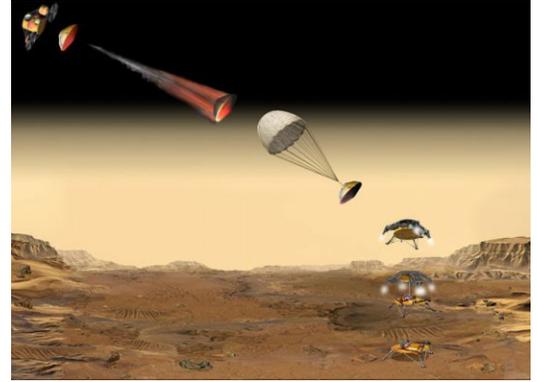# Chapter 2

Vectors

Courtesy NASA/JPL-Caltech

Summary   (see examples in Hw 1, 2, 3)

The history of vectors is complicated and intimately relates to co-developments in mechanics and mathematics. As mechanics became more sophisticated, the tools to solve mechanics problems developed.

- Discovery (Cardan 1545) and geometrical representation (Caspar Wessel 1799, Gauss 1800[+]) of complex numbers.
- Search for a geometry of position (Leibniz 1679).
- Idea of a parallelogram of forces or velocities (Newton 1687).
- Hamilton's 22-year development of quaternions (1843[+]) via lectures, 109 papers, and 2 immense books.
- Hamilton (1846) introduces the terms "scalar" and "vector" in a quaternion $\hat{q} = \underbrace{q_0}_{\text{scalar}} + \underbrace{q_1\hat{i} + q_2\hat{j} + q_3\hat{k}}_{\text{vector}}$

From 1881-1903, Gibbs developed modern **vectors** as a useful combination of magnitude and direction and invented their higher-dimensional counterparts **dyadics**, **triadics**, **polyadics**. Vectors are an important **geometrical tool** [useful for surveying, motion analysis, lasers, optics, computer graphics, animation, CAD/CAE (computer aided drawing/engineering), and FEA(finite element analysis), . . .].

Symbol	Description	Details
$\vec{0}$, $\widehat{\mathbf{u}}$	Zero vector and unit vector.	Sections 2.3, 2.4
$+ - *$	Vector addition, negation, subtraction, and multiplication/division with a scalar.	Sections 2.6 - 2.8
$\cdot$ $\times$	Vector dot product and cross product.	Sections 2.9, 2.10
$\frac{{}^F d}{dt}$	Vector differentiation.	Chapters 6, 7

2.1   Examples of scalars, vectors, and dyadics

- A **scalar** is a non-directional quantity (e.g., a real number).   Examples include:

time	density	volume	mass	moment of inertia	temperature
distance	speed	angle	weight	potential energy	kinetic energy

- A **vector** is a quantity that has magnitude and **one** associated direction. For example, a **velocity vector** has speed (how fast something is moving) and direction (which way it is going). A **force vector** has magnitude (how hard something is pushed) and direction (which way it is shoved).   Examples include:

position vector	velocity	acceleration	linear momentum	force
impulse	angular velocity	angular acceleration	angular momentum	torque

- A **dyad** is a quantity with magnitude and **two** associated directions. For example, **stress** associates with area and force (both regarded as vectors). A **dyadic** is the **sum of dyads**. For example, an **inertia dyadic** (Chapter 14) is the sum of dyads associated with moments and products of inertia.

2.2 Definition of a vector

A **_vector_** is defined as a quantity having **_magnitude_** and **_direction_**.[a]
Vectors are represented pictorially with straight or curved arrows (examples below).
Vectors are typeset with an arrow and bold-faced font, e.g., $\vec{\mathbf{v}}$ denotes a vector.

right / left   **up/down**   **out-from / into page**   **inclined at 45°**

Certain vectors have additional special properties. For example, a **_position vector_** $\vec{\mathbf{r}}$ is associated with two points and has units of distance.

Courtesy Bro. Claude
Rheaume LaSalette.

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[a] A vector's **_magnitude_** is a real non-negative number. A vector's **_direction_** can be resolved into **_orientation_** and **_sense_**. For example, a highway has an orientation (e.g., east-west) and a vehicle traveling east has a sense. Knowing both the orientation of a line and the sense on the line gives direction. Changing a vector's orientation or sense changes its direction.

Example of a vector: Consider the statement "the car is moving East at $5\ \frac{\text{m}}{\text{s}}$".
It is **_convenient_** to represent the car's speed and direction with the velocity vector
$\vec{\mathbf{v}} = 5 * \widehat{\mathbf{E}}\text{ast} = 5\,\widehat{\mathbf{E}}\text{ast}$ (a hat designates the direction $\widehat{\mathbf{E}}\text{ast}$ as a **_unit vector_**). The car's
speed is always a real non-negative number equal to $|\vec{\mathbf{v}}|$, the **_magnitude_** of $\vec{\mathbf{v}}$.
The combination of **_magnitude_** and **_direction_** is a **_vector_**.

