

**Homework 1. Chapters 2.**  
**Basis independent vectors and their properties**

Show work – except for ♣ fill-in-blanks-problems (print .pdf from [www.MotionGenesis.com](http://www.MotionGenesis.com) ⇒ [Textbooks](#) ⇒ [Resources](#)).

1.1 ♣ **Solving problems – what engineers do.**

Understanding dynamics results from doing problems. Many problems are guided to help you synthesize processes (imitation). You are encouraged to work by yourself or with colleagues/instructors and use the textbook’s reference theory and other resources.

**Confucius 500 B.C.**

“I hear and I forget.  
I see and I remember.  
I        and I understand.”

“By three methods we may learn wisdom:  
First, by reflection, which is noblest;  
Second, by imitation, which is easiest;  
Third by experience, which is the bitterest.”



1.2 ♣ **What is a vector?** (Section 2.2)

Two properties (attributes) of a vector are                                    and                                   .

1.3 ♣ **What is a zero vector?** (Section 2.3)

A zero vector  $\vec{0}$  has a magnitude of  $0/1/2/\infty$ .

A zero vector  $\vec{0}$  has no direction. **True/False.**

1.4 ♣ **Unit vectors.** (Section 2.4)

A unit vector has a magnitude of  $0/1/2/\infty$ .

All unit vectors are equal. **True/False.**

1.5 ♣ **Draw the following vectors:** (Section 2.2)

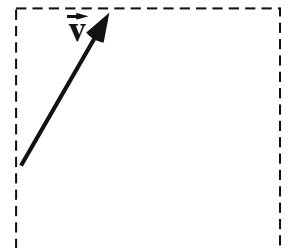
- Long, horizontally-right vector  $\vec{a}$
- Short, vertically-upward vector  $\vec{b}$
- Outwardly-directed unit vector  $\vec{c}$



1.6 ♣ **Vector magnitude and direction (orientation and sense).** (Section 2.2)

The figure to the right shows a vector  $\vec{v}$ . Draw the following vectors.

- $\vec{a}$ : Same magnitude and same direction as  $\vec{v}$  ( $\vec{a} = \vec{v}$ ).
- $\vec{b}$ : Same magnitude and orientation as  $\vec{v}$ , but different sense.
- $\vec{c}$ : Same direction as  $\vec{v}$ , but different magnitude.
- $\vec{d}$ : Same magnitude as  $\vec{v}$ , but different direction (orientation).
- $\vec{e}$ : Different magnitude and different direction (orientation) as  $\vec{v}$ .



1.7 ♣ **Magnitude of a vector.** (Section 2.2)

Consider a real number  $x$  and a horizontally-right pointing unit vector  $\hat{i}$ .

The *magnitude* of the vector  $-x\hat{i}$  is (circle one): positive negative non-negative non-positive.

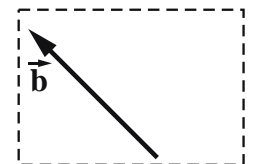
1.8 ♣ **Negating a vector.** (Section 2.8)

Complete the figure to the right by drawing the vector  $-\vec{b}$ .

Negating the vector  $\vec{b}$  results in a vector with different (circle all that apply):

magnitude                  direction                  orientation                  sense

Historical note: Negative numbers (e.g.,  $-3$ ) were not widely accepted until 1800 A.D.



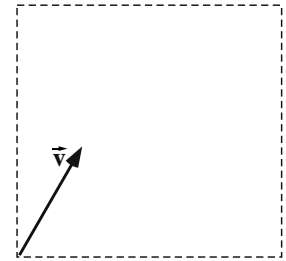
1.9 ♣ **Multiplying a vector by a scalar.** (Section 2.7)

Complete the figure to the right by drawing the vectors  $2\vec{v}$  and  $-2\vec{v}$ .

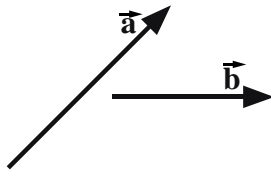
The following statements involve a vector  $\vec{v}$  and a real non-zero scalar  $s$  ( $s \neq 0$ ).

If a statement is true, provide a numerical value for  $s$  that supports your answer

- $s\vec{v}$  can have a different *magnitude* than  $\vec{v}$ . True/False  $s =$
- $s\vec{v}$  can have a different *direction* than  $\vec{v}$ . True/False  $s =$
- $s\vec{v}$  can have a different *sense* than  $\vec{v}$ . True/False  $s =$
- $s\vec{v}$  can have a different *orientation* than  $\vec{v}$ . True/False  $s =$



1.10 ♣ **Graphical vector addition/subtraction - draw.** (Sections 2.6,2.8)



Draw  $\vec{a} + \vec{b}$



Draw  $\vec{a} - \vec{b}$



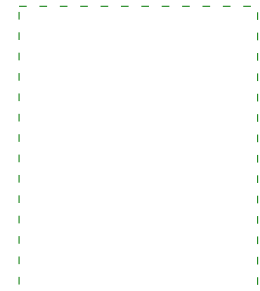
1.11 ♣ **Visual representation of a vector dot-product.** (Section 2.9)

Write the *definition* of the dot-product of a vector  $\vec{a}$  with a vector  $\vec{b}$ .

Include a *sketch* with *each symbol* in the right-hand-side of your definition clearly labeled. The sketch should include  $\vec{a}$ ,  $\vec{b}$ ,  $|\vec{a}|$ ,  $|\vec{b}|$ , ...

**Result:**

$$\vec{a} \cdot \vec{b} \triangleq \text{$$



1.12 ♣ **Visual representation of a vector cross-product.** (Section 2.10)

Write the *definition* of the cross-product of a vector  $\vec{a}$  with a vector  $\vec{b}$ .

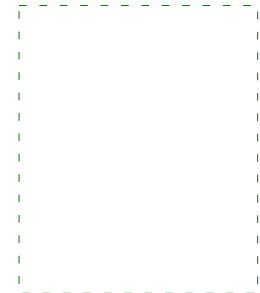
Include a *sketch* with *each symbol* in your definition labeled and described.

**Result:**

$$\vec{a} \times \vec{b} \triangleq \text{$$

where  $\hat{u}$  is

and  $\theta$  is



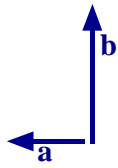
1.13 ♣ **Properties of vector dot-products and cross-products.** (Sections 2.9.1 and 2.10.1)

When $\vec{a}$ is parallel to $\vec{b}$ :	$\vec{a} \cdot \vec{b} = 0$	True/False	$\vec{a} \times \vec{b} = \vec{0}$	True/False
When $\vec{a}$ is perpendicular to $\vec{b}$ :	$\vec{a} \cdot \vec{b} = 0$	True/False	$\vec{a} \times \vec{b} = \vec{0}$	True/False
For arbitrary vectors $\vec{a}$ and $\vec{b}$ :	$\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$	True/False	$\vec{a} \times \vec{b} = \vec{b} \times \vec{a}$	True/False

1.14 ♣ **Calculating vector dot-products and cross-products via definitions.** (Sections 2.9 and 2.10)

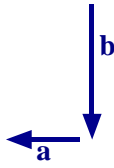
Draw a unit vector  $\hat{\mathbf{k}}$  outward-normal to the plane of the paper.

Knowing vector  $\vec{\mathbf{a}}$  has magnitude 2 and vector  $\vec{\mathbf{b}}$  has magnitude 4, calculate the following dot-products and cross-products via their definitions ( $2^+$  significant digits).



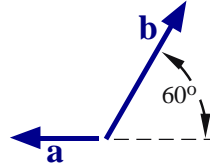
$$\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = \boxed{\phantom{000}}$$

$$\vec{\mathbf{a}} \times \vec{\mathbf{b}} = \boxed{\phantom{000}}$$



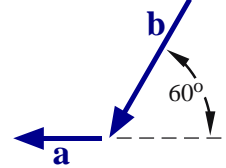
$$\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = \boxed{\phantom{000}}$$

$$\vec{\mathbf{a}} \times \vec{\mathbf{b}} = \boxed{\phantom{000}}$$



$$\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = \boxed{\phantom{000}}$$

$$\vec{\mathbf{a}} \times \vec{\mathbf{b}} = \boxed{\phantom{000}}$$



$$\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = \boxed{\phantom{000}}$$

$$\vec{\mathbf{a}} \times \vec{\mathbf{b}} = \boxed{\phantom{000}}$$

1.15 ♣ **Property of scalar triple product.** (Section 2.11).

For arbitrary non-zero vectors  $\vec{\mathbf{a}}, \vec{\mathbf{b}}, \vec{\mathbf{c}}$ :  $\vec{\mathbf{a}} \cdot (\vec{\mathbf{b}} \times \vec{\mathbf{c}}) = (\vec{\mathbf{a}} \times \vec{\mathbf{b}}) \cdot \vec{\mathbf{c}}$  Never/Sometimes/Always  
 A property of the *scalar triple product* is  $\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} \times \vec{\mathbf{a}} = 0$ . True/False.

