20.5 MG road-maps for efficient statics and dynamics

A modern way to efficiently form static or dynamic equations with FBDs is to:[1]

- Choose ***scalar variables*** that describe the relevant ***unknown*** configuration, motion, or forces.
- Complete the associated ***MG road-maps*** and ***free-body diagrams***.[2]
- Complete the calculations specified by the ***MG road-map equation***.

MG road-map for efficient statics and dynamics.

Variable	Translate/ Rotate	Direction (unit vector)	System S	FBD of S	About point	***MG road-map equation***	Additional Unknowns
				Draw			**?**
***** If applicable: Additional ***constraint equations*** and their time-derivatives (e.g., closed linkages or **rolling**).							

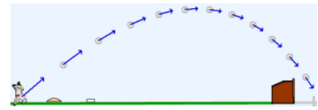
Column	Enter the following information
1	**Unknown scalar variable** (e.g., a position, velocity, force, or torque variable).
2	Type of motion associated with the variable: **translate** or **rotate**.
3	Vector direction (e.g., **unit vector** $\hat{\mathbf{u}}$) associated with the direction of motion.
4	List of objects whose motion (e.g., velocity or angular velocity) is directly effected by the variable in column 1 ("freeze" any variable other than the variable in column 1 and decide what objects **must** move). This picks a **system** S that reduces/eliminates constraint forces. Note: If the variable in column 1 is a force measure, treat it as a velocity measure and determine what objects move. If it is a torque measure, treat it as an angular velocity measure and determine what objects move.
5	**Draw** a ***free-body diagram*** of **system** S (**draw** relevant contact/distance forces). Note: See force/torque models for gravity, springs, dampers, air-resistance, etc., in Chapter 19.
6	If column 2 was **rotate**, choose a point O (or line L) about which moments are to be taken. Note: Choose point O to eliminate moments of unknown forces (e.g., contact forces on S) – look at FBD. Note: To facilitate calculations, you can slide the "about point" along the line L parallel to $\hat{\mathbf{u}}$. This is because $\hat{\mathbf{u}} \cdot \vec{\mathbf{M}}^{S/O} = \hat{\mathbf{u}} \cdot \vec{\mathbf{M}}^{S/P}$ if both points O and P are on line L (proved in Section 17.1.3).
7a	If column 2 was **translate**, use: (N is a Newtonian reference frame) $\hat{\mathbf{u}} \cdot (\vec{\mathbf{F}}^S \underset{\textbf{Dynamics}}{=} m^S * {}^N\vec{\mathbf{a}}^{S_{\text{cm}}})$ or $\hat{\mathbf{u}} \cdot \vec{\mathbf{F}}^S \underset{\textbf{Statics}}{=} 0$ To calculate $m^S * {}^N\vec{\mathbf{a}}^{S_{\text{cm}}}$ for a system S of objects A, B, C: $m^S * {}^N\vec{\mathbf{a}}^{S_{\text{cm}}} \underset{(11.3)}{=} m^A * {}^N\vec{\mathbf{a}}^{A_{\text{cm}}} + m^B * {}^N\vec{\mathbf{a}}^{B_{\text{cm}}} + m^C * {}^N\vec{\mathbf{a}}^{C_{\text{cm}}}$
7b	If column 2 was **rotate**, use: $\hat{\mathbf{u}} \cdot (\vec{\mathbf{M}}^{S/O} \underset{(20.4)}{=} \frac{{}^N d \; {}^N\vec{\mathbf{H}}^{S/O}}{dt} + \ldots)$ or $\hat{\mathbf{u}} \cdot \vec{\mathbf{M}}^{S/O} \underset{\textbf{Statics}}{=} 0$ For a system S with a particle Q and rigid body B: ${}^N\vec{\mathbf{H}}^{S/O} = {}^N\vec{\mathbf{H}}^{Q/O} + {}^N\vec{\mathbf{H}}^{B/O}$ For particle Q: ${}^N\vec{\mathbf{H}}^{Q/O} \underset{(10.3)}{=} \vec{\mathbf{r}}^{Q/O} \times m^Q \; {}^N\vec{\mathbf{v}}^Q$ For rigid body B: ${}^N\vec{\mathbf{H}}^{B/O} \underset{(15.4)}{=} {}^N\vec{\mathbf{H}}^{B/B_p} + \vec{\mathbf{r}}^{B_p/O} \times m^B \; {}^N\vec{\mathbf{v}}^{B_{\text{cm}}}$ where: ${}^N\vec{\mathbf{H}}^{B/B_p} \underset{(15.1)}{=} \overrightarrow{\overrightarrow{\mathbf{I}}}^{B/B_p} \cdot {}^N\vec{\boldsymbol{\omega}}^B + \vec{\mathbf{r}}^{B_{\text{cm}}/B_p} \times m^B \; {}^N\vec{\mathbf{v}}^{B_p}$
8*	Additional constraint force/torque variables may appear in MG road-maps. If applicable, append configuration or motion ***constraints*** (e.g., closed linkages or **rolling**) that interrelate Column 1 position/velocity variables.

See MG road-map examples in next sections and at **www.MotionGenesis.com** ⇒ **Textbooks** ⇒ **Resources**.

[1]Mitiguy and Fregly invented ***MG road-maps*** to efficiently form static and dynamic equations. ***MG road-maps*** combine the simplicity of a spreadsheet with the physical insights of free-body diagrams. ***MG road-maps*** were inspired by the cleverness of D'Alembert and the mathematical rigor of Kane and Euler/Lagrange. ***MG road-maps*** are easy to teach/learn and approximate the efficiency of Kane's method for most systems (except embedded constraints like rolling/gears).

[2]***Free-body diagrams (FBDs)*** are a means to an end, not an end in itself. ***MG road-maps*** help determine **which FBDs** to draw and **what to do** with them, – which differs significantly from knowing **how** to draw FBDs.