$5\frac{\text{m}}{\text{s}}$
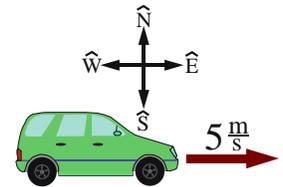
A car moving West with speed 5 has velocity $\vec{\mathbf{v}} = \text{–}5\,\widehat{\mathbf{E}}\text{ast}$. The negative sign in $\text{–}5\,\widehat{\mathbf{E}}\text{ast}$ is associated with the vector's direction (a vector's magnitude is inherently non-negative). When writing $\vec{\mathbf{v}} = \dot{x}\,\widehat{\mathbf{E}}\text{ast}$ where $\dot{x}$ is a scalar that can be **positive** <u>or</u> zero <u>or</u> **negative**, $\dot{x}$ is called the $\widehat{\mathbf{E}}\text{ast}$ **_measure_** of the vector $\vec{\mathbf{v}}$, ($\vec{\mathbf{v}}$'s non-negative **_magnitude_** is abs($\dot{x}$)).

2.3 Zero vector $\vec{\mathbf{0}}$ – a vector whose magnitude is zero

Addition with a zero vector:	$\text{any}\vec{\mathbf{V}}\text{ector} + \vec{\mathbf{0}} = \text{any}\vec{\mathbf{V}}\text{ector}$	
Dot product with a zero vector:	$\text{any}\vec{\mathbf{V}}\text{ector} \cdot \vec{\mathbf{0}} \underset{(2)}{=} 0$	$\vec{\mathbf{0}}$ is **_perpendicular_** to all vectors
Cross product with a zero vector:	$\text{any}\vec{\mathbf{V}}\text{ector} \times \vec{\mathbf{0}} \underset{(5)}{=} \vec{\mathbf{0}}$	$\vec{\mathbf{0}}$ is **_parallel_** to all vectors
Derivative of the **_zero vector_**:	$\dfrac{^F d\vec{\mathbf{0}}}{dt} = \vec{\mathbf{0}}$	F is any reference frame

Vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ are said to be "**_perpendicular_**" if $\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = 0$ whereas $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ are "**_parallel_**" if $\vec{\mathbf{a}} \times \vec{\mathbf{b}} = \vec{\mathbf{0}}$.
Note: Some say $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ are "**_parallel_**" only if $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ have the same direction and "anti-parallel" if $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ have opposite directions.[1]

2.4 Unit $\widehat{\text{v}}$ectors – a vector whose magnitude is 1 (typeset with a special $\widehat{\text{h}}$at)

Unit vectors are "**_sign posts_**" (e.g., unit vectors $\widehat{\mathbf{N}}$, $\widehat{\mathbf{S}}$, $\widehat{\mathbf{W}}$, $\widehat{\mathbf{E}}$ for local Earth directions) chosen to simplify communication and mathematics Other useful "sign posts" are:
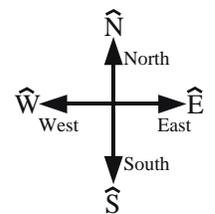
- Unit vector directed from one point to another point
- Unit vector directed locally vertical
- Unit vector parallel to the edge of an object
- Unit vector tangent to a curve or perpendicular to a surface

---

[1] The direction of a **_zero vector_** $\vec{\mathbf{0}}$ is arbitrary and may be regarded as having **any** direction so that $\vec{\mathbf{0}}$ is **_parallel_** to all vectors, $\vec{\mathbf{0}}$ is **_perpendicular_** to all vectors, all zero vectors are equal, and one may use the definite pronoun "the" instead of the indefinite "a" e.g., "<u>the</u> zero vector". It is improper to say the **_zero vector_** has no direction as a vector is **defined** to have both magnitude and direction. It is also improper to say a **_zero vector_** has all directions as a vector is defined to have a magnitude and **a** direction (as contrasted with a dyad which has 2 directions or triad which has 3 directions).

A unit vector can be defined so it has the same direction as an arbitrary non-zero vector $\vec{v}$ by dividing $\vec{v}$ by $|\vec{v}|$ (the magnitude of $\vec{v}$).

To avoid divide-by-zero problems during numerical computation, approximate the unit vector with a "small" positive real number ϵ in the denominator.

$$\mathbf{unit\vec{V}ector} = \frac{\vec{v}}{|\vec{v}|}$$
$$\approx \frac{\vec{v}}{|\vec{v}| + \epsilon} \tag{1}$$

## 2.5 Equal vectors (=) vectors with the same magnitude and direction

Shown right are three **equal vectors**. Although each has a different location, the vectors are equal because they have the same magnitude and direction.
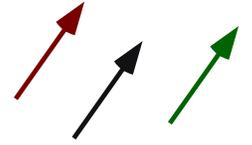
Some vectors have additional properties. For example, a position vector is associated with two points. Two position vectors are **equal position vectors** when, they have the same magnitude, same direction, and are associated with the same points. Two force vectors are **equal force vectors** when they have the same magnitude, direction, and point of application.

