Basis independent vector operations: $-\vec{b} \quad 5\,\vec{b} \quad \vec{a}+\vec{b} \quad \angle(\vec{a},\vec{b}) \quad \vec{a}\cdot\vec{b} \quad \vec{a}\times\vec{b}$

Show work – except for ♣ fill-in-blanks (print .pdf from www.MotionGenesis.com ⇒ Textbooks ⇒ Resources).

1.1 ♣ Solving problems – what physicists and engineers do.

Understanding dynamics results from **doing** problems. Many problems herein guide you to help you synthesize processes (imitation). Please **do** these problems by yourself or with colleagues/instructors and use the textbook and other resources.

Confucius 500 B.C.

"I hear and I forget.
I see and I remember.
I ____ and I understand."

"By three methods we may learn wisdom:
1^{st} by reflection, which is noblest;
2^{nd} by imitation, which is easiest;
3^{rd} by experience, which is the bitterest."

1.2 ♣ What is a vector (as defined by Gibbs circa 1897)? (Section 2.2)

Two properties (attributes) of a vector are ________ and ________ (fill in the blanks).

1.3 ♣ What is a zero vector? (Section 2.3)

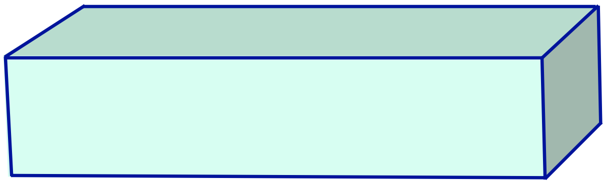
A zero vector $\vec{0}$ has a magnitude of 0 ($|\vec{0}| = 0$). **True/False** (circle true or false).
A zero vector $\vec{0}$ has a direction. **True/False**
any$\vec{\textbf{V}}$**ector** $+\ \vec{0}\ =$ **any**$\vec{\textbf{V}}$**ector** **True/False**

1.4 ♣ Unit vectors. (Section 2.4)

All unit vectors have a magnitude of 1 (e.g., $|\hat{i}| = 1$, $|\hat{j}| = 1$, $|\hat{k}| = 1$).	**True/False**
Typically, a unit vector is denoted with a hat, e.g., as $\hat{k}$ rather than $\vec{k}$.	**True/False**
All unit vectors are equal.	**True/False**
A unit vector $\hat{u}$ in the direction of the non-zero vector $\vec{v}$ is $\hat{u} = \frac{\vec{v}}{|\vec{v}|}$.	**True/False**

1.5 ♣ Draw the vectors $\vec{a}$, $\vec{b}$, $\hat{c}$, $\hat{d}$ (Section 2.2)

$\vec{a}$	Horizontally-right vector.
$\vec{b}$	Vertically-upward vector.
$\hat{c}$	Outwardly-directed **unit** vector.
$\hat{d}$	Inwardly-directed **unit** vector.

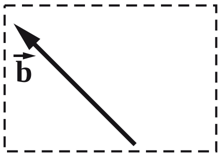


1.6 ♣ Negating a vector. (Section 2.8)

Draw the vector $-\vec{b}$. Negating the vector $\vec{b}$ results in a vector with different:

magnitude   direction   orientation   sense   (circle all that apply)

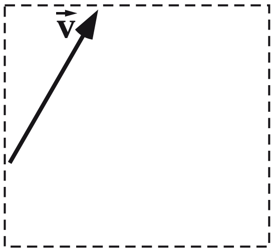
Historical note: Negative numbers (e.g., -3) were not widely accepted until 1800 A.D.



1.7 ♣ Vector magnitude and direction (orientation and sense). (Section 2.2)

The figure to the right shows a vector $\vec{v}$. **Draw** the vectors $\vec{a}$, $\vec{b}$, $\vec{c}$, $\vec{d}$, $\vec{e}$.

$\vec{a}$	Same magnitude and direction as $\vec{v}$ ($\vec{a} = \vec{v}$).
$\vec{b}$	Same magnitude as $\vec{v}$, with $\vec{b} = -\vec{v}$ (**antiparallel**).
$\vec{c}$	Same magnitude as $\vec{v}$, but different direction with $\vec{c} \neq -\vec{v}$.
$\vec{d}$	Smaller magnitude than $\vec{v}$, but same direction as $\vec{v}$.
$\vec{e}$	Different magnitude and different direction than $\vec{v}$.