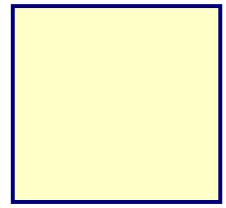
20.5.1 MG road-map: Projectile motion (2D)



A baseball (particle Q) flies over Earth N (a Newtonian reference frame). Aerodynamic forces on the baseball are modeled as $-b\,\vec{\mathbf{v}}$ ($\vec{\mathbf{v}}$ is Q's velocity in N).
$\hat{\mathbf{n}}_x$ is horizontally-right, $\hat{\mathbf{n}}_y$ is vertically-upward, and N_o is home-plate (point fixed in N).

MG road-map for projectile motion x and y ($\hat{\mathbf{n}}_x$, $\hat{\mathbf{n}}_y$ measures of Q's position vector from N_o)

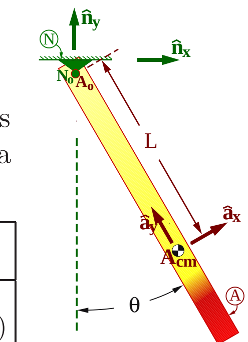
Variable	Translate/ Rotate	Direction (unit vector)	System S	FBD of S	About point	***MG road-map equation***
x	Translate	$\hat{\mathbf{n}}_x$	Q	**Draw**	Not applicable	$\hat{\mathbf{n}}_x \cdot (\vec{\mathbf{F}}^Q \underset{(20.1)}{=} m^Q\,{}^N\vec{\mathbf{a}}^Q)$
y	Translate	$\hat{\mathbf{n}}_y$	Q	**Draw**	Not applicable	$\hat{\mathbf{n}}_y \cdot (\vec{\mathbf{F}}^Q \underset{(20.1)}{=} m^Q\,{}^N\vec{\mathbf{a}}^Q)$



Draw FBD

Solution and simulation link at **www.MotionGenesis.com** ⇒ **Textbooks** ⇒ **Resources**.

20.5.2 MG road-map: Rigid body pendulum (2D)



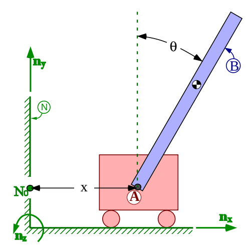
A non-uniform density rigid rod A is attached at point A_o of A by a frictionless revolute/pin joint to Earth N (Newtonian reference frame). The rod swings with a "pendulum angle" θ in a vertical plane that is perpendicular to unit vector $\hat{\mathbf{a}}_z$.

Variable	Translate/ Rotate	Direction (unit vector)	System S	FBD of S	About point	***MG road-map equation***
θ	Rotate	$\hat{\mathbf{a}}_z = \hat{\mathbf{n}}_z$	A	**Draw**	A_o	$\hat{\mathbf{a}}_z \cdot (\vec{\mathbf{M}}^{A/A_o} \underset{(20.4)}{=} \frac{{}^N d\;{}^N\vec{\mathrm{H}}^{A/A_o}}{dt})$

$${}^N\vec{\mathrm{H}}^{A/A_o} \underset{(15.3)}{=} \mathrm{I}_{zz}^{A/A_o}\;{}^N\vec{\boldsymbol{\omega}}^A$$

Solution and simulation link at **www.MotionGenesis.com** ⇒ **Textbooks** ⇒ **Resources**.

20.5.3 MG road-map: Inverted pendulum on cart (x and θ) (2D)



A rigid rod B is pinned to a massive cart A (modeled as a particle) that translates horizontally in a Newtonian reference frame N. The cart's position vector from a point N_o fixed in N is $x\,\hat{\mathbf{n}}_x$ ($\hat{\mathbf{n}}_x$ is horizontally-right). B's swinging motion in N is in a vertical plane perpendicular to $\hat{\mathbf{n}}_z$ (a unit vector fixed in both B and N).

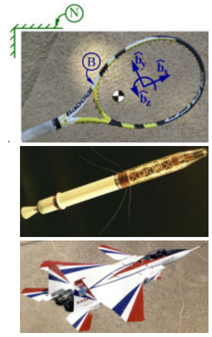
Variable	Translate/ Rotate	Direction (unit vector)	System S	FBD of S	About point	***MG road-map equation***
x	Translate	$\hat{\mathbf{n}}_x$	A, B	**Draw**	Not applicable	$\hat{\mathbf{n}}_x \cdot (\vec{\mathbf{F}}^S \underset{(20.1)}{=} m^S * {}^N\vec{\mathbf{a}}^{S_{cm}})$
θ	Rotate	$\hat{\mathbf{b}}_z = \hat{\mathbf{n}}_z$	B	**Draw**	A	$\hat{\mathbf{b}}_z \cdot (\vec{\mathbf{M}}^{B/A} \underset{(20.4)}{=} \frac{{}^N d\;{}^N\vec{\mathrm{H}}^{B/A}}{dt} + \ldots)$

Homework 15.8 and Chapter 24 complete these calculations.

Note: $m^S * {}^N\vec{\mathbf{a}}^{S_{cm}} \underset{(11.3)}{=} m^A * {}^N\vec{\mathbf{a}}^A + m^B * {}^N\vec{\mathbf{a}}^{B_{cm}}$ and $\frac{{}^N d\;{}^N\vec{\mathrm{H}}^{B/A}}{dt} + \ldots \underset{(20.6)}{=} \mathrm{I}_{zz}^{B/A} * {}^N\vec{\boldsymbol{\alpha}}^B + m^B * \vec{\mathbf{r}}^{B_{cm}/A} \times {}^N\vec{\mathbf{a}}^A$.

20.5.4 MG road-map: Rotating rigid body (3D)

Shown right is a rotating rigid body B (e.g., tennis racquet, spacecraft, or aircraft) in a Newtonian reference frame N. Right-handed orthogonal unit vectors $\widehat{\mathbf{b}}_x$, $\widehat{\mathbf{b}}_y$, $\widehat{\mathbf{b}}_z$ are fixed in B.