2.6 Vector addition (+)

As graphically shown right, adding two vectors $\vec{a} + \vec{b}$ produces a vector. First, vector $\vec{b}$ is translated[a] so its tail is at the tip of $\vec{a}$. Next, the vector $\vec{a} + \vec{b}$ is drawn from the tail of $\vec{a}$ to the tip of the translated $\vec{b}$.
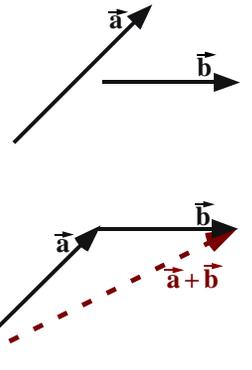
Properties of vector addition

Commutative law: $\quad \vec{a} + \vec{b} = \vec{b} + \vec{a}$
Associative law: $\quad (\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c}) = \vec{a} + \vec{b} + \vec{c}$
Addition of zero vector: $\quad \vec{a} + \vec{0} = \vec{a}$

---
[a] Translating $\vec{b}$ does *not* change the magnitude or direction of $\vec{b}$, and so produces an equal $\vec{b}$. Note: It is nonsensical to add vectors with different units, e.g., do not add a position vector with units of meters with a force vector with units of Newtons.

Example: Vector addition (+) algebra

Shown to the right is an example of how to add vector $\vec{w}$ to vector $\vec{v}$, each which is expressed in terms of orthogonal unit vectors $\hat{n}_x$, $\hat{n}_y$, $\hat{n}_z$.

$$\begin{aligned} \vec{v} &= 7\,\hat{n}_x + 5\,\hat{n}_y + 4\,\hat{n}_z \\ + \ \vec{w} &= 2\,\hat{n}_x + 3\,\hat{n}_y + 2\,\hat{n}_z \\ \hline \vec{v} + \vec{w} &= 9\,\hat{n}_x + 8\,\hat{n}_y + 6\,\hat{n}_z \end{aligned}$$
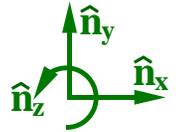
2.7 Vector multiplied or divided by a scalar (∗ or /)

- Multiplying a vector by a **positive** number (other than 1) changes the vector's magnitude.
- Multiplying a vector by a **negative** number changes the vector's magnitude **and** reverses the **sense** of the vector.
- Dividing a vector $\vec{a}$ by a scalar s is defined as $\dfrac{\vec{a}}{s} \triangleq \dfrac{1}{s} * \vec{a}$.
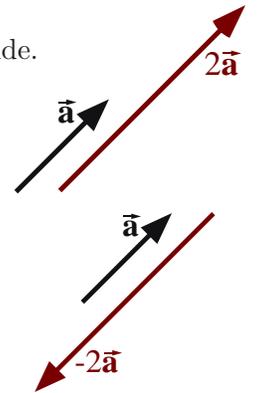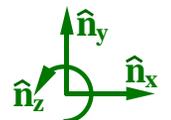
Properties of multiplication of a vector by a scalar s_1 or s_2

Commutative law: $\quad s_1\,\vec{a} = \vec{a}\,s_1$
Associative law: $\quad s_1\,(s_2\,\vec{a}) = (s_1\,s_2)\,\vec{a} = s_2\,(s_1\,\vec{a}) = s_1\,s_2\,\vec{a}$
Distributive law: $\quad (s_1 + s_2)\,\vec{a} = s_1\,\vec{a} + s_2\,\vec{a} \qquad s_1\,(\vec{a} + \vec{b}) = s_1\,\vec{a} + s_1\,\vec{b}$
Multiplication by zero: $\quad 0 * \vec{a} = \vec{0}$

Example: Vector scalar multiplication and division (∗ and /)

Given: $\quad \vec{v} = 7\,\hat{n}_x + 5\,\hat{n}_y + 4\,\hat{n}_z$
then: $\quad 5\,\vec{v} = 35\,\hat{n}_x + 25\,\hat{n}_y + 20\,\hat{n}_z$

and $\qquad \dfrac{\vec{v}}{-2} = -3.5\,\hat{n}_x - 2.5\,\hat{n}_y - 2\,\hat{n}_z$

Chapter 2: Vectors

## 2.8 Vector negation and subtraction (−)

Negation: As shown right, negating a vector (multiplying by –1) reverses the vector's **sense** (it points in the opposite direction).   Negation does not change the vector's magnitude or orientation.