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1.8 ♣ Vector magnitude and direction. (Section 2.2)

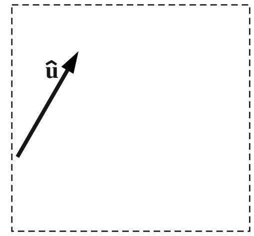
Knowing x is a real number (e.g., -3 or 0 or 7.8) and $\hat{\mathbf{u}}$ is a horizontal unit vector ⟶, complete ***magnitude*** with $\leq$ or $\geq$ and complete ***direction*** with $+\hat{\mathbf{u}}$ or $-\hat{\mathbf{u}}$.

Vector	with	Magnitude			Direction
$x\,\hat{\mathbf{u}}$	$x \geq 0$	$\lvert x\,\hat{\mathbf{u}} \rvert$	$\geq$	0	$+\hat{\mathbf{u}}$
$x\,\hat{\mathbf{u}}$	$x \leq 0$	$\lvert x\,\hat{\mathbf{u}} \rvert$		0	
$-x\,\hat{\mathbf{u}}$	$x \geq 0$	$\lvert -x\,\hat{\mathbf{u}} \rvert$		0	
$-x\,\hat{\mathbf{u}}$	$x \leq 0$	$\lvert -x\,\hat{\mathbf{u}} \rvert$		0	

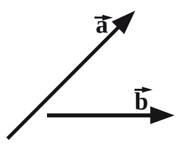
1.9 ♣ Multiplying a vector by a scalar. (Section 2.7)

The following statements involve a unit vector $\hat{\mathbf{u}}$ and a real scalar s ($s \neq 0$).
If a statement is **true**, provide any numerical value for s that supports your answer, and if **true** also **draw** a corresponding vector, i.e., $\vec{a}$ or $\vec{b}$ or $\vec{c}$.

$s\,\hat{\mathbf{u}}$ can have a different ***magnitude*** than $\hat{\mathbf{u}}$.	If **true** $s =$ ___,	**draw** $\vec{a}$.
$s\,\hat{\mathbf{u}}$ can have a different ***direction*** than $\hat{\mathbf{u}}$.	If **true** $s =$ ___,	**draw** $\vec{b}$.
$s\,\hat{\mathbf{u}}$ can have different ***magnitude and direction*** than $\hat{\mathbf{u}}$.	If **true** $s =$ ___,	**draw** $\vec{c}$.



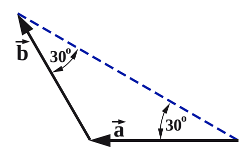
1.10 ♣ Graphical vector addition/subtraction. (Sections 2.6, 2.8)



Draw $\vec{a} + \vec{b}$ | **Draw** $\vec{b} + \vec{a}$ | **Draw** $\vec{a} + -\vec{b}$ | **Draw** $\vec{b} - \vec{a}$ | **Draw** $-\vec{a} - \vec{b}$

1.11 ♣ Angle $\angle(\vec{a}, \vec{b})$ between vectors. (Section 2.9)

For the figure shown right, determine the numerical value for the angle between vector $\vec{a}$ and vector $\vec{b}$.



Result: $\angle(\vec{a},\ \vec{b}) =$ ___ °

1.12 ♣ Visual representation of a vector dot-product. (Section 2.9)

Write the ***definition*** of the dot-product of a vector $\vec{a}$ with a vector $\vec{b}$.
Include a ***sketch*** with ***each symbol*** in your definition clearly labeled.

Result: $\vec{a} \cdot \vec{b} \triangleq$ ___ ___ ___

Knowing $\vec{a}$ and $\vec{b}$ are arbitrary vectors, complete the blanks with $\leq$, $=$, or $\geq$.