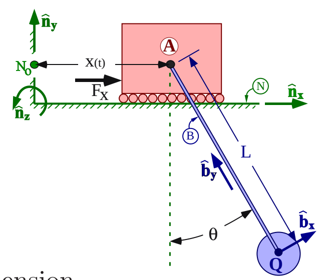


Variable	Translate/ Rotate	Direction (unit vector)	System S	FBD of S	About point	*MG road-map equation*
ω_x	Rotate	$\widehat{\mathbf{b}}_x$	B	Draw	B_{cm}	$\widehat{\mathbf{b}}_x \cdot (\vec{\mathbf{M}}^{B/B_{cm}} \underset{(20.4)}{=} \frac{{}^N d\ {}^N\vec{\mathrm{H}}^{B/B_{cm}}}{dt})$
ω_y	Rotate	$\widehat{\mathbf{b}}_y$	B	Draw	B_{cm}	$\widehat{\mathbf{b}}_y \cdot (\vec{\mathbf{M}}^{B/B_{cm}} \underset{(20.4)}{=} \frac{{}^N d\ {}^N\vec{\mathrm{H}}^{B/B_{cm}}}{dt})$
ω_z	Rotate	$\widehat{\mathbf{b}}_z$	B	Draw	B_{cm}	$\widehat{\mathbf{b}}_z \cdot (\vec{\mathbf{M}}^{B/B_{cm}} \underset{(20.4)}{=} \frac{{}^N d\ {}^N\vec{\mathrm{H}}^{B/B_{cm}}}{dt})$

Solution and simulation link at www.MotionGenesis.com ⇒ Textbooks ⇒ Resources.

Note: The "about point" is somewhat arbitrary. When B_{cm} is chosen: ${}^N\vec{\mathrm{H}}^{B/B_{cm}} \underset{(15.2)}{=} \vec{\vec{\mathbf{I}}}^{B/B_{cm}} \cdot {}^N\vec{\boldsymbol{\omega}}^B$.

20.5.5 MG road-map: Bridge crane equations of motion (2D)



A payload (particle) Q is welded to the end of a light rigid cable B which swings in a Newtonian reference frame N. Cable B is pinned to a massive trolley A. Trolley A moves horizontally along a smooth slot fixed in N with a **specified** (known) displacement $x(t)$ due to a force of measure F_x (a linear actuator connects A to a point N_o of N).

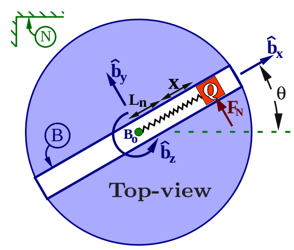
MG road-map for pendulum angle θ, actuator force F_x, and cable tension

Variable	Translate/ Rotate	Direction (unit vector)	System S	FBD of S	About point	*MG road-map equation*
θ	Rotate	$\widehat{\mathbf{n}}_z = \widehat{\mathbf{b}}_z$	B, Q	Draw	A	$\widehat{\mathbf{n}}_z \cdot (\vec{\mathbf{M}}^{S/A} = \frac{{}^N d\ {}^N\vec{\mathrm{H}}^{S/A}}{dt} + \ldots)$
F_x	Translate	$\widehat{\mathbf{n}}_x$	A, B, Q	Draw	Not applicable	$\widehat{\mathbf{n}}_x \cdot (\vec{\mathbf{F}}^S = m^S\ {}^N\vec{\mathbf{a}}^{S_{cm}})$
Tension	Translate	$\widehat{\mathbf{b}}_y$	A, B, Q	Draw	Not applicable	$\widehat{\mathbf{n}}_x \cdot (\vec{\mathbf{F}}^Q = m^Q\ {}^N\vec{\mathbf{a}}^Q)$

Student/Instructor version at www.MotionGenesis.com ⇒ Textbooks ⇒ Resources

Note: Only the θ road-map equation is needed to predict this system's motion. The others are shown for illustrative purposes.

20.5.6 MG road-map: Particle on spinning slot (2D)



A particle Q slides on a straight slot B. The slot is connected with a revolute joint to a Newtonian frame N at point B_o so that B rotates in a horizontal plane perpendicular to $\widehat{\mathbf{b}}_z$ ($\widehat{\mathbf{b}}_z$ is vertically-upward and fixed in both B and N).

Note: Homework 14.7 completes the MG road-map calculations for x and θ.

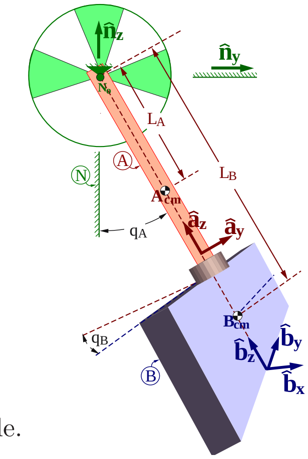
MG road-map for x, θ, and F_N ($\widehat{\mathbf{b}}_y$ measure of normal force on Q from B)

Variable	Translate/ Rotate	Direction (unit vector)	System S	FBD of S	About point	*MG road-map equation*
x	Translate	$\widehat{\mathbf{b}}_x$	Q	Draw	Not applicable	$\widehat{\mathbf{b}}_x \cdot (\vec{\mathbf{F}}^Q = m^Q\ {}^N\vec{\mathbf{a}}^Q)$
θ	Rotate	$\widehat{\mathbf{b}}_z$	B, Q	Draw	B_o	$\widehat{\mathbf{b}}_z \cdot (\vec{\mathbf{M}}^{S/B_o} = \frac{{}^N d\ {}^N\vec{\mathrm{H}}^{S/B_o}}{dt})$
F_N	Translate	$\widehat{\mathbf{b}}_y$	Q	Draw	Not applicable	$\widehat{\mathbf{b}}_y \cdot (\vec{\mathbf{F}}^Q = m^Q\ {}^N\vec{\mathbf{a}}^Q)$