Subtraction: As the drawing to the right shows, subtracting a vector $\vec{b}$ from a vector $\vec{a}$ is simply addition and negation.[a]

$$\vec{a} - \vec{b} \triangleq \vec{a} + -\vec{b}$$

After negating vector $\vec{b}$, it is translated so the tail of $-\vec{b}$ is at the tip of $\vec{a}$.
Next, vector $\vec{a} + -\vec{b}$ is drawn from the tail of $\vec{a}$ to the tip of the translated $-\vec{b}$.
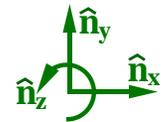
[a] In most/all mathematics, subtraction is defined as negation and addition.
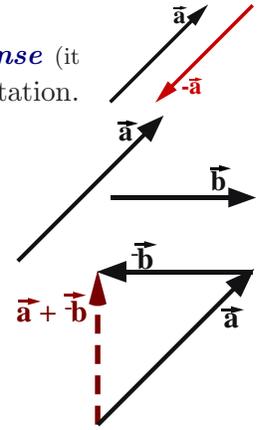
Example: Vector subtraction (−)

Shown right is an example of how to subtract vector $\vec{w}$ from vector $\vec{v}$, when each is expressed in terms of orthogonal unit vectors $\hat{n}_x$, $\hat{n}_y$, $\hat{n}_z$.

$$
\begin{aligned}
\vec{v} &= 7\,\hat{n}_x + 5\,\hat{n}_y + 4\,\hat{n}_z \\
-\ \vec{w} &= 2\,\hat{n}_x + 3\,\hat{n}_y + 2\,\hat{n}_z \\
\hline
\vec{v} - \vec{w} &= 5\,\hat{n}_x + 2\,\hat{n}_y + 2\,\hat{n}_z
\end{aligned}
$$
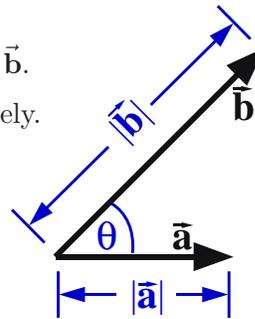
2.9 Vector dot product (·)

Equation (2) defines the **dot product** of vectors $\vec{a}$ and $\vec{b}$.

- $|\vec{a}|$ and $|\vec{b}|$ are the magnitudes of $\vec{a}$ and $\vec{b}$, respectively.
- θ is the smallest angle between $\vec{a}$ and $\vec{b}$ $(0 \le \theta \le \pi)$.

$$\vec{a} \cdot \vec{b} \triangleq |\vec{a}|\,|\vec{b}|\,\cos(\theta) \quad (2)$$

Equation (3) is a rearrangement of equation (2) that is useful for calculating the angle θ between two vectors.

Note: $\vec{a}$ and $\vec{b}$ are "**perpendicular**" when $\vec{a} \cdot \vec{b} = 0$.
Note: Dot-products encapsulate the **law of cosines**.

$$\cos(\theta) \underset{(2)}{=} \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|\,|\vec{b}|} \quad (3)$$

Use **acos** to calculate θ.

Equation (2) shows $\vec{v} \cdot \vec{v} = |\vec{v}|^2$. Hence, the dot product can calculate a vector's **magnitude** as shown for $|\vec{v}|$ in equation (4).

Equation (4) also defines **vector exponentiation** $\vec{v}^n$ (vector $\vec{v}$ raised to scalar power n) as a non-negative scalar.

Example: Kinetic energy $K = \frac{1}{2}\,m\,\vec{v}^2 \underset{(4)}{=} \frac{1}{2}\,m\,\vec{v} \cdot \vec{v}$

$$
\begin{aligned}
\vec{v}^2 &\triangleq |\vec{v}|^2 = \vec{v} \cdot \vec{v} \\
|\vec{v}| &= +\sqrt{\vec{v} \cdot \vec{v}} \\
\vec{v}^n &\triangleq |\vec{v}|^n = +(\vec{v} \cdot \vec{v})^{\frac{n}{2}}
\end{aligned} \quad (4)
$$

2.9.1 Properties of the dot-product (·)

Dot product with a zero vector	$\vec{a} \cdot \vec{0} = 0$
Dot product of **perpendicular** vectors	$\vec{a} \cdot \vec{b} = 0 \qquad \text{if } \vec{a} \perp \vec{b}$
Dot product of parallel vectors	$\vec{a} \cdot \vec{b} = \pm |\vec{a}|\,|\vec{b}| \qquad \text{if } \vec{a} \parallel \vec{b}$
Dot product with vectors scaled by s_1 and s_2	$s_1\,\vec{a} \cdot s_2\,\vec{b} = s_1\,s_2\,(\vec{a} \cdot \vec{b})$
Commutative law	$\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$
Distributive law	$\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$
Distributive law	$(\vec{a} + \vec{b}) \cdot (\vec{c} + \vec{d}) = \vec{a} \cdot \vec{c} + \vec{a} \cdot \vec{d} + \vec{b} \cdot \vec{c} + \vec{b} \cdot \vec{d}$

Note: The distributive law for dot-products and cross-products is proved in [33, pgs. 23-24, 32-34].