1.16 **Property of vector triple cross-product.** (Sections 2.10.1 and 2.11)

Complete the following equation:  $\vec{\mathbf{a}} \times (\vec{\mathbf{b}} \times \vec{\mathbf{c}}) = \vec{\mathbf{b}} (\boxed{\phantom{000}}) - \vec{\mathbf{c}} (\boxed{\phantom{000}})$

Circle true or false (show supporting work):  $\vec{\mathbf{a}} \times (\vec{\mathbf{b}} \times \vec{\mathbf{c}}) = (\vec{\mathbf{a}} \times \vec{\mathbf{b}}) \times \vec{\mathbf{c}} + \vec{\mathbf{b}} \times (\vec{\mathbf{a}} \times \vec{\mathbf{c}})$  True/False

1.17 ♣ **Optional: Proof of magnitude of vector cross product property.** (Sections 2.9 and 2.10)

Letting  $\hat{\boldsymbol{\lambda}}$  be a *unit vector* and  $\vec{\mathbf{v}}$  be *any vector*, prove<sup>1</sup>  $|\vec{\mathbf{v}} \times \hat{\boldsymbol{\lambda}}|^2 = \vec{\mathbf{v}} \cdot \vec{\mathbf{v}} - (\vec{\mathbf{v}} \cdot \hat{\boldsymbol{\lambda}})^2$ .

1.18 ♣ **Vector exponentiation:  $\vec{\mathbf{v}}^2$  and  $\vec{\mathbf{v}}^3$ .** Complete the 3-step proofs. (Section 2.9)

**Step 1:** Complete the **definition** of  $\vec{\mathbf{v}}^2$  in terms of  $|\vec{\mathbf{v}}|$ .

**Step 2:** Use the **definition** of the dot-product to show how  $\vec{\mathbf{v}} \cdot \vec{\mathbf{v}}$  can be expressed in terms of  $|\vec{\mathbf{v}}|$ .

**Step 3:** Combine these two definitions to provide an alternate way to calculate  $\vec{\mathbf{v}}^2$  with a vector dot-product.

**Result:**  $\vec{\mathbf{v}}^2 \triangleq |\vec{\mathbf{v}}|^{\boxed{\phantom{000}}}$   $\vec{\mathbf{v}} \cdot \vec{\mathbf{v}} \stackrel{(2.2)}{=} \boxed{\phantom{000}}$   $\vec{\mathbf{v}}^2 = \boxed{\phantom{000}} \cdot \boxed{\phantom{000}}$

Complete the 3-step proof that relates  $\vec{\mathbf{v}}^3$  to  $\vec{\mathbf{v}} \cdot \vec{\mathbf{v}}$  raised to a real number.

**Result:**  $\vec{\mathbf{v}}^3 \triangleq |\vec{\mathbf{v}}|^{\boxed{\phantom{000}}} \stackrel{(2.4)}{=} (\sqrt{\boxed{\phantom{000}}})^{\boxed{\phantom{000}}} = (\vec{\mathbf{v}} \cdot \vec{\mathbf{v}})^{\boxed{\phantom{000}}}$

1.19 ♣  **$|c\hat{\mathbf{a}}_x|$  Calculate vector magnitude with dot products.** (Section 2.9 and Hw 1.18)

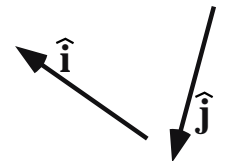
Show how the vector dot-product can be used to show that the magnitude of the vector  $c\hat{\mathbf{a}}_x$  ( $c$  is a positive or **negative** number and  $\hat{\mathbf{a}}_x$  is a unit vector) can be written solely in terms of  $c$  (without  $\hat{\mathbf{a}}_x$ ).

**Result:**  $|c\hat{\mathbf{a}}_x| = \sqrt{\boxed{\phantom{000}} \cdot \boxed{\phantom{000}}} = \sqrt{c^2 * \boxed{\phantom{000}} \cdot \boxed{\phantom{000}}} = \sqrt{\boxed{\phantom{000}}} = \text{abs}(c)$

1.20 † **Magnitude of the vector  $\vec{\mathbf{v}}$ .** Show work. (Section 2.9)

Knowing the angle between a unit vector  $\hat{\mathbf{i}}$  and unit vector  $\hat{\mathbf{j}}$  is  $110^\circ$ , calculate a numerical value for the magnitude of  $\vec{\mathbf{v}} = 3\hat{\mathbf{i}} + 4\hat{\mathbf{j}}$ .

**Result:**  $|\vec{\mathbf{v}}| \approx \boxed{4.098}$  Note: The answer is not 5.

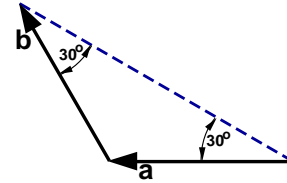


<sup>1</sup>One way to prove this is to write  $(\vec{\mathbf{v}} \times \hat{\boldsymbol{\lambda}})^2 = (\vec{\mathbf{v}} \times \hat{\boldsymbol{\lambda}}) \cdot (\vec{\mathbf{v}} \times \hat{\boldsymbol{\lambda}}) \stackrel{(2.10)}{=} \vec{\mathbf{v}} \cdot [\hat{\boldsymbol{\lambda}} \times (\vec{\mathbf{v}} \times \hat{\boldsymbol{\lambda}})]$  and then use the vector triple cross-product property  $\vec{\mathbf{a}} \times (\vec{\mathbf{b}} \times \vec{\mathbf{c}}) = \vec{\mathbf{b}}(\vec{\mathbf{a}} \cdot \vec{\mathbf{c}}) - \vec{\mathbf{c}}(\vec{\mathbf{a}} \cdot \vec{\mathbf{b}})$  from Section 2.10. Alternately, it is helpful to write  $\vec{\mathbf{v}} = \vec{\mathbf{v}}_\perp \hat{\boldsymbol{\lambda}}_\perp + \vec{\mathbf{v}}_\parallel \hat{\boldsymbol{\lambda}}$  where  $\vec{\mathbf{v}}_\perp \hat{\boldsymbol{\lambda}}_\perp$  is the component of  $\vec{\mathbf{v}}$  that is perpendicular to  $\hat{\boldsymbol{\lambda}}$  and  $\vec{\mathbf{v}}_\parallel \hat{\boldsymbol{\lambda}}$  is the component of  $\vec{\mathbf{v}}$  that is parallel to  $\hat{\boldsymbol{\lambda}}$ .

1.21 ♣ **Angle between vectors.** (Section 2.9)

Referring to the figure to the right, find the numerical value for the angle between vector  $\vec{a}$  and vector  $\vec{b}$ .

$$\angle(\vec{a}, \vec{b}) = \text{[ ]}^\circ$$



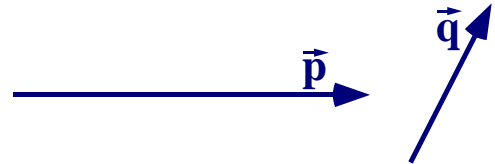
1.22 **Visual estimation of vector dot/cross-products.** Show work. (Sections 2.9 and 2.10)

Estimate (e.g., using your pinky) the magnitude of the vector  $\vec{p}$  shown below. Note: 1 inch  $\triangleq$  2.54 cm.

Estimate the angle between  $\vec{p}$  and  $\vec{q}$ ,  $\vec{p} \cdot \vec{q}$ , and the magnitude of  $\vec{p} \times \vec{q}$ . Show work.

**Result:** (Provide numerical results with 1 or more significant digits).

$ \vec{p}  \approx \text{[ ]} \text{ cm}$	$\angle(\vec{p}, \vec{q}) \approx \text{[ ]}^\circ$
$\vec{p} \cdot \vec{q} \approx \text{[ ]} \text{ cm}^2$	$ \vec{p} \times \vec{q}  \approx \text{[ ]} \text{ cm}^2$



1.23 ♣ **Form the unit vector  $\hat{u}$  having the same direction as  $c\hat{a}_x$ .** (Section 2.4)

**Result:**  $\hat{u} = \text{[ ]} \hat{a}_x$  Note:  $\hat{a}_x$  is a unit vector and  $c$  is a non-zero real number, e.g., 3 or -3

1.24 ♣ **Coefficient of  $\hat{u}$  in cross products – definitions and trig functions.** (Section 2.10)

The **cross product** of vectors  $\vec{a}$  and  $\vec{b}$  can be written in terms of a real scalar  $s$  as  $\vec{a} \times \vec{b} = s\hat{u}$  where  $\hat{u}$  is a unit vector perpendicular to both  $\vec{a}$  and  $\vec{b}$  in a direction defined by the **right-hand rule**. The coefficient  $s$  of the unit vector  $\hat{u}$  is inherently non-negative. **True/False.**

1.25 ♣ **Orthogonal vectors: Insights via drawing.** (Section 2.10)

Consider three unit vectors  $\hat{a}$ ,  $\hat{b}$ , and  $\hat{c}$ .  
 Vector  $\hat{a}$  is perpendicular to vector  $\hat{b}$ .  
 Vector  $\hat{b}$  is perpendicular to vector  $\hat{c}$ .  
 Vector  $\hat{a}$  is not parallel to vector  $\hat{c}$ .