Note: The F_N road-map equation is needed to predict motion **if** a friction force depends on $\mu\, F_N$.

$${}^N\vec{\mathrm{H}}^{S/B_o} = {}^N\vec{\mathrm{H}}^{B/B_o} + {}^N\vec{\mathrm{H}}^{Q/B_o} \quad \text{where} \quad {}^N\vec{\mathrm{H}}^{B/B_o} \underset{(15.3)}{=} \mathrm{I}_{zz}\ {}^N\vec{\boldsymbol{\omega}}^B \quad \text{and} \quad {}^N\vec{\mathrm{H}}^{Q/B_o} \underset{(10.3)}{=} \vec{\mathbf{r}}^{\,Q/B_o} \times m^Q\ {}^N\vec{\mathbf{v}}^Q.$$

20.5.7 MG road-map: Chaotic motion of a double pendulum (3D)



The schematic to the right shows a swinging babyboot attached by a shoelace to a rigid support. The mechanical model of the babyboot consists of a thin uniform rod A attached to a fixed support N by a revolute joint at point N_o and a uniform plate B connected to A with a second revolute joint at point B_o so B can rotate freely about A's axis.
Note: The revolute joints' axes are ***perpendicular***, not parallel.

Modeling considerations

- The plate, rod, and support are rigid.
- The revolute joints are ideal (massless, frictionless, no slop/flexibility).
- Earth is a Newtonian reference frame.
- Forces due to Earth's gravitation are uniform and constant.
- Other distance forces (electromagnetic and gravitational) and air-resistance are negligible.

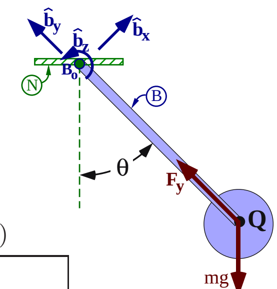
Right-handed sets of unit vectors $\hat{n}_x, \hat{n}_y, \hat{n}_z$; $\hat{a}_x, \hat{a}_y, \hat{a}_z$; $\hat{b}_x, \hat{b}_y, \hat{b}_z$ are fixed in N, A, B, respectively, with $\hat{n}_x = \hat{a}_x$ parallel to the revolute axis joining A to N, $\hat{n}_z$ vertically-upward, $\hat{a}_z = \hat{b}_z$ parallel to the rod's long axis (and the revolute axis joining B to A), and $\hat{b}_z$ perpendicular to plate B.

Complete the ***MG road-map*** for angles q_A (angle from $\hat{n}_z$ to $\hat{a}_z$ with $+\hat{n}_x$ sense) and q_B (angle from $\hat{a}_y$ to $\hat{b}_y$ with $+\hat{a}_z$ sense). Note: The "about point" can be shifted from B_o to B_{cm} since $\hat{b}_z \cdot \vec{M}^{B/B_{cm}} \underset{(17.4)}{=} \hat{b}_z \cdot \vec{M}^{B/B_o}$.

Variable	Translate/ Rotate	Direction (unit vector)	System S	FBD of S	About point	MG road-map equation
q_A	Rotate	$\hat{a}_x$	A, B	**Draw**	A_o	$\hat{a}_x \cdot \left(\vec{M}^{S/N_o} = \frac{^N d \, ^N\vec{H}^{S/N_o}}{dt} \right)$
q_B	Rotate	$\hat{b}_z$	B	**Draw**	B_{cm}	$\hat{b}_z \cdot \left(\vec{M}^{B/B_{cm}} = \frac{^N d \, ^N\vec{H}^{B/B_{cm}}}{dt} \right)$

Student/Instructor version at **www.MotionGenesis.com** ⇒ **Textbooks** ⇒ **Resources**

20.5.8 MG road-map: Classic particle pendulum (2D)



A particle Q is welded to the distal end of a light rigid rope B. The rope's other end attaches to a point B_o, fixed in a Newtonian reference frame N. The swinging motion of B and Q is in a vertical plane that is perpendicular to unit vector $\hat{b}_z$.

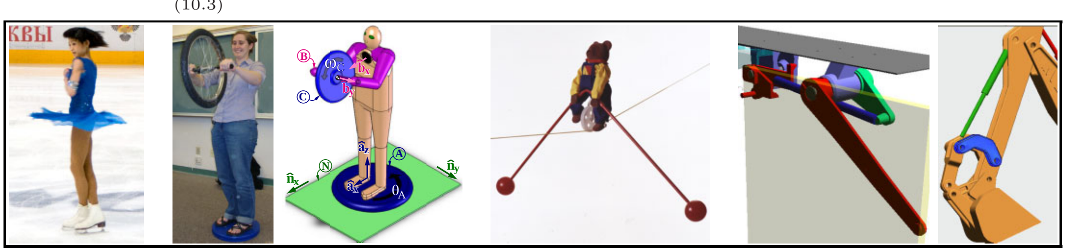
MG road-map for pendulum angle θ and tension F_y ($\hat{b}_y$ measure of force on Q from B)

Variable	Translate/ Rotate	Direction (unit vector)	System S	FBD of S	About point	***MG road-map equation***
θ	Rotate	$\hat{b}_z$	B, Q	**Draw**	B_o	$\hat{b}_z \cdot \left(\vec{M}^{S/B_o} \underset{(20.4)}{=} \frac{^N d \, ^N\vec{H}^{S/B_o}}{dt} \right)$
F_y	Translate	$\hat{b}_y$	Q	**Draw**	Not applicable	$\hat{b}_y \cdot \left(\vec{F}^Q \underset{(20.1)}{=} m^Q \, ^N\vec{a}^Q \right)$

Draw FBDs

Solution and simulation link at **www.MotionGenesis.com** ⇒ **Textbooks** ⇒ **Resources**.
Note: Only the θ road-map equation is needed to predict motion. The other is shown for illustrative purposes.