When the angle between $\vec{a}$ and $\vec{b}$ is 0°	$\vec{a} \cdot \vec{b}$	___	0	(**parallel**)
When the angle between $\vec{a}$ and $\vec{b}$ is 90°	$\vec{a} \cdot \vec{b}$	___	0	(**perpendicular**)
When the angle between $\vec{a}$ and $\vec{b}$ is 180°	$\vec{a} \cdot \vec{b}$	___	0	(**antiparallel**)
For arbitrary vectors $\vec{a}$ and $\vec{b}$,	$\vec{a} \cdot \vec{b}$	___	$\vec{b} \cdot \vec{a}$	

Sketch should include $\vec{a}$, $\vec{b}$, $|\vec{a}|$, $|\vec{b}|$, θ.

1.13 ♣ Visual representation of a vector cross-product. (Section 2.10)

Write the ***definition*** of the cross-product of a vector $\vec{a}$ with a vector $\vec{b}$.
Include a ***sketch*** with ***each symbol*** in your definition clearly labeled.

Result: $\vec{a} \times \vec{b} \triangleq$ ___ ___ ___$(\theta)\ \hat{\mathbf{u}}$

where $\hat{\mathbf{u}}$ is ______________________________

______________________ and θ is ______________________

Knowing $\vec{a}$ and $\vec{b}$ are non-zero vectors, complete the blanks with $=$ or $\neq$.

When the angle between $\vec{a}$ and $\vec{b}$ is 0°	$\vec{a} \times \vec{b}$	___	$\vec{0}$	(**parallel**)
When the angle between $\vec{a}$ and $\vec{b}$ is 90°	$\vec{a} \times \vec{b}$	___	$\vec{0}$	(**perpendicular**)
When the angle between $\vec{a}$ and $\vec{b}$ is 180°	$\vec{a} \times \vec{b}$	___	$\vec{0}$	(**antiparallel**)
For arbitrary vectors $\vec{a}$ and $\vec{b}$,	$\vec{a} \times \vec{b}$	___	$\vec{b} \times \vec{a}$	

Sketch should include $\vec{a}$, $\vec{b}$, $|\vec{a}|$, $|\vec{b}|$, θ, $\hat{\mathbf{u}}$.

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1.14 Properties of vector dot/cross-products **Draw/show work.** $\vec{a} \neq \vec{0}$, $\vec{b} \neq \vec{0}$. **(Sections 2.9.1, 2.10)**

When $\vec{a}$ is ***parallel*** to $\vec{b}$,	$\vec{a} \cdot \vec{b} = 0$ True/False	$\vec{a} \times \vec{b} = \vec{0}$ True/False
When $\vec{a}$ is ***perpendicular*** to $\vec{b}$,	$\vec{a} \cdot \vec{b} = 0$ True/False	$\vec{a} \times \vec{b} = \vec{0}$ True/False
For arbitrary vectors $\vec{a}$ and $\vec{b}$,	$\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$ True/False	$\vec{a} \times \vec{b} = \vec{b} \times \vec{a}$ True/False

1.15 Dot-products and cross-products via definitions. **Show work.** **(Sections 2.9, 2.10)**

- **Draw** a unit vector $\widehat{\mathbf{k}}$ outward-normal to the plane of the paper (perpendicular to $\vec{a}$ and $\vec{b}$).
- **Redraw** each figure to clarify $\angle(\vec{a}, \vec{b})$, the angle between $\vec{a}$ and $\vec{b}$ (useful for dot and cross-product).
- Knowing $|\vec{a}| = 2$ and $|\vec{b}| = 4$, calculate each expressions below (2^+ significant digits) using only the definitions of dot-product and cross-product.