2.9.2 Uses for the dot-product ($\cdot$)

- Calculating an **angle** between two vectors [see equation (3) and example in Section 3.3]
- Determining when two vectors are **perpendicular**, e.g., $\vec{a} \cdot \vec{b} = 0$.
- Calculating a vector's **magnitude** [see equation (4) and **distance** examples in Sections 3.2 and 3.3].
- Changing a **vector equation** into a **scalar equation** (see Homework 2.31).
- Calculating a **unit vector** in the direction of a vector $\vec{v}$ [see equation (1)] $\boxed{\mathbf{unit\,\vec{V}ector} \underset{(1)}{=} \dfrac{\vec{v}}{|\vec{v}|}}$

- **Projection** of a vector $\vec{v}$ in the direction of $\vec{b}$, defined as: $\boxed{\vec{v} \cdot \dfrac{\vec{b}}{|\vec{b}|}}$
  See Section 4.2 for **projections**, **measures**, **coefficients**, **components**.

2.9.3 Special case: Dot-products with orthogonal unit vectors

When $\widehat{\mathbf{n}}_x$, $\widehat{\mathbf{n}}_y$, $\widehat{\mathbf{n}}_z$ are **orthogonal unit** vectors, it can be shown (see Homework 2.4)

$$(a_x\,\widehat{\mathbf{n}}_x + a_y\,\widehat{\mathbf{n}}_y + a_z\,\widehat{\mathbf{n}}_z) \cdot (b_x\,\widehat{\mathbf{n}}_x + b_y\,\widehat{\mathbf{n}}_y + b_z\,\widehat{\mathbf{n}}_z) = a_x\,b_x + a_y\,b_y + a_z\,b_z$$
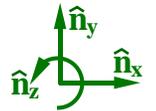
2.9.4 Examples: Vector dot-products ($\cdot$)

Shown below is how to use dot-products when vectors $\vec{v}$ and $\vec{w}$ $\vec{v} = 7\,\widehat{\mathbf{n}}_x + 5\,\widehat{\mathbf{n}}_y + 4\,\widehat{\mathbf{n}}_z$
are expressed in terms of orthogonal unit vectors $\widehat{\mathbf{n}}_x$, $\widehat{\mathbf{n}}_y$, $\widehat{\mathbf{n}}_z$. $\vec{w} = 2\,\widehat{\mathbf{n}}_x + 3\,\widehat{\mathbf{n}}_y + 2\,\widehat{\mathbf{n}}_z$

$\widehat{\mathbf{n}}_x$ measure of $\vec{v}$	$\vec{v} \cdot \widehat{\mathbf{n}}_x = 7$ (**measures** how much of $\vec{v}$ is in the $\widehat{\mathbf{n}}_x$ direction).
$\vec{v} \cdot \vec{v} = 7^2 + 5^2 + 4^2 = 90$	$|\vec{v}| = \sqrt{90} \approx 9.4868$
$\vec{w} \cdot \vec{w} = 2^2 + 3^2 + 2^2 = 17$	$|\vec{w}| = \sqrt{17} \approx 4.1231$
Unit vector in the direction of $\vec{v}$:	$\dfrac{\vec{v}}{|\vec{v}|} = \dfrac{7\,\widehat{\mathbf{n}}_x + 5\,\widehat{\mathbf{n}}_y + 4\,\widehat{\mathbf{n}}_z}{\sqrt{90}} \approx 0.738\,\widehat{\mathbf{n}}_x + 0.527\,\widehat{\mathbf{n}}_y + 0.422\,\widehat{\mathbf{n}}_z$
Unit vector in the direction of $\vec{w}$:	$\dfrac{\vec{w}}{|\vec{w}|} = \dfrac{2\,\widehat{\mathbf{n}}_x + 3\,\widehat{\mathbf{n}}_y + 2\,\widehat{\mathbf{n}}_z}{\sqrt{17}} \approx 0.485\,\widehat{\mathbf{n}}_x + 0.728\,\widehat{\mathbf{n}}_y + 0.485\,\widehat{\mathbf{n}}_z$
$\vec{v} \cdot \vec{w} = 7*2 + 5*3 + 4*2 = 37$	$\angle(\vec{v}, \vec{w}) = \mathrm{acos}\left(\frac{37}{\sqrt{90}\,\sqrt{17}}\right) \approx 0.33\,\text{rad} \approx 18.93°$

2.9.5 Dot-products to change vector equations to scalar equations (see Hw 1.31)

One way to form up to three linearly independent scalar equations from the vector equation
$\vec{v} = \vec{0}$ is by dot-multiplying $\vec{v} = \vec{0}$ with three orthogonal unit vectors $\widehat{\mathbf{a}}_1$, $\widehat{\mathbf{a}}_2$, $\widehat{\mathbf{a}}_3$, i.e.,

$$\text{if} \quad \vec{v} = \vec{0} \quad \Rightarrow \quad \boxed{\vec{v} \cdot \widehat{\mathbf{a}}_1 = 0 \qquad \vec{v} \cdot \widehat{\mathbf{a}}_2 = 0 \qquad \vec{v} \cdot \widehat{\mathbf{a}}_3 = 0}$$
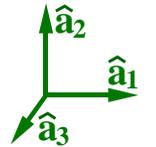
Section 2.11.2 describes another way to form three **different** scalar equations from $\vec{v} = \vec{0}$.
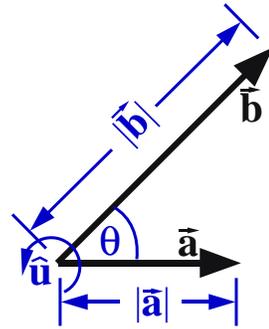
2.10 Vector cross product (×)

The ***cross product*** of a vector $\vec{a}$ with a vector $\vec{b}$ is defined in equation (5).