Note: $^N\vec{H}^{S/N_o} = {^N\vec{H}^{Q/N_o}} \underset{(10.3)}{=} \vec{r}^{\,Q/N_o} \times m^Q \, ^N\vec{v}^Q$. Section 23.3.2 completes all MG road-map calculations for θ.



Many additional MG road-map examples at **www.MotionGenesis.com** ⇒ **Textbooks** ⇒ **Resources**.

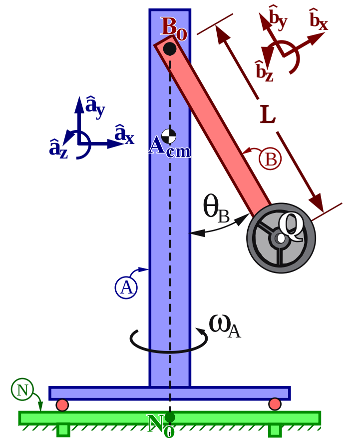
20.5.9 MG road-map: Dynamicist on a turntable (ice-skater)

A dynamics instructor stands on a spinning turntable and swings a heavy dumb-bell Q inward and outward to change his spin-rate (similar to the ice-skater).
Q is modeled as a particle rigidly attached (welded) to the end of the instructor's hands.

The schematic (below-right) shows a rigid body A (modeling the instructor's legs, torso, and head) that rotates (without friction) relative to Earth (a Newtonian reference frame N) about a vertical axis that is fixed in both A and N and which passes through both point N_o of N and point A_{cm} (A's center of mass).

A massless rigid arm B (modeling the instructor's arms and hands) attaches to A by a revolute motor (shoulder/muscles) whose revolute axis is horizontal and located at point B_o of B (B_o lies on the vertical axis connecting N_o and A_{cm}).

The motor (muscles) **specifies** B's angle θ_B relative to A to change in a known (prescribed) manner from 0 to π rad in 4 seconds ($\theta_B = \pi \frac{t}{4}$).



Right-handed orthogonal unit vectors $\hat{\mathbf{a}}_x$, $\hat{\mathbf{a}}_y$, $\hat{\mathbf{a}}_z$ and $\hat{\mathbf{b}}_x$, $\hat{\mathbf{b}}_y$, $\hat{\mathbf{b}}_z$ are fixed in A and B, respectively, with $\hat{\mathbf{a}}_y$ vertically-upward, $\hat{\mathbf{b}}_z = \hat{\mathbf{a}}_z$ parallel to the revolute motor's axis, and $\hat{\mathbf{b}}_y$ directed from Q to B_o.

Quantity	Symbol	Type	Value
Earth's gravitational constant	g	Constant	$9.8\ \frac{m}{s^2}$
Distance between Q and B_o	L	Constant	0.7 m
Mass of Q	m	Constant	12 kg
A's moment of inertia about line $\overline{A_{cm} B_o}$	I_{yy}	Constant	$0.6\ \text{kg m}^2$
Angle from $\hat{\mathbf{a}}_y$ to $\hat{\mathbf{b}}_y$ with $^+\hat{\mathbf{a}}_z$ sense	θ_B	**Specified**	**$0.25\,\pi\,t$** rad
$\hat{\mathbf{a}}_y$ measure of A's angular velocity in N	ω_A	Variable	

Complete the ***MG road-map*** for the turntable's "spin-rate" ω_A (Note: The "about point" is not unique)

Variable	Translate/ Rotate	Direction (unit vector)	System S	FBD of S	About point	*MG road-map equation*
ω_A	Rotate	$\hat{\mathbf{a}}_y$	A, B, Q	**Draw**	B_o	$\hat{\mathbf{a}}_y \cdot (\vec{\mathbf{M}}^{S/B_o} = \frac{^N d\ ^N\vec{\mathbf{H}}^{S/B_o}}{dt})$

Student/Instructor version at **www.MotionGenesis.com** ⇒ **Textbooks** ⇒ **Resources**

20.5.10 MG road-map: Instructor on turntable with spinning wheel (3D)

The pictures to the right shows dynamicist Dr. G standing on a spinning turntable and holding a spinning bicycle wheel.

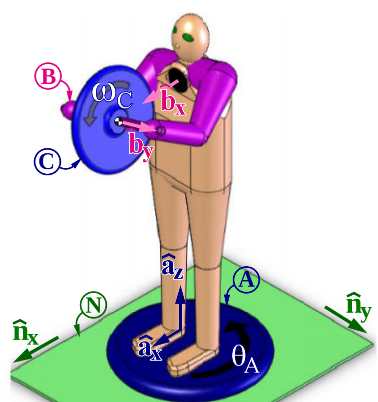
The mechanical model (below right) has a rigid body A (modeling the turntable, legs, torso, and head) that can freely rotate relative to Earth (Newtonian reference frame N) about a vertical axis that is fixed in both A and N and which passes through the center of the turntable (point N_o) and A_{cm} (A's center of mass).

A light (massless) rigid frame B (modeling the shoulders, arms, hands, and a portion of the bicycle wheel's axle) is attached to A by a revolute motor at point B_o of B (B_o lies on the vertical axis passing through A_{cm}). The motor's revolute axis passes through points B_o and C_{cm}, is horizontal, and is parallel to $\widehat{\mathbf{b}}_x = \widehat{\mathbf{a}}_x$.

A rigid bicycle wheel C is attached to B by a frictionless revolute joint whose axis passes through C_{cm} (C's center of mass) and is parallel to $\widehat{\mathbf{b}}_y$.