In **all** cases,  $\hat{a}$  is perpendicular to  $\hat{c}$ . **True/False.**

Explain your answer by drawing  $\hat{a}$ ,  $\hat{b}$ ,  $\hat{c}$  and relevant angles.



1.26 **Calculating distance between a point and a line via cross-products.** (Section 2.10.2)

Draw a horizontally-right unit vector  $\hat{a}_x$  and vertically-upward unit vector  $\hat{a}_y$ .

Draw a point  $Q$  whose position vector from a point  $P$  is  $\vec{r}^{Q/P} = 5\hat{a}_x$ .

Draw a line  $L$  that passes through point  $P$  and is parallel to  $\hat{u} = \frac{3}{5}\hat{a}_x + \frac{4}{5}\hat{a}_y$ .

Calculate the **distance**  $d$  between  $Q$  and  $L$  using both formulas in equation (2.9).

**Result:**  $d \stackrel{(2.9)}{=} \text{[ ]} = \text{[4]}$        $d \stackrel{(2.9)}{=} \text{[ ]} = \text{[4]}$



1.27 ♣ **Ranges of angles from dot-product and cross-product calculations.** (Sections 2.9 and 2.10)

**Given:** Unit vectors  $\hat{a}$  and  $\hat{b}$  and the numerical values of  $c \triangleq \hat{a} \cdot \hat{b}$  and  $s \triangleq |\hat{a} \times \hat{b}|$ .

Complete the following table with an appropriate range of numerical values.

Quantity	Range of values
$c$	$\text{[ ]} \leq c \leq \text{[ ]}$
$s$	$\text{[ ]} \leq s \leq \text{[ ]}$
Angle $\theta_c$ between $\hat{a}$ and $\hat{b}$ that can be uniquely determined <b>solely</b> from $c$	$\text{[ ]}^\circ \leq \theta_c \leq \text{[ ]}^\circ$
Angle $\theta_s$ between $\hat{a}$ and $\hat{b}$ that can be uniquely determined <b>solely</b> from $s$	$\text{[ ]}^\circ \leq \theta_s \leq \text{[ ]}^\circ$
Angle $\theta$ between $\hat{a}$ and $\hat{b}$	$\text{[ ]}^\circ \leq \theta \leq \text{[ ]}^\circ$

Note: The range of  $\theta_s$  differs from the (correct) range for  $\theta$ . Hence,  $s$  and  $\theta_s$  are insufficient to calculate  $\theta$ .

1.28 ♣ **Vector operations and units.** (Chapter 2)

Circle the vector operations below (scalar multiplication, addition, dot-product, etc.) that are **defined** for a position vector  $\vec{a}$  (with **units** of m) and a velocity vector  $\vec{b}$  (with **units** of  $\frac{m}{s}$ ).

$-\vec{a}$        $5\vec{a}$        $\vec{a}/5$        $\vec{a} + \vec{b}$        $\vec{a} \cdot \vec{b}$        $\vec{a} \times \vec{b}$

1.29 ♣ **Using vector identities to simplify expressions** (refer to Homework 1.13).

One reason to treat vectors as **basis-independent** quantities is to simplify vector expressions **without** resolving the vectors into orthogonal “ $\vec{x}, \vec{y}, \vec{z}$ ” or “ $\vec{i}, \vec{j}, \vec{k}$ ” components. Simplify the following vector expressions using various properties of dot-products and cross-products.

Express your results in terms of dot-products  $\cdot$  and cross-products  $\times$  of the arbitrary vectors  $\vec{u}, \vec{v}, \vec{w}$  (i.e.,  $\vec{u}, \vec{v}, \vec{w}$  are not orthogonal).

Vector expression	Simplified vector expression
$(3\vec{u} - 2\vec{v}) \times (\vec{u} + \vec{v})$	$\square \vec{u} \times \vec{v}$
$(3\vec{u} - 2\vec{v}) \cdot (\vec{u} + \vec{v})$	$\square \vec{u}^2 - \square \vec{v}^2 + \square \vec{u} \cdot \vec{v}$
$(\vec{u} - \vec{v}) \cdot (\vec{u} + \vec{v})$	$\square - \square$
$(3\vec{u} - 2\vec{v}) \times (\vec{u} + \vec{v}) \cdot (2\vec{u} - 7\vec{v})$	$\square$
$(\vec{u} + \vec{v}) \times (\vec{v} + 2\vec{w}) \cdot (\vec{w} + 2\vec{u})$	$\square \vec{u} \times \vec{v} \cdot \vec{w}$

1.30 ♣ **Vector concepts: Solving a vector equation.** (Section 2.9.5)

Consider the vector equation to the right and the process that follows that solves for  $\dot{\theta}$  ( $\hat{a}_x$  is a unit vector and  $v_x, \dot{\theta}, R$  are scalars).

$$v_x \hat{a}_x = \dot{\theta} R \hat{a}_x$$

$$\dot{\theta} = \frac{v_x \hat{a}_x}{R \hat{a}_x} = \frac{v_x}{R}$$

This process is a valid way to solve for  $\dot{\theta}$ . **True/False.**

Explain:

1.31 **Changing a vector equation to scalar equations.** Show work. (Section 2.9.5)

Draw three mutually orthogonal unit vectors  $\hat{p}, \hat{q}, \hat{r}$ .

- (a) Use a vector operation (e.g., +, −, \*, ·, ×) to transform the following **vector** equation into **one scalar** equation and subsequently solve the scalar equation.

$$(2x - 4) \hat{p} = \vec{0} \quad \stackrel{??}{\Rightarrow} \quad x = 2$$

- (b) Show **every** vector operation (e.g., +, −, \*, ·, or ×) that transforms the following **vector** equation into **three scalar** equations and subsequently solve the scalar equations for  $x, y, z$ .

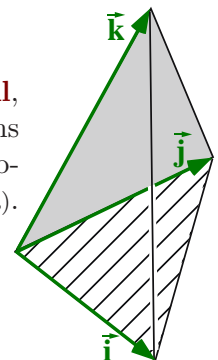
$$(2x - 4) \hat{p} + (3y - 9) \hat{q} + (4z - 16) \hat{r} = \vec{0}$$

**Result:**  $x = 2$        $y = 3$        $z = \square$

- (c) †**Optional:** The figure to the right shows three **non-zero, non-orthogonal, non-coplanar** vectors  $\vec{i}, \vec{j}, \vec{k}$ . Show **every** vector operation that transforms the following **vector** equation into **three uncoupled scalar** equations and subsequently solve those scalar equations for  $x, y, z$  (think  $\times \cdot$ , not matrix algebra).

$$(2x - 4) \vec{i} + (3y - 9) \vec{j} + (4z - 16) \vec{k} = \vec{0}$$

**Result:**  $x = 2$        $y = 3$        $z = \square$



1.32 ♣ **Number of independent scalar equations from one vector equation.** (Section 2.9.5)

Consider the **vector** equation shown to the right that can be useful for static analyses of any system  $S$ .

$$\vec{F}^S = \vec{0}$$

Complete the blanks in the table to the right with **all** integers that could be equal to the number of **independent scalar** equations produced by the previous vector equation for any system  $S$ .

System type	Integer(s)
1D (line)	0, <input type="text"/>
2D (planar)	0, <input type="text"/>
3D (spatial)	0, <input type="text"/>

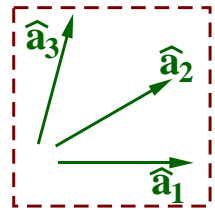
Hint: See Homework 1.31 for ideas.

Note: Regard 1D/linear as meaning  $\vec{F}^S$  can be expressed in terms of a single unit vector  $\hat{i}$  whereas 2D/planar means  $\vec{F}^S$  can be expressed in terms of two non-parallel unit vectors  $\hat{i}$  and  $\hat{j}$ , and 3D/spatial means  $\vec{F}^S$  can be expressed in terms of three non-coplanar unit vectors  $\hat{i}$ ,  $\hat{j}$ ,  $\hat{k}$ .