- $|\vec{a}|$ and $|\vec{b}|$ are the magnitudes of $\vec{a}$ and $\vec{b}$, respectively
- θ is the smallest angle between $\vec{a}$ and $\vec{b}$ $(0 \leq \theta \leq \pi)$.
- $\hat{u}$ is the unit vector **perpendicular** to both $\vec{a}$ and $\vec{b}$.
  The direction of $\hat{u}$ is determined by the ***right-hand rule***.
  The right-hand rule is a convention like driving on the right-hand side of the road
  in North America. Until 1965, the Soviet Union used the left-hand rule.

  Note: $|\vec{a}|\,|\vec{b}|\,\sin(\theta)$ [the coefficient of $\hat{u}$ in equation (5)] is inherently non-negative
  because $\sin(\theta) \geq 0$ since $0 \leq \theta \leq \pi$. Hence, $|\vec{a} \times \vec{b}| = |\vec{a}|\,|\vec{b}|\,\sin(\theta)$.

$$\vec{a} \times \vec{b} \triangleq |\vec{a}|\,|\vec{b}|\,\sin(\theta)\,\hat{u} \qquad (5)$$

Properties of the cross-product (×)

Cross product with a zero vector $\qquad \vec{a} \times \vec{0} = \vec{0}$

Cross product of a vector with itself $\qquad \vec{a} \times \vec{a} = \vec{0}$

Cross product of ***parallel*** vectors $\qquad \vec{a} \times \vec{b} = \vec{0} \qquad$ if $\vec{a} \parallel \vec{b}$

Cross product of scaled vectors $\qquad s_1\,\vec{a} \times s_2\,\vec{b} = s_1\,s_2\,(\vec{a} \times \vec{b})$

Distributive law $\qquad \vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c} \qquad (\vec{a} + \vec{b}) \times \vec{c} = \vec{a} \times \vec{c} + \vec{b} \times \vec{c}$

Cross products are ***not*** associative $\qquad \vec{a} \times (\vec{b} \times \vec{c}) \neq (\vec{a} \times \vec{b}) \times \vec{c}$

$\qquad$ Cross products are ***not*** commutative $\qquad \boxed{\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}} \qquad\qquad (6)$

$\qquad$ ***Vector triple cross product*** $\qquad \boxed{\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}\,(\vec{a} \cdot \vec{c}) - \vec{c}\,(\vec{a} \cdot \vec{b})} \qquad (7)$

A mnemonic for eqn (7) $\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}\,(\vec{a} \cdot \vec{c}) - \vec{c}\,(\vec{a} \cdot \vec{b})$ is "**bac**k **cab**" - as in were you born in the <u>back</u> of a <u>cab</u>?
Many proofs of this formula resolve $\vec{a}$, $\vec{b}$, and $\vec{c}$ into orthogonal unit vectors (e.g., $\hat{n}_x$, $\hat{n}_y$, $\hat{n}_z$) and equate components.

2.10.1 Uses for the cross-product (×) in geometry, statics, motion analysis, ...

- ***Perpendicular*** vectors, e.g., $\vec{a} \times \vec{b}$ is perpendicular to both $\vec{a}$ and $\vec{b}$
- ***Moment*** of a force or linear momentum, e.g., $\vec{r} \times \vec{F} \qquad$ and $\qquad \vec{r} \times m\,\vec{v}$
- ***Velocity/acceleration*** formulas, e.g., $\vec{v} = \vec{\omega} \times \vec{r} \qquad$ and $\qquad \vec{a} = \vec{\alpha} \times \vec{r} + \vec{\omega} \times (\vec{\omega} \times \vec{r})$
- ***Area of a triangle*** whose sides have length $|\vec{a}|$ and $|\vec{b}|$
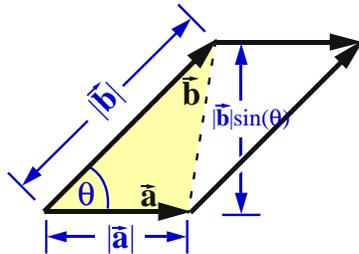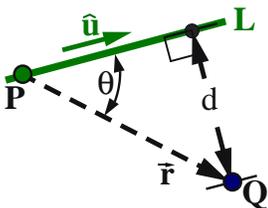
The ***area of a parallelogram*** having sides of length $|\vec{a}|$ and $|\vec{b}|$ is $|\vec{a} \times \vec{b}|$.
The ***area of a triangle*** Δ is half the area of a parallelogram.

$$\boxed{\text{Vector area } \vec{\Delta}(\vec{a}, \vec{b}) = \tfrac{1}{2}\,\vec{a} \times \vec{b} \qquad \text{Scalar area } \Delta(\vec{a}, \vec{b}) = \tfrac{1}{2}\,|\vec{a} \times \vec{b}|} \qquad (8)$$

Hw 2.14 and Section 3.3 show how cross-products calculate area (***surveying***).