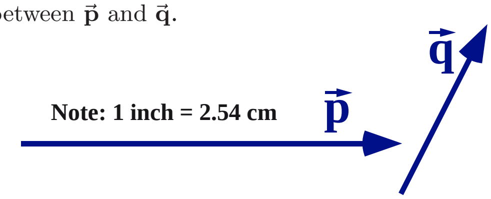
$\angle(\vec{a}, \vec{b}) = $ ___°	$\angle(\vec{a}, \vec{b}) = $ ___°	$\angle(\vec{a}, \vec{b}) = $ ___°	$\angle(\vec{a}, \vec{b}) = $ ___°
$\vec{a} \cdot \vec{b} = $ ___	$\vec{a} \cdot \vec{b} = $ ___	$\vec{a} \cdot \vec{b} = $ ___	$\vec{a} \cdot \vec{b} = $ ___
$\vec{a} \times \vec{b} = -8\,\widehat{\mathbf{k}}$	$\vec{a} \times \vec{b} = $ ___	$\vec{a} \times \vec{b} = $ ___	$\vec{a} \times \vec{b} = $ ___

1.16 Visual estimation of vector dot/cross-products. **Show work.** **(Sections 2.9, 2.10)**

Estimate the magnitude of the vector $\vec{q}$ shown below, the angle between $\vec{p}$ and $\vec{q}$, $\vec{p} \cdot \vec{q}$, and the magnitude of $\vec{p} \times \vec{q}$. **Show work** and **redraw** to clarify the angle between $\vec{p}$ and $\vec{q}$.

Result: (Provide numerical results with 1 or more significant digits).

$|\vec{p}| \approx$ 4.0 cm	$|\vec{q}| \approx$ ___ cm	$\angle(\vec{p}, \vec{q}) \approx$ ___°
$\vec{p} \cdot \vec{q} \approx$ ___ cm^2		$|\vec{p} \times \vec{q}| \approx$ ___ cm^2



1.17 ♣ Vector operations and units. **(Chapter 2)**

Each vector operation below involves a position vector $\vec{r}$ (with **units** of m) and/or a velocity vector $\vec{v}$ (with **units** of $\frac{m}{s}$). Determine whether the operation produces a well-defined scalar or vector or is **undefined**. If well-defined, determine the associated units.

Operation:	$-\vec{r}$	$5\,\vec{v}$	$5\,\frac{m}{s} + \vec{v}$	$\vec{r} + 2\,\vec{r}$	$\vec{r} + \vec{v}$	$5\,\frac{m}{s} \cdot \vec{v}$	$\vec{r} \cdot \vec{v}$	$\vec{r} \times \vec{v}$
Produces:	vector							
Units:	meters							

1.18 ♣ Vector exponentiation: $\vec{v}^2 = \vec{v} \cdot \vec{v}$ and $\vec{v}^3$. **(Section 2.9)**

The following is a reasonable proof that $\vec{v}^2 = \vec{v} \cdot \vec{v}$. **True/False** (if **False**, provide a proof).

$$\vec{v}^2 \triangleq |\vec{v}|^2 \qquad \vec{v} \cdot \vec{v} \underset{(2.2)}{\triangleq} |\vec{v}|\,|\vec{v}| \cos(0^\circ) = |\vec{v}|^2 \qquad \vec{v}^2 = \vec{v} \cdot \vec{v}$$

Complete the proof that relates $\vec{v}^3$ to $\vec{v} \cdot \vec{v}$ raised to a real number.

Result:

$$|\vec{v}| \underset{(2.4)}{=} \sqrt{\square \cdot \square} \qquad \vec{v}^3 \triangleq |\vec{v}|^{\square} = \left(\sqrt{\square \cdot \square}\right)^{\square} = (\vec{v} \cdot \vec{v})^{\frac{3}{2}}$$

1.19 ♣ $|c\,\widehat{\mathbf{a}}_x|$ Calculate vector magnitude with dot products. **(Section 2.9 and Hw 1.18)**

Show how the vector dot-product can be used to show that the magnitude of the vector $c\,\widehat{\mathbf{a}}_x$ (c is a positive or **negative** number and $\widehat{\mathbf{a}}_x$ is a unit vector) can be written solely in terms of c (without $\widehat{\mathbf{a}}_x$).

Result:

$$|c\,\widehat{\mathbf{a}}_x| = {}^{+}\sqrt{\square \cdot \square} = {}^{+}\sqrt{c^2 * \square \cdot \square} = {}^{+}\sqrt{c^2} = \text{abs}(c)$$

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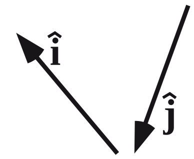
1.20 †(Challenge) **Magnitude of the vector $\vec{\mathbf{v}}$.** Show work. (Section 2.9)



Knowing the angle between a unit vector $\hat{\mathbf{i}}$ and unit vector $\hat{\mathbf{j}}$ is 120°, calculate a numerical value for the magnitude of $\vec{\mathbf{v}} = 3\hat{\mathbf{i}} + 4\hat{\mathbf{j}}$.