1.33 ♣ **Vector concepts: Solving a vector equation (just circle true or false and fill-in the blank).**

Consider the following vector equation written in terms of the scalars  $x, y, z$  and three unique non-orthogonal **coplanar** unit vectors  $\hat{a}_1, \hat{a}_2, \hat{a}_3$ .

$$(2x - 4)\hat{a}_1 + (3y - 9)\hat{a}_2 + (4z - 16)\hat{a}_3 = \vec{0}$$



The **unique** solution to this vector equation is  $x = 2, y = 3, z = 4$ . **True/False.**

**Explain:**  $\hat{a}_2$  can be expressed in terms of  $\hat{a}_1$  and  $\hat{a}_3$  (i.e.,  $\hat{a}_2$  is a linear combination of  $\hat{a}_1$  and  $\hat{a}_3$ ). Hence the vector equation produces  linearly independent scalar equations.

1.34 ♣ **A vector revolution in geometry.** (Chapter 2)

The relatively new invention of vectors (Gibbs  $\approx$  1900 AD) has revolutionized Euclidean geometry (Euclid  $\approx$  300 BC). For each geometrical quantity below, circle the vector operation(s) (either the dot-product, cross-product, or both) that is **most** useful for their calculation.

Length:    •    ×	Angle:    •    ×
Area:    •    ×	Volume: <input checked="" type="checkbox"/> <input checked="" type="checkbox"/>

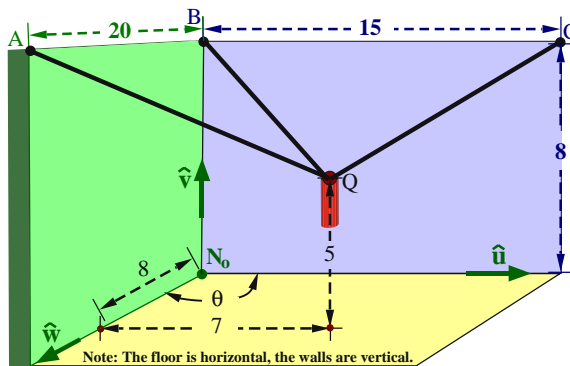
1.35 † **Microphone cable lengths (non-orthogonal walls) “It’s just geometry”.** **Show work.**

A microphone  $Q$  is attached to three pegs  $A, B, C$  by three cables. Knowing the peg locations, microphone location, and the angle  $\theta$  between the vertical walls, express  $L_A, L_B, L_C$  solely in terms of numbers and  $\theta$ . Next, complete the table by calculating  $L_B$  when  $\theta = 120^\circ$ .

Hint: To do this **efficiently**, use only unit vectors  $\hat{u}, \hat{v}, \hat{w}$ , and do **not** introduce an **orthogonal** set of unit vectors.

Hint: Use the distributive property of the vector dot-product as shown in Section 2.9.1 and Homework 2.4.

Note: Synthesis problems are difficult. Engineers solve problems. Think, talk, draw, sleep, walk, get help, ...



Distance between A and B	20 m
Distance between B and C	15 m
Distance between $N_o$ and B	8 m
Distance along back wall (see picture)	7 m
$Q$ 's height above $N_o$	5 m
Distance along side wall (see picture)	8 m
$L_A$ : Length of cable joining A and Q	<input type="text"/> m
$L_B$ : Length of cable joining B and Q	<input type="text"/> m
$L_C$ : Length of cable joining C and Q	<input type="text"/> m

$$\vec{r}^{Q/N_o} = 7\hat{u} + 5\hat{v} + 8\hat{w}$$

**Result:**

$$L_A = \sqrt{202 - 168 \cos(\theta)}$$

$$L_B = \text{$$

$$L_C = \sqrt{137 - \text{$$

Vector addition, dot products, and cross products

Show work – except for ♣ fill-in-blanks-problems (print .pdf from [www.MotionGenesis.com](http://www.MotionGenesis.com) ⇒ [Textbooks](#) ⇒ [Resources](#)).

2.1 ♣ Right-handed, orthogonal, unitary, vector basis. (Section 4.1)

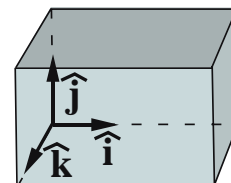
**Draw** a right-handed orthogonal (mutually perpendicular) vector basis consisting of the unit vectors  $\hat{a}_x, \hat{a}_y, \hat{a}_z$ .



2.2 ♣ Adding and subtracting vectors with bases. (Sections 2.6 and 2.8)

Shown right are right-handed orthogonal unit vectors  $\hat{i}, \hat{j}, \hat{k}$  and vectors  $\vec{u}, \vec{v}$ , and  $\vec{w}$ . Form the vector sums and differences below.

$$\begin{aligned}\vec{u} &= 2\hat{i} + 3\hat{j} + 4\hat{k} \\ \vec{v} &= x\hat{i} + y\hat{j} + z\hat{k} \\ \vec{w} &= 5\hat{i} - 6\hat{j} + 7\hat{k}\end{aligned}$$



**Result:**

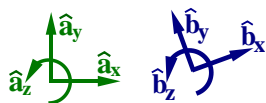
$$\vec{u} + \vec{v} = (2+x)\hat{i} + \boxed{\phantom{000}}\hat{j} + \boxed{\phantom{000}}\hat{k} \quad \vec{u} - \vec{v} = (2-x)\hat{i} + \boxed{\phantom{000}}\hat{j} + \boxed{\phantom{000}}\hat{k}$$

2.3 ♣ Column matrices and vectors (Hint: What is a vector – Hw 1.2). (Section 2.1)

The vector  $\hat{a}_x + 2\hat{a}_y + 3\hat{a}_z$  is equal to the column matrix  $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ . **True/False.**  
 Note:  $\hat{a}_x, \hat{a}_y, \hat{a}_z$  are orthogonal unit vectors as drawn in Hw 2.1.

Adding the following vectors and column matrices produce equivalent results. **True/False.**

Note:  $\hat{a}_x, \hat{a}_y, \hat{a}_z$  and  $\hat{b}_x, \hat{b}_y, \hat{b}_z$  are sets of orthogonal unit vectors.



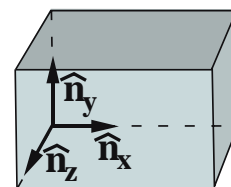
$$\hat{a}_x + 2\hat{a}_y + 3\hat{a}_z + 4\hat{b}_x + 5\hat{b}_y + 6\hat{b}_z = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \\ 9 \end{bmatrix}$$

**Explain:**

2.4 ♣ Calculating vector dot products with bases. (Sections 2.9 and 2.9.3)

**Given:** Right-handed orthogonal unit vectors  $\hat{n}_x, \hat{n}_y, \hat{n}_z$  and:

$$\begin{aligned}\vec{u} &= 2\hat{n}_x + 3\hat{n}_y + 4\hat{n}_z \\ \vec{v} &= x\hat{n}_x + y\hat{n}_y + z\hat{n}_z \\ \vec{w} &= 5\hat{n}_x - 6\hat{n}_y + 7\hat{n}_z\end{aligned}$$



(a) Use the distributive law for dot products to write  $\vec{u} \cdot \vec{v}$  in terms of  $\hat{n}_x \cdot \hat{n}_x, \hat{n}_x \cdot \hat{n}_y$ , etc.

**Result:**

$$\begin{aligned}\vec{u} \cdot \vec{v} &= 2x\hat{n}_x \cdot \hat{n}_x + 2y\hat{n}_x \cdot \hat{n}_y + 2z\hat{n}_x \cdot \hat{n}_z \\ &+ 3x\hat{n}_y \cdot \hat{n}_x + 3y\boxed{\phantom{00}} \cdot \boxed{\phantom{00}} + \boxed{\phantom{000}} \\ &+ \boxed{\phantom{000}} + \boxed{\phantom{000}} + \boxed{\phantom{000}}\end{aligned}$$

(b) Use the definition of the dot product in equation (2.2) to calculate  $\hat{n}_x \cdot \hat{n}_x, \hat{n}_x \cdot \hat{n}_y$ , etc.

**Result:**

$$\begin{aligned}\hat{n}_x \cdot \hat{n}_x &= \boxed{\phantom{00}} & \hat{n}_x \cdot \hat{n}_y &= \boxed{\phantom{00}} & \hat{n}_x \cdot \hat{n}_z &= \boxed{\phantom{00}} \\ \hat{n}_y \cdot \hat{n}_x &= \boxed{\phantom{00}} & \hat{n}_y \cdot \hat{n}_y &= \boxed{\phantom{00}} & \hat{n}_y \cdot \hat{n}_z &= 0 \\ \hat{n}_z \cdot \hat{n}_x &= \boxed{\phantom{00}} & \hat{n}_z \cdot \hat{n}_y &= \boxed{\phantom{00}} & \hat{n}_z \cdot \hat{n}_z &= 1\end{aligned}$$

(c) In view of your previous two results, calculate  $\vec{u} \cdot \vec{v}$ .