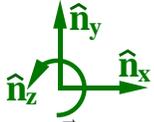- ***Distance*** d between a line L and a point Q can be calculated as shown below.

$$\boxed{\text{d} = \left|\vec{r}^{\,Q/P} \times \hat{u}\right| \underset{(5)}{=} \left|\vec{r}^{\,Q/P}\right| \sin(\theta)} \qquad (9)$$

The line L (shown left) passes through point P and is parallel to the unit vector $\hat{u}$.
Note: See example in Hw 1.26. Other distance calculations are in Sections 3.2 and 3.3.

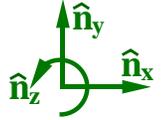## 2.10.2 Determinants and cross-products (with right-handed unit vectors)

When $\widehat{\mathbf{n}}_x$, $\widehat{\mathbf{n}}_y$, $\widehat{\mathbf{n}}_z$ are **orthogonal unit** vectors, it can be shown (Homework 2.13) the cross product of two vectors happens to be equal to the **determinant** of an associated matrix.

$$\left.\begin{array}{l} \vec{\mathbf{a}} = a_x\,\widehat{\mathbf{n}}_x + a_y\,\widehat{\mathbf{n}}_y + a_z\,\widehat{\mathbf{n}}_z \\ \vec{\mathbf{b}} = b_x\,\widehat{\mathbf{n}}_x + b_y\,\widehat{\mathbf{n}}_y + b_z\,\widehat{\mathbf{n}}_z \end{array}\right\} \qquad \vec{\mathbf{a}} \times \vec{\mathbf{b}} = \det\begin{bmatrix} \widehat{\mathbf{n}}_x & \widehat{\mathbf{n}}_y & \widehat{\mathbf{n}}_z \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{bmatrix} = \begin{array}{l} (a_y\,b_z - a_z\,b_y)\,\widehat{\mathbf{n}}_x \\ -\;(a_x\,b_z - a_z\,b_x)\,\widehat{\mathbf{n}}_y \\ +\;(a_x\,b_y - a_y\,b_x)\,\widehat{\mathbf{n}}_z \end{array} \qquad (10)$$

Examples: Vector cross-products (×) with determinants.

The following shows how to use cross-products with the vectors $\vec{\mathbf{v}}$ and $\vec{\mathbf{w}}$, each which is expressed in terms of the orthogonal unit vectors $\widehat{\mathbf{n}}_x$, $\widehat{\mathbf{n}}_y$, $\widehat{\mathbf{n}}_z$ shown to the right.

$$\left.\begin{array}{l} \vec{\mathbf{v}} = 7\,\widehat{\mathbf{n}}_x + 5\,\widehat{\mathbf{n}}_y + 4\,\widehat{\mathbf{n}}_z \\ \vec{\mathbf{w}} = 2\,\widehat{\mathbf{n}}_x + 3\,\widehat{\mathbf{n}}_y + 2\,\widehat{\mathbf{n}}_z \end{array}\right\} \qquad \vec{\mathbf{v}} \times \vec{\mathbf{w}} = \det\begin{bmatrix} \widehat{\mathbf{n}}_x & \widehat{\mathbf{n}}_y & \widehat{\mathbf{n}}_z \\ 7 & 5 & 4 \\ 2 & 3 & 2 \end{bmatrix} = \text{-}2\,\widehat{\mathbf{n}}_x - 6\,\widehat{\mathbf{n}}_y + 11\,\widehat{\mathbf{n}}_z$$

Area from vectors $\vec{\mathbf{v}}$ and $\vec{\mathbf{w}}$: $\quad \Delta(\vec{\mathbf{v}}, \vec{\mathbf{w}}) = \dfrac{1}{2}\,|\vec{\mathbf{v}} \times \vec{\mathbf{w}}| = \dfrac{1}{2}\sqrt{(\text{-}2)^2 + (\text{-}6)^2 + 11^2} = \dfrac{\sqrt{161}}{2} \approx 6.344$

Scalar triple product: $\quad (2\,\widehat{\mathbf{n}}_x + 3\,\widehat{\mathbf{n}}_y + 4\,\widehat{\mathbf{n}}_z) \cdot (\vec{\mathbf{v}} \times \vec{\mathbf{w}}) = \det\begin{bmatrix} 2 & 3 & 4 \\ 7 & 5 & 4 \\ 2 & 3 & 2 \end{bmatrix} = 22$

Chapter 2: Vectors

2.11 Optional: Scalar triple product ($\cdot \times$ or $\times \cdot$)

The ***scalar triple product*** of vectors $\vec{\mathbf{a}}, \vec{\mathbf{b}}, \vec{\mathbf{c}}$ is the scalar defined in the various ways shown below.