Result: $|\vec{\mathbf{v}}| = \sqrt{13}$ Note: The answer is ***not*** $\sqrt{25} = 5$.

1.21 ♣ **Property of scalar triple product.** (Section 2.11)

For arbitrary non-zero vectors $\vec{\mathbf{a}}, \vec{\mathbf{b}}, \vec{\mathbf{c}}$: $\vec{\mathbf{a}} \cdot (\vec{\mathbf{b}} \times \vec{\mathbf{c}}) = (\vec{\mathbf{a}} \times \vec{\mathbf{b}}) \cdot \vec{\mathbf{c}}$ **Never/Sometimes/Always**

A property of the ***scalar triple product*** is $\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} \times \vec{\mathbf{a}} = 0$. **True/False**.

1.22 ♣ **Property of vector triple cross-product.** (Sections 2.10, 2.11)

Complete the following equation: $\vec{\mathbf{a}} \times (\vec{\mathbf{b}} \times \vec{\mathbf{c}}) = \vec{\mathbf{b}}(\_\_\_\_) - \vec{\mathbf{c}}(\_\_\_\_)$

For arbitrary vectors $\vec{\mathbf{a}}, \vec{\mathbf{b}}, \vec{\mathbf{c}}$: $\vec{\mathbf{a}} \times (\vec{\mathbf{b}} \times \vec{\mathbf{c}}) = (\vec{\mathbf{a}} \times \vec{\mathbf{b}}) \times \vec{\mathbf{c}} + \vec{\mathbf{b}} \times (\vec{\mathbf{a}} \times \vec{\mathbf{c}})$ **True/False** (show work).

1.23 ♣ **Form the *unit* vector $\hat{\mathbf{u}}$ having the same direction as $c\,\hat{\mathbf{a}}_x$.** (Section 2.4)

Result: $\hat{\mathbf{u}} = \dfrac{\_\_\_\_}{\_\_\_\_}\,\hat{\mathbf{a}}_x$ Note: $\hat{\mathbf{a}}_x$ is a unit vector and c is a non-zero real number, e.g., 3 or -3.

1.24 ♣ **Coefficient of $\hat{\mathbf{u}}$ in cross products – definitions and trig functions.** (Section 2.10)

The ***cross product*** of vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ can be written in terms of a real scalar s as $\vec{\mathbf{a}} \times \vec{\mathbf{b}} = s\,\hat{\mathbf{u}}$ where $\hat{\mathbf{u}}$ is a unit vector perpendicular to both $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ in a direction defined by the **right-hand rule**. The coefficient s of the unit vector $\hat{\mathbf{u}}$ is inherently non-negative. **True/False**.

1.25 ♣ **Ranges of angles from dot-product and cross-product calculations.** (Sections 2.9, 2.10)

Quantity	Numerical range of values
$c = \hat{\mathbf{a}} \cdot \hat{\mathbf{b}}$ (assume $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ are known so a numerical value for $\hat{\mathbf{a}} \cdot \hat{\mathbf{b}}$ can be calculated).	____ $\leq c \leq$ ____
$s = \lvert\hat{\mathbf{a}} \times \hat{\mathbf{b}}\rvert$ (assume $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ are known so a numerical value for $\lvert\hat{\mathbf{a}} \times \hat{\mathbf{b}}\rvert$ can be calculated).	____ $\leq s \leq$ ____
Angle θ_c between $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ that can be uniquely determined **solely** from c.	____° $\leq \theta_c \leq$ ____°
Angle θ_s between $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ that can be uniquely determined **solely** from s.	____° $\leq \theta_s \leq$ ____°
Angle θ between $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$, i.e., $\theta = \angle(\hat{\mathbf{a}}, \hat{\mathbf{b}})$	____° $\leq \theta \leq$ ____°

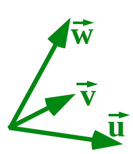
Note: The range of θ_s is smaller than the range for θ. Hence, s and θ_s are insufficient to correctly calculate θ.
What this means: Use the **dot-product** $\cdot$ to calculate an angle θ from two given/known vectors $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$.