**Result:**

$$\vec{u} \cdot \vec{v} = \boxed{\phantom{000}}$$

- (d) As shown below for the arbitrary vectors  $\vec{a}$  and  $\vec{b}$  (also shown in Section 2.9.3), the dot product  $\vec{a} \cdot \vec{b}$  is relatively easy to calculate when  $\hat{n}_x, \hat{n}_y, \hat{n}_z$  are **orthogonal unit** vectors.

$$\left. \begin{aligned} \vec{a} &= a_x \hat{n}_x + a_y \hat{n}_y + a_z \hat{n}_z \\ \vec{b} &= b_x \hat{n}_x + b_y \hat{n}_y + b_z \hat{n}_z \end{aligned} \right\} \Rightarrow \vec{a} \cdot \vec{b} = a_x b_x + a_y b_y + a_z b_z$$

Use this **orthogonal-short-cut** to calculate:

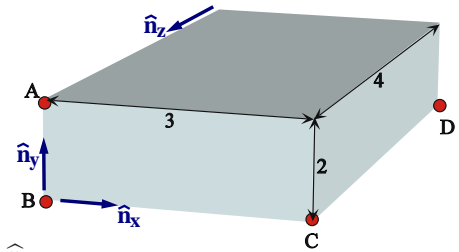
$$\vec{u} \cdot \vec{v} = 2x + 3y + \boxed{\phantom{00}}z \quad \vec{u} \cdot \vec{w} = \boxed{\phantom{00}} \quad \vec{v} \cdot \vec{w} = \boxed{\phantom{00}}$$

**2.5 ♣ Perpendicular vectors.** (Note:  $\hat{i}, \hat{j}, \hat{k}$  are orthogonal unit vectors). (Section 2.9)

When vector  $\vec{v} = x\hat{i} + 2\hat{j} + 3\hat{k}$  is perpendicular to vector  $\vec{w} = 4\hat{i} + 5\hat{j} + 6\hat{k}$ ,  $x = \boxed{\phantom{00}}$ .

**2.6 Dot product for calculating angles.** (Sections 2.9 and 3.3)

The figure to the right shows a rectangular parallelepiped (block) of sides 2, 3, and 4 with points  $A, B, C$  located at corners.  $\hat{n}_x, \hat{n}_y, \hat{n}_z$  are right-handed orthogonal unit vectors with  $\hat{n}_x$  directed from  $B$  to  $C$  and  $\hat{n}_y$  from  $B$  to  $A$ .



- (a) Express  $\vec{r}^{C/A}$  ( $C$ 's position vector from  $A$ ) in terms of  $\hat{n}_x, \hat{n}_y, \hat{n}_z$ .

Find a numerical value for  $|\vec{r}^{C/A}|^2 = \vec{r}^{C/A} \cdot \vec{r}^{C/A}$ .

Next, use equation (2.4) to calculate the magnitude of  $\vec{r}^{C/A}$  (the distance from  $A$  to  $C$ ).

**Result:**  $\vec{r}^{C/A} = \boxed{\phantom{00}}\hat{n}_x - \boxed{\phantom{00}}\hat{n}_y$   $|\vec{r}^{C/A}|^2 = \vec{r}^{C/A} \cdot \vec{r}^{C/A} = \boxed{\phantom{00}}$   $|\vec{r}^{C/A}| = \boxed{\phantom{00}}$

- (b) Using equation (2.1), calculate the unit vector  $\hat{u}$  directed from  $A$  to  $C$  in terms of  $\hat{n}_x, \hat{n}_y, \hat{n}_z$ . Next, find the unit vector  $\hat{v}$  directed from  $A$  to  $D$  in terms of  $\hat{n}_x, \hat{n}_y, \hat{n}_z$ .

**Result:**

$$\hat{u} = \frac{3\hat{n}_x - 2\hat{n}_y}{\sqrt{13}} \quad \hat{v} = \frac{\boxed{\phantom{00}}}{\boxed{\phantom{00}}}$$

- (c) Calculate  $\angle BAC$ , the angle between line  $\overline{AB}$  and line  $\overline{AC}$ .

Next, calculate  $\angle CAD$ , the angle between line  $\overline{AC}$  and line  $\overline{AD}$ .

**Result:**  $\angle BAC = \boxed{\phantom{00}}^\circ$   $\angle CAD = \boxed{47.97^\circ}$

**2.7 ♣ Vector components: Sine and cosine.** (Section 1.4)

Trigonometry plays a central role in **rotation matrices**.

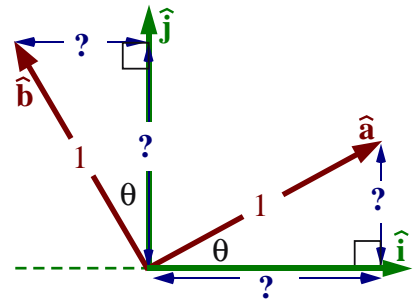
**Replace** each  $?$  in the figure to the right with  $\sin(\theta)$  or  $\cos(\theta)$ .

Express unit vectors  $\hat{a}$  and  $\hat{b}$  in terms of  $\sin(\theta), \cos(\theta), \hat{i}, \hat{j}$ .

**Result:**  $\hat{a} = \boxed{\phantom{00}}\hat{i} + \boxed{\phantom{00}}\hat{j}$

**SohCahToa**

$$\hat{b} = \boxed{\phantom{00}}\hat{i} + \cos(\theta)\hat{j}$$



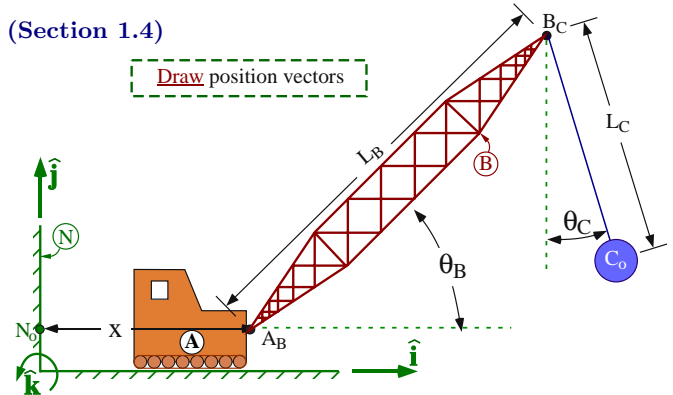


**2.8 ♣ Vector components for a crane-boom.** (Section 1.4)

Shown right is a crane whose cab  $A$  supports a boom  $B$  that swings a wrecking ball  $C_o$ .

$\hat{i}$ ,  $\hat{j}$ ,  $\hat{k}$  are right-handed orthogonal unit vectors with  $\hat{i}$  horizontally-right,  $\hat{j}$  vertically-upward, and  $\hat{k}$  perpendicular to the plane containing points  $N_o$ ,  $A_B$ ,  $B_C$ ,  $C_o$ .

**Draw** each position vector listed below and then use your knowledge of sine/cosine to resolve these vectors into  $\hat{i}$  and  $\hat{j}$  components.

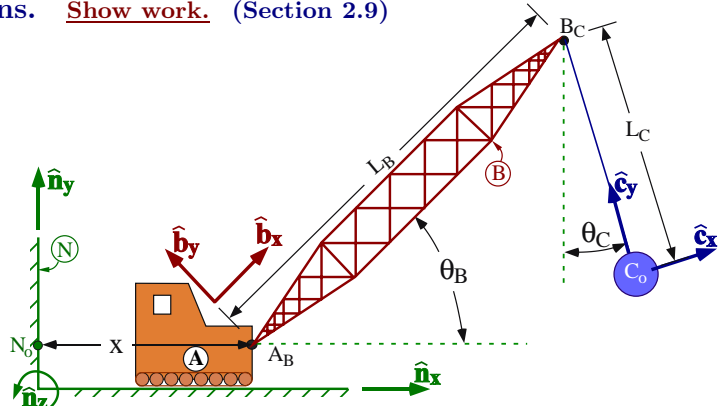


- $A_B$ 's position vector from  $N_o$      $\vec{r}^{A_B/N_o} =$       $\hat{i} +$       $\hat{j}$
- $B_C$ 's position vector from  $A_B$      $\vec{r}^{B_C/A_B} =$       $\hat{i} +$       $\hat{j}$
- $C_o$ 's position vector from  $B_C$      $\vec{r}^{C_o/B_C} =$       $\hat{i} +$       $\hat{j}$
- $B_C$ 's position vector from  $N_o$      $\vec{r}^{B_C/N_o} =$       $\hat{i} +$       $\hat{j}$
- $N_o$ 's position vector from  $C_o$      $\vec{r}^{N_o/C_o} =$    $\hat{i} + [-L_B \sin(\theta_B) + L_C \cos(\theta_C)] \hat{j}$

**2.9 Dot products and distance calculations.** **Show work.** (Section 2.9)

Shown right is a crane whose cab  $A$  supports a boom  $B$  that swings a wrecking ball  $C_o$ . To prevent the wrecking ball from hitting a car at point  $N_o$ , the distance between  $N_o$  and the tip of the boom (point  $B_C$ ) must be controlled.