Right-handed orthogonal unit vectors $\widehat{\mathbf{a}}_x$, $\widehat{\mathbf{a}}_y$, $\widehat{\mathbf{a}}_z$ and $\widehat{\mathbf{n}}_x$, $\widehat{\mathbf{n}}_y$, $\widehat{\mathbf{n}}_z$ are fixed in A and N, respectively. Initially $\widehat{\mathbf{a}}_i = \widehat{\mathbf{n}}_i$ $(i = x, y, z)$, and then rigid body A is subjected to a right-handed rotation characterized by $\theta_A\, \widehat{\mathbf{a}}_z$ where $\widehat{\mathbf{a}}_z = \widehat{\mathbf{n}}_z$ is directed vertically-upward and $\widehat{\mathbf{a}}_x$ points from Dr. G's back to front (parallel to the axis of the revolute motor connecting A and B).

Unit vectors $\widehat{\mathbf{b}}_x$, $\widehat{\mathbf{b}}_y$, $\widehat{\mathbf{b}}_z$ are fixed in B. Initially $\widehat{\mathbf{b}}_i = \widehat{\mathbf{a}}_i$ $(i = x, y, z)$, then B is subjected to a $\theta_B\, (\widehat{\mathbf{a}}_x = \widehat{\mathbf{b}}_x)$ right-handed rotation in A where $\widehat{\mathbf{b}}_y$ is directed along the wheel's axle from Dr. G's right-to-left hand. Dr. G changes θ_B in a **specified** sinusoid manner with amplitude 30° and period 4 seconds.



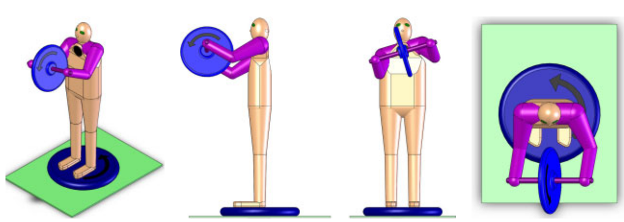
Courtesy Doug Schwandt
Purchase turntable/bicycle wheel at Arbor-scientific

Quantity	Symbol and type		Value
Mass of C	m^C	Constant	2 kg
Distance between B_o and C_{cm}	L_x	Constant	0.5 m
A's moment of inertia about B_o for $\widehat{\mathbf{a}}_z$	I^A_{zz}	Constant	0.64 kg m^2
C's moment of inertia about C_{cm} for $\widehat{\mathbf{b}}_x$	I^C	Constant	0.12 kg m^2
C's moment of inertia about C_{cm} for $\widehat{\mathbf{b}}_y$	J^C	Constant	0.24 kg m^2
Angle from $\widehat{\mathbf{n}}_x$ to $\widehat{\mathbf{a}}_x$ with $^+\widehat{\mathbf{n}}_z$ sense	θ_A	Variable	
Angle from $\widehat{\mathbf{a}}_y$ to $\widehat{\mathbf{b}}_y$ with $^+\widehat{\mathbf{a}}_x$ sense	θ_B	**Specified**	$\frac{\pi}{6}\sin(\frac{\pi}{2}\,t)$
$\widehat{\mathbf{b}}_y$ measure of C's angular velocity in B	ω_C	Variable	

Complete the ***MG road-map*** for θ_A and ω_C (the "about points" are not unique).

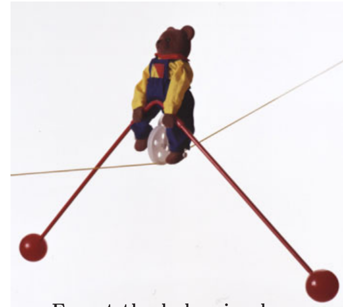
Variable	Translate/ Rotate	Direction (unit vector)	System S	FBD of S	About point	MG road-map equation
θ_A	Rotate	$\widehat{\mathbf{a}}_z$	A, B, C	**Draw**	B_o	$\widehat{\mathbf{a}}_z \cdot (\vec{\mathrm{M}}^{S/B_o} = \frac{^N d\ ^N\vec{\mathrm{H}}^{S/B_o}}{dt})$
ω_C	Rotate	$\widehat{\mathbf{b}}_y$	C	**Draw**	C_{cm}	$\widehat{\mathbf{b}}_y \cdot (\vec{\mathrm{M}}^{C/C_{cm}} = \frac{^N d\ ^N\vec{\mathrm{H}}^{C/C_{cm}}}{dt})$

Student/Instructor version at **www.MotionGenesis.com** ⇒ **Textbooks** ⇒ **Resources**



20.5.11 MG road-map: Bear riding a unicycle on a high-wire (3D)

The figures to the right show a (massless) pulley-wheel B that **rolls** along a taut (rigid) cable N (fixed on Earth, a Newtonian frame). Rigid body C (seat, rider, and balancing poles) attach to B with an ideal revolute motor at B_o (B's centroid). The motor axis is aligned with B's symmetry axis.

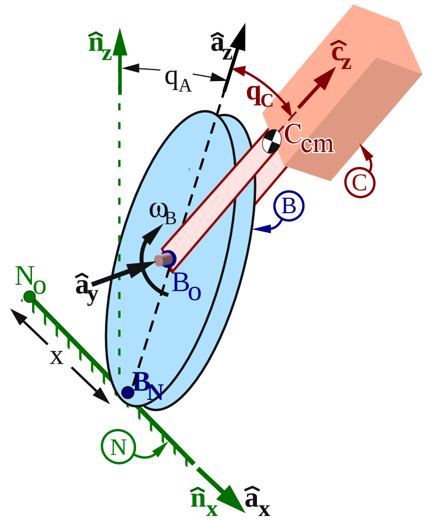


Ernest the balancing bear
Purchase at Schylling toy

Right-handed orthogonal unit vectors $\hat{\mathbf{n}}_x$, $\hat{\mathbf{n}}_y$, $\hat{\mathbf{n}}_z$ are fixed in N with $\hat{\mathbf{n}}_z$ vertically-upward and $\hat{\mathbf{n}}_x$ directed horizontally along the cable from a point N_o (fixed in N) to B_N (B's rolling point of contact with N).

