $\frac{u}{v}$ with respect to $t$ can be implemented with the ***product rule*** and exponents (without memorizing special ***quotient-rule*** formulas).

$$\frac{\partial}{\partial t}\left(\frac{u}{v}\right) \;=\; \frac{\partial u}{\partial t}\, v^{-1} \;-\; u\, v^{-2}\, \frac{\partial v}{\partial t} \qquad (13)$$

### 1.6.8 Chain rule for derivatives

When the variable $x$ depends on the variable $t$, the derivative of the function $f(x)$ with respect to $t$ can be written via the ***chain rule for differentiation*** as shown in equation (14).

$$\frac{\partial\, f(x)}{\partial t} \;=\; \frac{\partial\, f(x)}{\partial x}\, \frac{\partial x}{\partial t} \qquad (14)$$

### 1.6.9 Implicit differentiation: A useful tool for calculating derivatives

**Example:** In general, it is difficult to solve the nonlinear equation below to find $y$ explicitly in terms of $t$. However, ***implicit differentiation*** calculates $\frac{dy}{dt}$ **without** first solving for $y$, e.g.,

$$y^2 + \sin(y) = \cos(t) \qquad \Rightarrow \qquad 2\,y\,\frac{dy}{dt} + \cos(y)\,\frac{dy}{dt} = -\sin(t) \qquad \Rightarrow \qquad \frac{dy}{dt} = \frac{-\sin(t)}{2\,y + \cos(y)}$$

**Example:** The use of implicit differentiation in conjunction with ***natural logarithms*** is useful for calculating the ordinary time-derivative of $y = c^t$ ($c$ is a constant and $t$ is time), as shown below.

$$y = c^t \qquad \Rightarrow \qquad \ln(y) = t\,\ln(c) \qquad \Rightarrow \qquad d\,[\ln(y)] = \ln(c)\,dt \qquad \Rightarrow \qquad \frac{1}{y}\,dy = \ln(c)\,dt$$

$$\frac{dy}{dt} = \ln(c)\,y = \ln(c)\,c^t$$

Note: When $c = e = 2.718281828$, $\frac{dy}{dt} = y$.
This plays a **central role** in solving ordinary differential equations.

## 1.7 Integration and a short table of integrals

An ***integral*** can be regarded as either an ***anti-derivative*** or as a ***sum*** (e.g., **area under a curve**).

| Function | Integral of $F(t)$ | |
|---|---|---|
| $F(t) = t^n$ | $\int F(t)\,dt = \frac{t^{n+1}}{n+1} + C$ | ($n$ is a number other than $-1$) |
| $F(t) = t^{-1}$ | $\int F(t)\,dt = \ln(t) + C$ | |
| $F(t) = e^t$ | $\int F(t)\,dt = e^t + C$ | |
| $F(t) = \sin(t)$ | $\int F(t)\,dt = -\cos(t) + C$ | |
| $F(t) = \cos(t)$ | $\int F(t)\,dt = \sin(t) + C$ | |

The website www.WolframResearch.com is a valuable resource for calculating integrals.

**History:** In 1675, Leibniz invented the integral notation $\int$ (Latin abbreviation for summa - sum) and its natural extension to double and triple integrals. Newton's integral notation was so defective, it was never popular – even in England. Euler was the first to use a symbol for an integral's limits, and its modern notation, e.g., $\int_a^b x\,dx$, was invented by Fourier in 1820.

**Math helps predict planetary motion, seasons, and climate change.**
**Courtesy Claude Rheaume LaSalette Enfield NH.** (Lower Shaker Village)

## 1.8 Solutions of *polynomial* equations (roots)

***Polynomial equations*** are a special class of nonlinear algebraic equations. A special polynomial equation is the ***quadratic equation***, which is a polynomial equation of degree **2**. Shown below is a quadratic equation in $x$ and its **2** ***roots*** (solutions).

***Quadratic equation***

$$a\,x^2 + b\,x + c = 0$$

***Solution to quadratic equation***

$$x = \frac{-b + \sqrt{b^2 - 4\,a\,c}}{2\,a} \quad \text{and} \quad x = \frac{-b - \sqrt{b^2 - 4\,a\,c}}{2\,a}$$

Two other polynomial equations with "closed-form solutions" are the ***cubic*** and ***quartic*** equations

$$x^3 + c_2\,x^2 + c_1\,x + c_0 = 0 \qquad \text{and} \qquad x^4 + c_3\,x^3 + c_2\,x^2 + c_1\,x + c_0 = 0$$

The ***Fundamental Theorem of Algebra*** states that any polynomial of degree $n$ with complex coefficients has $n$ complex roots.[8] In 1824, Abel proved that no general closed-form solution for $5^{th}$-order (or higher) polynomials exist. Numerical methods are useful for calculating roots of polynomials of any order.

## 1.9 Computer solutions of algebraic and differential equations

The invention of computers radically changed human ability to form and solve equations governing space, matter, and time – and visualize their results (plots, animation, virtual reality, ...).

| Equation type | Example |
|---|---|
| Linear algebraic | $A\,x = B$ |
| Polynomial | $x^6 + 3\,x^3 + x + 9 = 0$ |
| Eigen | $A\,v = \lambda v$ |
| Nonlinear algebraic | $x^2 - \cos^2(x) = 0$ |
| Differential | $\ddot{\theta} + 4\,\sin(\theta) = \cos(t)$ |

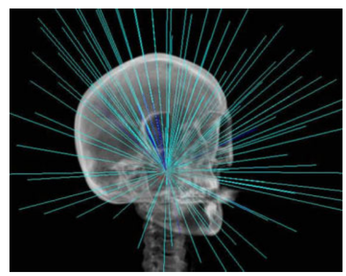
## 1.10 Optional: Continuous solutions of *nonlinear* algebraic equations

One way to find a continuous solution for $x$ in the range $0 \le t \le 8$ for

$$x^2 - \cos^2(x) = 0.3\,\sin(t)$$

is to differentiate this ***nonlinear*** equation with respect to $t$ and then solve the derivative equation that is ***linear*** in $\dot{x}$ as

$$2\,x\,\dot{x} + 2\,\cos(x)\,\sin(x)\,\dot{x} = 0.3\,\sin(t) \quad \Rightarrow \quad \dot{x} = \frac{0.3\,\cos(t)}{2\,x + 2\,\cos(x)\,\sin(x)}$$



**Courtesy Accuray Inc.**

Solving the nonlinear equation **once** at $t = 0$ gives $x(t=0) \approx 0.74$. With this initial value for $x$ and continuous formula for $\dot{x}$, ODE techniques can numerically integrate $\dot{x}(t)$ to solve for $x(t)$.

**Math helps design structures, robots, and satellites and predicts weather and saves lives**

[8]The proof of the ***Fundamental Theorem of Algebra*** is difficult and was presented with various rigor between 1608 and 1981 by great mathematicians including, Roth (1608) Girard (1629), Leibniz (1702), Bernoulli (1742), d'Alembert (1746), Euler (1749), Lagrange (1772), Laplace (1795), Gauss (1799), Argand (**1806**), Gauss (again in 1816 and 1849), Cauchy (1821), Weierstrauss (1891), Hellmuth Kneser (1940), and his son Martin Kneser (**1981**).