To start this problem, express  $B_C$ 's position vector from  $N_o$  in terms of  $x$ ,  $L_B$ , and the unit vectors  $\hat{n}_x$  and  $\hat{b}_x$ .



**Result:**  $\vec{r}^{B_C/N_o} =$    $\hat{n}_x +$    $\hat{b}_x$

- (a) **Without** resolving  $\vec{r}^{B_C/N_o}$  into  $\hat{n}_x$  and  $\hat{n}_y$  components (done in the next step), use equation (2.4) and the distributive property to calculate the distance between  $N_o$  and  $B_C$  in terms of  $x$ ,  $L_B$ ,  $\theta_B$ . Then calculate its numerical value when  $x = 20$  m,  $L_B = 10$  m,  $\theta_B = 30^\circ$ .

**Result:** (If necessary, complete the footnote hint below).<sup>1</sup>

Distance between  $N_o$  and  $B_C$ :  $|\vec{r}^{B_C/N_o}| = \sqrt{\text{input} + \text{input} + \text{input}} = 29.1$  m

- (b) Two colleagues are confused by your use of *mixed-bases* vectors (i.e.,  $x\hat{n}_x + L_B\hat{b}_x$ ), and ask you to verify  $B$ 's position vector from  $N_o$  can be expressed in the *uniform-basis* as shown below. Use this uniform-basis expression to verify your previous result for  $|\vec{r}^{B_C/N_o}|$ .

Note: This inefficient uniform-basis approach requires the simplifying trigonometric identity  $\sin^2(\theta_B) + \cos^2(\theta_B) = 1$ .

**Result:**  $\vec{r}^{B_C/N_o} = [x + L_B \cos(\theta_B)] \hat{n}_x + L_B \sin(\theta_B) \hat{n}_y$      $|\vec{r}^{B_C/N_o}|$  simplifies to previous result.

- (c) **Optional:** Calculate the distance between  $N_o$  and  $C_o$  in terms of  $x$ ,  $L_B$ ,  $L_C$ ,  $\theta_B$ , and  $\theta_C$ .

**Result:**  $|\vec{r}^{C_o/N_o}| = \sqrt{x^2 + L_B^2 + L_C^2 + 2xL_B \cos(\theta_B) + 2xL_C \sin(\theta_C) - 2L_B L_C \sin(\theta_B - \theta_C)}$

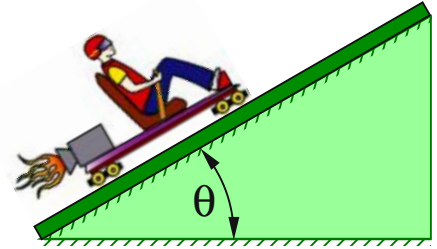
<sup>1</sup>Hint: Use the distributive property to express  $\vec{r}^{B_C/N_o} \cdot \vec{r}^{B_C/N_o}$  in terms of  $x$ ,  $L_B$ , and  $\hat{n}_x \cdot \hat{b}_x$ . Next, use the *dot-product definition* in equation (2.2) to calculate  $\hat{n}_x \cdot \hat{b}_x =$  , and then rewrite  $\vec{r}^{B_C/N_o} \cdot \vec{r}^{B_C/N_o} =$   <sup>2</sup> +  <sup>2</sup> +  $2xL_B$  (  $\cdot$  ) =  <sup>2</sup> +  <sup>2</sup> +  $2xL_B \cos$ ().

Note: The distributive property for vector dot-multiplication is  $(\vec{a} + \vec{b}) \cdot (\vec{c} + \vec{d}) = \vec{a} \cdot \vec{c} + \vec{a} \cdot \vec{d} + \vec{b} \cdot \vec{c} + \vec{b} \cdot \vec{d}$ .

## 2.10 Vector components, free-body diagram (FBD), and motion graphs for a rocket-sled.

The following figure shows a rocket-sled moving along smooth (**frictionless**) inclined rails.

Description	Symbol	Type
Mass of rocket-sled and rider	$m$	Constant
Earth's gravitational acceleration	$g$	Constant
Angle between horizontal and inclined-rails	$\theta$	Constant
$\hat{i}$ measure of thrust force on sled	$F_T$	Specified
$\hat{j}$ measure of normal force on sled	$F_N$	Variable
$\hat{i}$ measure of rocket-sled position	$x$	Variable



**Draw** a unit vector  $\hat{i}$  upward-right and parallel to the rails.

**Draw** a unit vector  $\hat{j}$  outward-normal to the rails (perpendicular to  $\hat{i}$ ) and in the plane of the paper.

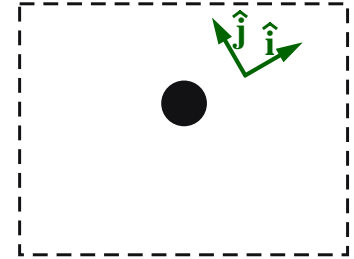
**Free-body diagram**

**Draw** a particle representing the rocket-sled.

**Draw** the thrust, normal, and gravity forces on the rocket-sled.

Express the net force on the rocket-sled in terms of  $\hat{i}$  and  $\hat{j}$ .

**Result:**  $\vec{F}_{\text{Net}} = \vec{F}_{\text{Thrust}} + \vec{F}_{\text{Normal}} + \vec{F}_{\text{gravity}}$   
 $= \text{[ ]} \hat{i} + \text{[ ]} \hat{j}$



**$\vec{F} = m \vec{a}$**  The rocket-sled's acceleration can be calculated as  $\vec{a} = \frac{d^2 x}{dt^2} \hat{i} = \ddot{x} \hat{i}$ .

Substitute the right-hand sides of  $\vec{F}_{\text{Net}}$  and  $\vec{a}$  into  $\vec{F}_{\text{Net}} = m \vec{a}$  and solve for  $\ddot{x}$  and  $F_N$ .

**Result:**  $\text{[ ]} \hat{i} + \text{[ ]} \hat{j} = m \ddot{x} \hat{i}$   
 $\hat{i}: \ddot{x} = \frac{\text{[ ]} - \text{[ ]}}{m}$        $\hat{j}: F_N = \text{[ ]}$

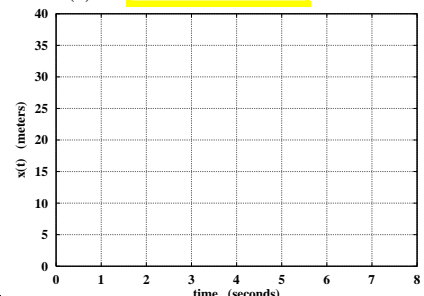
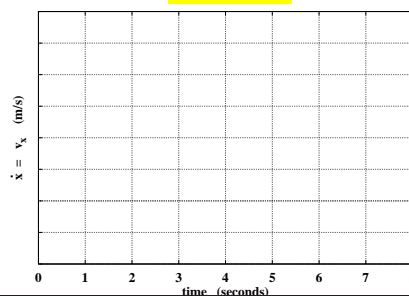
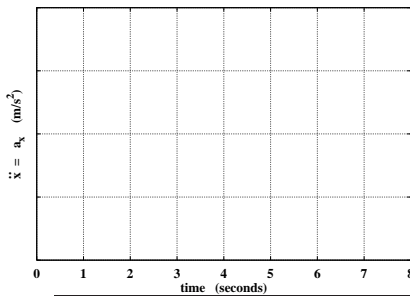
**Solve for acceleration:** Knowing  $m = 100 \text{ kg}$ ,  $g = 10 \frac{\text{m}}{\text{s}^2}$ ,  $\theta = 30^\circ$ ,  $F_T = 700 \text{ N}$ , **graph**  $\ddot{x}$ .

**Solve for velocity and position:** Knowing the rocket-sled starts at  $x = 8 \text{ m}$ , and is initially moving **downward-left** along the rail at  $4 \frac{\text{m}}{\text{s}}$ , solve and **sketch**  $\dot{x}(t)$  and  $x(t)$  for  $0 \leq t \leq 8$ .