Right-handed orthogonal unit vectors $\hat{\mathbf{a}}_x$, $\hat{\mathbf{a}}_y$, $\hat{\mathbf{a}}_z$ are directed with $\hat{\mathbf{a}}_x = \hat{\mathbf{n}}_x$, $\hat{\mathbf{a}}_y$ parallel to the motor axis, and $\hat{\mathbf{a}}_z$ from B_N to B_o.

Right-handed unit vectors $\hat{\mathbf{c}}_x$, $\hat{\mathbf{c}}_y$, $\hat{\mathbf{c}}_z$ are parallel to C's principal inertia axes about C_{cm} (C's center of mass), with $\hat{\mathbf{c}}_y = \hat{\mathbf{a}}_y$ and $\hat{\mathbf{c}}_z$ from B_o to C_{cm} (with balancing poles, C_{cm} is below B_o and L_C is negative).



Quantity	Symbol	Type	Value
Earth's gravitational constant	g	Constant	$9.8\ \text{m/s}^2$
Radius of B	r_B	Constant	30 cm
$\hat{\mathbf{c}}_z$ measure of C_{cm}'s position vector from B_o	L_C	Constant	–35 cm
Mass of C	m^C	Constant	2 kg
C's moment of inertia about C_{cm} for $\hat{\mathbf{c}}_x$	I	Constant	$3.4\ \text{kg m}^2$
C's moment of inertia about C_{cm} for $\hat{\mathbf{c}}_y$	J	Constant	$3.2\ \text{kg m}^2$
C's moment of inertia about C_{cm} for $\hat{\mathbf{c}}_z$	K	Constant	$2.8\ \text{kg m}^2$
$\hat{\mathbf{a}}_y$ measure of motor torque on B from C	T_y	**Specified**	below
Angle from $\hat{\mathbf{n}}_z$ to $\hat{\mathbf{a}}_z$ with $-\hat{\mathbf{n}}_x$ sense	q_A	Variable	
$\hat{\mathbf{a}}_y$ measure of ${}^A\vec{\boldsymbol{\omega}}^B$ (${}^A\vec{\boldsymbol{\omega}}^B = \omega_B\,\hat{\mathbf{a}}_y$)	ω_B	Variable	
Angle from $\hat{\mathbf{a}}_z$ to $\hat{\mathbf{c}}_z$ with $+\hat{\mathbf{a}}_y$ sense	q_C	Variable	
$\hat{\mathbf{n}}_x$ measure of $\vec{\mathbf{r}}^{\,B_N/N_o}$	x	Variable	

Form a complete set of ***MG road-maps*** for this systems's equations of motion (solution is not unique).
If necessary, add more ***MG road-maps*** so there are the same number of equations as unknowns.

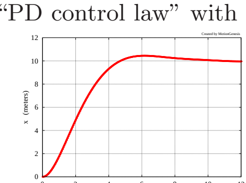
Variable	Translate/ Rotate	Direction (unit vector)	System S	FBD of S	About point	***MG road-map equation***	Additional Unknowns
q_A	Rotate	$\hat{\mathbf{a}}_x$	A,B,C	**Draw**	B_N	$\hat{\mathbf{a}}_x \cdot (\ \vec{\mathbf{M}}^{S/B_N} \underset{(20.4)}{=} \frac{{}^N d\ {}^N\vec{\mathbf{H}}^{S/B_N}}{dt} + \ldots)$	
ω_B	Rotate	$\hat{\mathbf{a}}_y$	$B,\ C$	**Draw**	B_N	$\hat{\mathbf{a}}_y \cdot (\ \vec{\mathbf{M}}^{S/B_N} \underset{(20.4)}{=} \frac{{}^N d\ {}^N\vec{\mathbf{H}}^{S/B_N}}{dt} + \ldots)$	
q_C	Rotate	$\hat{\mathbf{a}}_y$	C	**Draw**	B_o	$\hat{\mathbf{a}}_y \cdot (\ \vec{\mathbf{M}}^{C/B_o} \underset{(20.4)}{=} \frac{{}^N d\ {}^N\vec{\mathbf{H}}^{C/B_o}}{dt} + \ldots)$	
x	Translate	$\hat{\mathbf{a}}_x$	A,B,C	**Draw**	Not applicable	$\hat{\mathbf{a}}_x \cdot (\ \vec{\mathbf{F}}^S \underset{(20.1)}{=} m^S * {}^N\vec{\mathbf{a}}^{S_{cm}}\ )$	* F_x

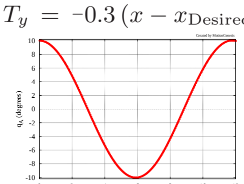
* Additional scalar constraint equation(s): $\dot{x} - r\,\omega_B = 0$

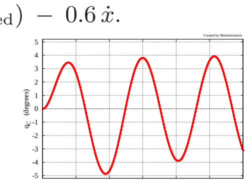
Instructor notes: One way to eliminate F_x from this analysis is to solve $\dot{x} = r\,\omega_B$ and eliminate the MG road-map for x.
Alternate ω_B MG road-map (that allows for sliding): "System" B, "About point" B_o, "Additional unknown" F_x.

To move the unicycle to $x_{\text{Desired}} = 10$ m, use a "PD control law" with $T_y = -0.3\,(x - x_{\text{Desired}}) - 0.6\,\dot{x}$.