$$\text{ScalarTripleProduct} \triangleq \boxed{\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} \times \vec{\mathbf{c}} \;=\; \vec{\mathbf{a}} \times \vec{\mathbf{b}} \cdot \vec{\mathbf{c}}} \;=\; \vec{\mathbf{b}} \cdot \vec{\mathbf{c}} \times \vec{\mathbf{a}} \;=\; \vec{\mathbf{b}} \times \vec{\mathbf{c}} \cdot \vec{\mathbf{a}} \qquad (11)$$

Although parentheses make equation (11) clearer, i.e., $\text{ScalarTripleProduct} \triangleq \vec{\mathbf{a}} \cdot (\vec{\mathbf{b}} \times \vec{\mathbf{c}})$, the parentheses are unnecessary because the cross product $\vec{\mathbf{b}} \times \vec{\mathbf{c}}$ **must** be performed before the dot product for a sensible result to be produced.

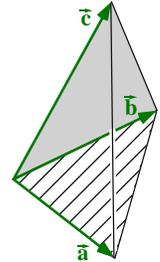## 2.11.1 Scalar triple product and the volume of a tetrahedron

For a tetrahedron whose sides are described by the vectors $\vec{\mathbf{a}}, \vec{\mathbf{b}}, \vec{\mathbf{c}}$ (sides of length $|\vec{\mathbf{a}}|, |\vec{\mathbf{b}}|, |\vec{\mathbf{c}}|$), a geometrical interpretation of $\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} \times \vec{\mathbf{c}}$ is the ***volume of the parallelepiped***.

This formula helps calculate mass and volume (e.g., highway cut/fill calculations).

$$\text{Tetrahedron Volume} = \tfrac{1}{6} \, \vec{\mathbf{a}} \cdot \vec{\mathbf{b}} \times \vec{\mathbf{c}}$$

2.11.2 ($\times \cdot$) to change vector equations to scalar equations (see Hw 1.31)

Section 2.9.5 showed one method to form scalar equations from the vector equation $\vec{\mathbf{v}} = \vec{\mathbf{0}}$. A 2nd method expresses $\vec{\mathbf{v}}$ in terms of three non-coplanar (but not necessarily orthogonal or unit) vectors $\vec{\mathbf{a}}_1, \vec{\mathbf{a}}_2, \vec{\mathbf{a}}_3$, and writes the equally valid (but generally different) set of linearly independent scalar equations shown below.

Method 2: if $\vec{\mathbf{v}} = v_1 \, \vec{\mathbf{a}}_1 + v_2 \, \vec{\mathbf{a}}_2 + v_3 \, \vec{\mathbf{a}}_3 = \vec{\mathbf{0}} \;\Rightarrow\; \boxed{v_1 = 0 \qquad v_2 = 0 \qquad v_3 = 0}$

Note: The proof that $v_i = 0 \;\; (i = 1, 2, 3)$ follows directly by substituting $\vec{\mathbf{v}} = \vec{\mathbf{0}}$ into equation (4.2).
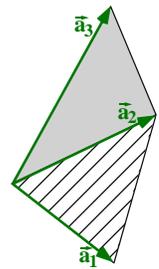
2.12 Optional: Vectors vs. column matrices in the context of $\vec{\mathbf{F}} = m\,\vec{\mathbf{a}}$

In the context of matrices, the word "vector" describes a column matrix – which does **not** have direction. In the context of $\vec{\mathbf{F}} = m\,\vec{\mathbf{a}}$, the word "vector" means something different. To associate direction to a column matrix, attach a basis as shown below.

$$\widehat{\mathbf{a}}_{\text{x}} + 2\,\widehat{\mathbf{a}}_{\text{y}} + 3\,\widehat{\mathbf{a}}_{\text{z}} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} \widehat{\mathbf{a}}_{\text{x}} \\ \widehat{\mathbf{a}}_{\text{y}} \\ \widehat{\mathbf{a}}_{\text{z}} \end{bmatrix} = \begin{bmatrix} \widehat{\mathbf{a}}_{\text{x}} & \widehat{\mathbf{a}}_{\text{y}} & \widehat{\mathbf{a}}_{\text{z}} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}_{\widehat{\mathbf{a}}_{\text{xyz}}}$$

where $\widehat{\mathbf{a}}_{\text{x}}, \widehat{\mathbf{a}}_{\text{y}}, \widehat{\mathbf{a}}_{\text{z}}$ are orthogonal unit vectors

Note: Although a vector can be represented with a column matrix and orthogonal unit vectors, it is not always desirable or efficient. Postponing resolution of vectors into orthogonal components allows maximum use of simplifying vector properties and avoids simplifications such as $\sin^2(\theta) + \cos^2(\theta) = 1$ (see Homework 2.9). Words such as "vector" require context. Some English words are contranyms (opposite meanings) such as "fast" and "bolt" (move quickly or fasten).

Courtesy Accuray Inc.. Vectors are widely useful, e.g., with medical robotics or cut/fill calculations for highway construction.