**Result:**  $\ddot{x}(t) = 2 \text{ m/s}^2$

$\dot{x}(t) = \text{[ ]} \text{ m/s}$

$x(t) = \text{[ ]} \text{ meters}$

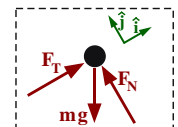


Include friction between the rails and rocket-sled, modeled via a coefficient of kinetic friction  $\mu_k$ . Express  $\ddot{x}$  and  $F_N$  in terms of some/all of  $\mu_k$  and symbols in the table. Knowing  $\mu_k \approx 0.115$ ,  $x(0) = 8 \text{ m}$ , and the rocket-sled initially moves **upward-right** at  $4 \frac{\text{m}}{\text{s}}$ , find  $\dot{x}(t)$  and  $x(t)$ .

**Result:**  $\ddot{x}(t) = \text{[ ]} \approx 1 \frac{\text{m}}{\text{s}^2}$        $F_N = \text{[ ]}$        $\dot{x}(t) \approx \text{[ ]}$   
 $x(t) \approx \text{[ ]}$

Calculate the minimum thrust (redraw FBDs) to:

	Result
a. Keep the rocket-sled moving uphill at constant speed	$F_T \approx \text{[ ]} \text{ N}$
b. Keep the rocket-sled moving downhill at constant speed	$F_T \approx \text{[ ]} \text{ N}$

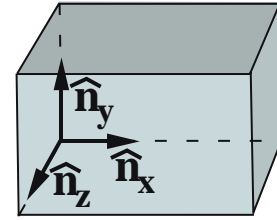


For part b, assume the rocket is initially moving **downward-left** at  $4 \text{ m/s}$ .

Solution at [www.MotionGenesis.com](http://www.MotionGenesis.com)  $\Rightarrow$  [Get Started](#)  $\Rightarrow$  [Rocket sled](#).

2.11 ♣ Construct a unit vector  $\hat{u}$  in the direction of each vector given below. (Section 2.9.2)

Vector	Unit vector $\hat{u}$
$3\hat{n}_x$	$\hat{n}_x$
$-3\hat{n}_x$	<input type="text"/>
$3\hat{n}_x - 4\hat{n}_y$	<input type="text"/>
$3\hat{n}_x - 4\hat{n}_y + 12\hat{n}_z$	<input type="text"/>
$c\hat{n}_x$	<input type="text"/> or <input type="text"/>
$c$ is a real non-zero number	



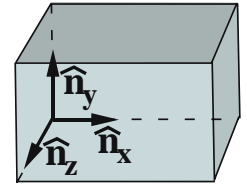
Note:  $\hat{n}_x, \hat{n}_y, \hat{n}_z$  are orthogonal unit vectors.

Note: Ensure your last answer agrees with your first two answers, e.g., if  $c = 3$  or  $c = -3$ .

2.12 ♣ Calculating vector cross products with bases. (Section 2.10)

Given: Right-handed orthogonal unit vectors  $\hat{n}_x, \hat{n}_y, \hat{n}_z$  and:

$$\begin{aligned}\vec{u} &= 2\hat{n}_x + 3\hat{n}_y + 4\hat{n}_z \\ \vec{v} &= x\hat{n}_x + y\hat{n}_y + z\hat{n}_z \\ \vec{w} &= 5\hat{n}_x - 6\hat{n}_y + 7\hat{n}_z\end{aligned}$$



(a) Use the distributive law for cross products to write  $\vec{u} \times \vec{v}$  in terms of  $\hat{n}_x \times \hat{n}_x, \hat{n}_x \times \hat{n}_y,$  etc.

Result: 
$$\begin{aligned}\vec{u} \times \vec{v} &= 2x\hat{n}_x \times \hat{n}_x + 2y\hat{n}_x \times \hat{n}_y + 2z\hat{n}_x \times \hat{n}_z \\ &+ \text{[ ]} + \text{[ ]} + \text{[ ]} \\ &+ \text{[ ]} + \text{[ ]} + \text{[ ]}\end{aligned}$$

(b) Use the definition of the cross product to calculate  $\hat{n}_x \times \hat{n}_x, \hat{n}_x \times \hat{n}_y,$  etc.

Result: 
$$\begin{aligned}\hat{n}_x \times \hat{n}_x &= \vec{0} & \hat{n}_x \times \hat{n}_y &= \hat{n}_z & \hat{n}_x \times \hat{n}_z &= -\hat{n}_y \\ \hat{n}_y \times \hat{n}_x &= \text{[ ]} & \hat{n}_y \times \hat{n}_y &= \text{[ ]} & \hat{n}_y \times \hat{n}_z &= \text{[ ]} \\ \hat{n}_z \times \hat{n}_x &= \text{[ ]} & \hat{n}_z \times \hat{n}_y &= \text{[ ]} & \hat{n}_z \times \hat{n}_z &= \text{[ ]}\end{aligned}$$

(c) In view of your previous two results, calculate  $\vec{u} \times \vec{v}$ .

Result: 
$$\vec{u} \times \vec{v} = \text{[ ]}\hat{n}_x + \text{[ ]}\hat{n}_y + \text{[ ]}\hat{n}_z$$

2.13 Cross products and determinants. (Section 2.10.3)

Given **right-handed orthogonal unit** vectors  $\hat{n}_x, \hat{n}_y, \hat{n}_z$  and two arbitrary vectors  $\vec{a}$  and  $\vec{b}$  expressed as shown to the right, show that calculating  $\vec{a} \times \vec{b}$  with the distributive property of the cross product happens to be equal to the determinant of the matrix shown to the right.

$$\vec{a} = a_x\hat{n}_x + a_y\hat{n}_y + a_z\hat{n}_z$$

$$\vec{b} = b_x\hat{n}_x + b_y\hat{n}_y + b_z\hat{n}_z$$

$$\vec{a} \times \vec{b} = \det \begin{bmatrix} \hat{n}_x & \hat{n}_y & \hat{n}_z \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{bmatrix}$$

Next use this determinant method to calculate the following cross products (refer to Hw 2.12).

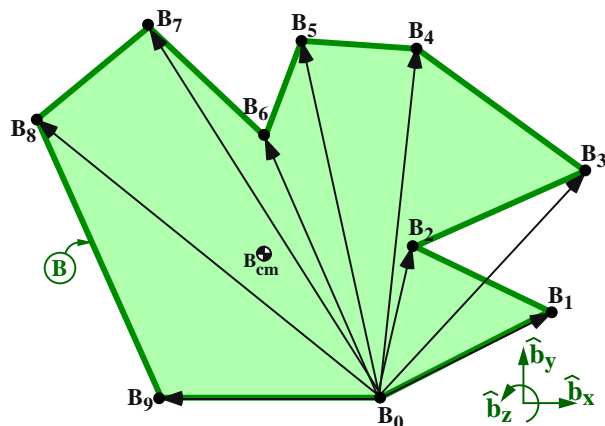
Result: 
$$\vec{u} \times \vec{v} = (3z - 4y)\hat{n}_x + (4x - 2z)\hat{n}_y + (2y - 3x)\hat{n}_z$$

Optional: 
$$\vec{u} \times \vec{w} = 45\hat{n}_x + 6\hat{n}_y - 27\hat{n}_z$$

Optional: 
$$\vec{v} \times \vec{w} = \text{[ ]}\hat{n}_x + \text{[ ]}\hat{n}_y - \text{[ ]}\hat{n}_z$$

2.14 ♣ **Cross products: Commercial algorithm for area calculations (surveying).** (Section 2.10.2)

One reason triangles are important is that complex **planar objects** such as the polygon  $B$  below can be decomposed into triangles. Planar measurements are important in various professions, including farming acreage, building costs, and mass and area properties of 2D objects.



$$\begin{aligned} \vec{r}^{B_1/B_0} &= 2.0 \hat{b}_x + 2.0 \hat{b}_y \\ \vec{r}^{B_2/B_0} &= 0.5 \hat{b}_x + 2.5 \hat{b}_y \\ \vec{r}^{B_3/B_0} &= 3.0 \hat{b}_x + 4.0 \hat{b}_y \\ \vec{r}^{B_4/B_0} &= 0.2 \hat{b}_x + 6.0 \hat{b}_y \\ \vec{r}^{B_5/B_0} &= -0.5 \hat{b}_x + 7.0 \hat{b}_y \\ \vec{r}^{B_6/B_0} &= -1.0 \hat{b}_x + 5.0 \hat{b}_y \\ \vec{r}^{B_7/B_0} &= -2.0 \hat{b}_x + 7.0 \hat{b}_y \\ \vec{r}^{B_8/B_0} &= -4.0 \hat{b}_x + 5.0 \hat{b}_y \\ \vec{r}^{B_9/B_0} &= -2.0 \hat{b}_x + 0.0 \hat{b}_y \end{aligned}$$

A commercial algorithm for calculating the area of the polygon  $B$  shown above is to:

- Label a vertex  $B_0$  and number the remaining vertices sequentially in a counter-clockwise fashion.
- Form  $\vec{r}^{B_i/B_0}$ , the position vector of vertex  $B_i$  ( $i = 1, 2, \dots$ ) from vertex  $B_0$
- Calculate  $\vec{A}_2$  and  $\vec{A}_4$ , the “vector-areas” of the triangles defined by vertices  $B_0 B_2 B_3$ , and  $B_0 B_4 B_5$ , respectively. Formulas for the vector-areas of each triangle are given below along with the vector sum of these areas  $\vec{A}$  and the polygon’s area (the magnitude of  $\vec{A}$ ).