Optional simulation:
Plot x, q_A, q_C for $0 \le t \le 12$ sec.
Use initial values:
$x = 0$ m $\quad q_A = 10°\quad q_C = 0°$
$\dot{x} = 0 \quad \dot{q}_A = 0 \quad \dot{q}_C = 0$

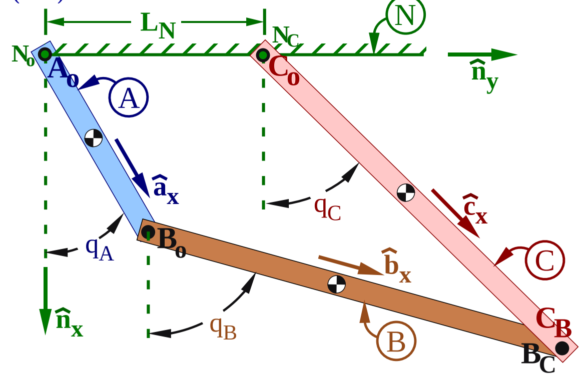






Solution at **www.MotionGenesis.com** ⇒ **Get Started** ⇒ **Bear on rolling unicycle**.

20.5.12 MG road-map: Four-bar linkage statics (2D)



The figure to the right shows a planar four-bar linkage consisting of frictionless-pin-connected uniform rigid links A, B, and C and ground N.

- Link A connects to N and B at points A_o and A_B
- Link B connects to A and C at points B_o and B_C
- Link C connects to N and B at points C_o and C_B
- Point N_o of N is coincident with A_o
- Point N_C of N is coincident with C_o

Right-handed orthogonal unit vectors $\hat{\mathbf{a}}_i$, $\hat{\mathbf{b}}_i$, $\hat{\mathbf{c}}_i$, $\hat{\mathbf{n}}_i$ $(i = x, y, z)$ are fixed in A, B, C, N, with:

- $\hat{\mathbf{a}}_x$ directed from A_o to A_B
- $\hat{\mathbf{b}}_x$ directed from B_o to B_C
- $\hat{\mathbf{c}}_x$ directed from C_o to C_B
- $\hat{\mathbf{n}}_x$ vertically-downward
- $\hat{\mathbf{n}}_y$ directed from N_o to N_C
- $\hat{\mathbf{a}}_z = \hat{\mathbf{b}}_z = \hat{\mathbf{c}}_z = \hat{\mathbf{n}}_z$ parallel to pin axes

As in Hw 8.7, create the following "***loop equation***" and dot-product with $\hat{\mathbf{n}}_x$ and $\hat{\mathbf{n}}_y$.

$$L_A\,\hat{\mathbf{a}}_x + L_B\,\hat{\mathbf{b}}_x - L_C\,\hat{\mathbf{c}}_x - L_N\,\hat{\mathbf{n}}_y = \vec{\mathbf{0}}$$

Quantity	Symbol	Value
Length of link A	L_A	1 m
Length of link B	L_B	2 m
Length of link C	L_C	2 m
Distance between N_o and N_C	L_N	1 m
Mass of A	m^A	10 kg
Mass of B	m^B	20 kg
Mass of C	m^C	20 kg
Earth's gravitational acceleration	g	9.81 $\frac{\text{m}}{\text{s}^2}$
$\hat{\mathbf{n}}_y$ measure of force applied to C_B	H	200 N
Angle from $\hat{\mathbf{n}}_x$ to $\hat{\mathbf{a}}_x$ with $^+\hat{\mathbf{n}}_z$ sense	q_A	Variable
Angle from $\hat{\mathbf{n}}_x$ to $\hat{\mathbf{b}}_x$ with $^+\hat{\mathbf{n}}_z$ sense	q_B	Variable
Angle from $\hat{\mathbf{n}}_x$ to $\hat{\mathbf{c}}_x$ with $^+\hat{\mathbf{n}}_z$ sense	q_C	Variable

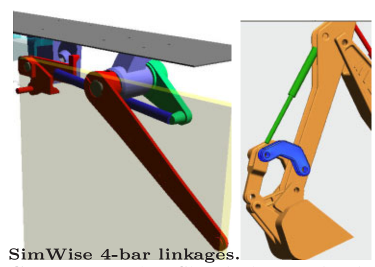
Complete the following ***MG road-map*** to determine this systems's ***static configuration***.

Variable	Translate/ Rotate	Direction (unit vector)	System S	FBD of S	About point	***MG road-map equation***	Additional Unknowns
q_A	Rotate	$\hat{\mathbf{n}}_z$	A, B	**Draw**	A_o	$\hat{\mathbf{a}}_x \cdot \vec{\mathbf{M}}^{S/A_o} = 0$	F^C_x, F^C_y
q_B	Rotate	$\hat{\mathbf{n}}_z$	B	**Draw**	B_o	$\hat{\mathbf{a}}_y \cdot \vec{\mathbf{M}}^{B/B_o} = 0$	F^C_x, F^C_y
q_C	Rotate	$\hat{\mathbf{n}}_z$	C	**Draw**	C_o	$\hat{\mathbf{a}}_y \cdot \vec{\mathbf{M}}^{C/C_o} = 0$	F^C_x, F^C_y
* Additional scalar constraint equation:						$-L_A\,\sin(q_A)\,\dot{q}_A - L_B\,\sin(q_B)\,\dot{q}_B + L_C\,\sin(q_C)\,\dot{q}_C = 0$	
* Additional scalar constraint equation:						$L_A\,\cos(q_A)\,\dot{q}_A + L_B\,\cos(q_B)\,\dot{q}_B - L_C\,\cos(q_C)\,\dot{q}_C = 0$	

Determine the ***static equilibrium*** values of q_A, q_B, q_C.
Use your intuition (**guess**), circle the ***stable*** solution.

Solution 1	$q_A \approx 20.0°$	$q_B \approx 71.7°$	$q_C = 38.3°$
Solution 2	$q_A \approx 249.3°$	$q_B \approx 140.2°$	$q_C = 199.1°$
Solution 3	$q_A \approx 30.7°$	$q_B \approx 226.1°$	$q_C = 254.7°$

Solution at **www.MotionGenesis.com** ⇒ **Get Started** ⇒ **Four-bar linkage**



SimWise 4-bar linkages.
Courtesy Design Simulation Technology