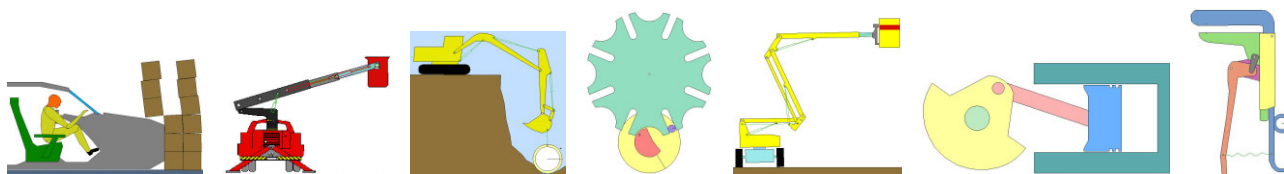
**Result:** (Just fill in the blanks. You only need to calculate  $\vec{A}_2$ ,  $\vec{A}_4$ , and  $\mathbf{A}$ )

$\vec{A}_1 = \frac{1}{2} * \vec{r}^{B_1/B_0} \times \vec{r}^{B_2/B_0} = 2 \hat{b}_z$	
$\vec{A}_2 = \frac{1}{2} * \vec{r}^{B_2/B_0} \times \vec{r}^{B_3/B_0} =$	<input type="text"/>
$\vec{A}_3 = \dots = 8.6 \hat{b}_z$	
$\vec{A}_4 = \dots =$	<input type="text"/>
$\vec{A}_5 = \dots = 2.25 \hat{b}_z$	
$\vec{A}_6 = \dots = 1.5 \hat{b}_z$	
$\vec{A}_7 = \dots = 9 \hat{b}_z$	
$\vec{A}_8 = \frac{1}{2} * \vec{r}^{B_8/B_0} \times \vec{r}^{B_9/B_0} = 5 \hat{b}_z$	
$\mathbf{A} = \sum_{i=1}^8 \vec{A}_i =$	<input type="text"/>
Area = $ \vec{A} $	= 27.8

Accounting for overlapped areas is done with **positive** and **negative** signs on vectors.



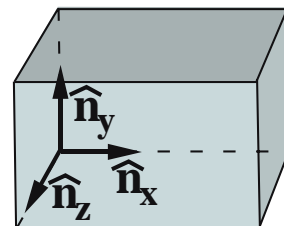
Note: Compute cross products with the distributive property  $(\vec{a} + \vec{b}) \times (\vec{c} + \vec{d}) = \vec{a} \times \vec{c} + \vec{a} \times \vec{d} + \vec{b} \times \vec{c} + \vec{b} \times \vec{d}$  and the **cross-product-definition with the right-hand rule** (not determinants or special formulas in a book). Also, use the fact that  $\hat{b}_x, \hat{b}_y, \hat{b}_z$  are orthogonal unit vectors.



Planar objects, courtesy of Working Model and Design-Simulation Technologies

2.15 ♣ **Scalar triple product with bases** (Section 2.11).

The figure shows right-handed orthogonal unit vectors  $\hat{n}_x, \hat{n}_y, \hat{n}_z$ .



Given:  $\vec{u} = 2\hat{n}_x + 3\hat{n}_y + 4\hat{n}_z$   
 $\vec{v} = x\hat{n}_x + y\hat{n}_y + z\hat{n}_z$   
 $\vec{w} = 5\hat{n}_x - 6\hat{n}_y + 7\hat{n}_z$

Calculate:  $\vec{u} \times \vec{v} \cdot \vec{u}$   
 $\vec{u} \times \vec{v} \cdot \vec{w}$   
 $\vec{u} \cdot \vec{v} \times \vec{w}$

**Result:** (Although the order of operations in  $\vec{u} \times \vec{v} \cdot \vec{u}$  is unambiguous, parentheses may clarify your work.)

$\vec{u} \times \vec{v} \cdot \vec{u} = \square$

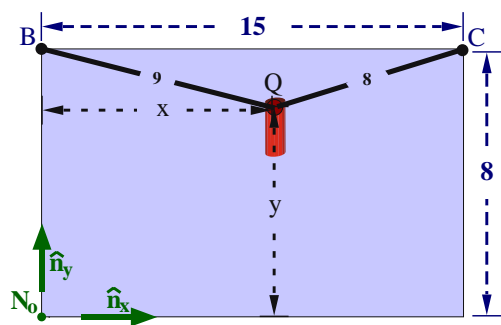
$\vec{u} \times \vec{v} \cdot \vec{w} = \square$

$\vec{u} \cdot \vec{v} \times \vec{w} = 27z - 45x - 6y$

In view of your last two results,  $\vec{u} \times \vec{v} \cdot \vec{w}$  is **equal/not equal** (circle one) to  $\vec{u} \cdot \vec{v} \times \vec{w}$ .  
 It is **OK/not OK** to switch the  $\cdot$  and  $\times$  in the scalar triple product.

2.16 **Locating a microphone (2D).** Show work. (Section 1.4)

A microphone  $Q$  is attached to two pegs  $B$  and  $C$  by two cables. Knowing the peg locations, cable lengths, and that points  $B, C, Q$ , and  $N_o$  all lie in the same plane, determine the distance between  $Q$  and  $N_o$ . Try to do the problem first using Euclidean geometry - and then try vectors.<sup>2</sup>

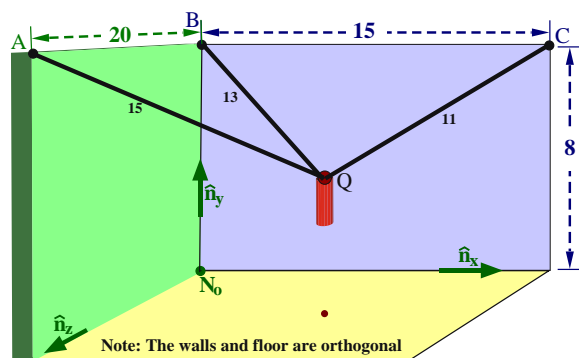


Quantity	Value
Distance from $B$ to $C$	15 m
Distance from $N_o$ to $B$	8 m
Length of cable joining $B$ and $Q$	9 m
Length of cable joining $C$ and $Q$	8 m
Distance between $N_o$ and $Q$	<b>9.01 m</b>
If $Q$ is above ceiling, distance $\approx 12$ m	

Note: Although there are two “mathematical” answers to this problem, one is above the ceiling and requires the cables to be in compression.

2.17 † **Locating a microphone (3D).**

A microphone  $Q$  is attached to three pegs  $A, B$ , and  $C$  by three cables. Knowing the peg locations and cable lengths, determine the distance between  $Q$  and point  $N_o$ . Show work. (If needed, hint below).<sup>3</sup>



Quantity	Value
Distance from $A$ to $B$	20 m
Distance from $B$ to $C$	15 m
Distance from $N_o$ to $B$	8 m
Length of cable joining $A$ and $Q$	15 m
Length of cable joining $B$ and $Q$	13 m
Length of cable joining $C$ and $Q$	11 m
Distance between $N_o$ and $Q$	<b>13.3 m</b>
If $Q$ is above ceiling, distance $\approx 17$ m	

Note: This is part of the process of a camera targeting a football/baseball in a stadium or laser targeting cancer or ...

2.18 **Optional: FBD and statics “review” problems from Homework 12.**

The following problems require an understanding of dot-products, force, and *free-body diagrams*.

Hw 12.1	Hw 12.8	Hw 12.9	Hw 12.10	Hw 12.12(a)
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<sup>2</sup>Hint: For Euclidean geometry, use the law of cosines and SohCahToa. For vectors, see Homework 1.35.

<sup>3</sup>Hint: See Homework 1.35 or Section 3.3. Introduce unknowns  $x, y, z$  so  $Q$ 's position vector from  $N_o$  is  $x\hat{n}_x + y\hat{n}_y + z\hat{n}_z$ . Although nonlinear algebraic equations are usually solved with a computer, these can also be solved “by-hand”.  
 Solution at [www.MotionGenesis.com](http://www.MotionGenesis.com). Alternately, [www.WolframAlpha.com](http://www.WolframAlpha.com) solves sets of nonlinear equations, e.g., type Solve  $x^2 + (-20+z)^2 + (-8+y)^2 = 225, \quad x^2 + z^2 + (-8+y)^2 = 169, \quad z^2 + (-15+x)^2 + (-8+y)^2 = 121$