1.26 ♣ **Using vector identities to simplify expressions** (refer to Homework 1.14)

One reason to treat vectors as ***basis-independent*** quantities is to simplify vector expressions ***without*** resolving the vectors into orthogonal "$\vec{\mathbf{x}}, \vec{\mathbf{y}}, \vec{\mathbf{z}}$" or "$\vec{\mathbf{i}}, \vec{\mathbf{j}}, \vec{\mathbf{k}}$" components. Simplify the following vector expressions using mathematical properties of dot-products and cross-products.

Express results in terms of dot-products $\cdot$ and cross-products $\times$ of the arbitrary vectors $\vec{\mathbf{u}}, \vec{\mathbf{v}}, \vec{\mathbf{w}}$.

$\vec{\mathbf{u}}, \vec{\mathbf{v}}, \vec{\mathbf{w}}$ are not necessarily orthogonal or co-planar.



Vector expression	Simplified vector expression
$(3\,\vec{\mathbf{u}} - 2\,\vec{\mathbf{v}}) \times (\vec{\mathbf{u}} + \vec{\mathbf{v}})$	____ $\vec{\mathbf{u}} \times \vec{\mathbf{v}}$
$(3\,\vec{\mathbf{u}} - 2\,\vec{\mathbf{v}}) \cdot (\vec{\mathbf{u}} + \vec{\mathbf{v}})$	____ $\vec{\mathbf{u}}^2$ − ____ $\vec{\mathbf{v}}^2$ + ____ $\vec{\mathbf{u}} \cdot \vec{\mathbf{v}}$
$(\vec{\mathbf{u}} - \vec{\mathbf{v}}) \cdot (\vec{\mathbf{u}} + \vec{\mathbf{v}})$	____ − ____
$(3\,\vec{\mathbf{u}} - 2\,\vec{\mathbf{v}}) \times (\vec{\mathbf{u}} + \vec{\mathbf{v}}) \cdot (2\,\vec{\mathbf{u}} - 7\,\vec{\mathbf{v}})$	____
$(\vec{\mathbf{u}} + \vec{\mathbf{v}}) \times (\vec{\mathbf{v}} + 2\,\vec{\mathbf{w}}) \cdot (\vec{\mathbf{w}} + 2\,\vec{\mathbf{u}})$	____ $\vec{\mathbf{u}} \times \vec{\mathbf{v}} \cdot \vec{\mathbf{w}}$

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1.27 ♣ Vector concepts: Solving a vector equation? **(Section 2.9.3)**

Shown right is a vector equation and a questionable process that solves for v_x ($\widehat{\mathbf{a}}_{\text{x}}$ is a unit vector and v_x, $\dot{\theta}$, R are scalars).

$$v_x\,\widehat{\mathbf{a}}_{\text{x}} \;=\; \dot{\theta}\,R\,\widehat{\mathbf{a}}_{\text{x}}$$

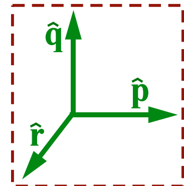
$$v_x \;=\; \dot{\theta}\,R\,\frac{\widehat{\mathbf{a}}_{\text{x}}}{\widehat{\mathbf{a}}_{\text{x}}} \;=\; \dot{\theta}\,R$$

This is a valid process to solve for v_x. **True/False**.

Explain: ______

1.28 Change a vector equation to scalar equations. **Show work.** **(Section 2.9.3)**

Shown right are three mutually orthogonal unit vectors $\widehat{\mathbf{p}}$, $\widehat{\mathbf{q}}$, $\widehat{\mathbf{r}}$.



Use a vector operation (e.g., $+$, $*$, $\cdot$, $\times$) to change the **vector** equation $(2x-4)\,\widehat{\mathbf{p}} \;=\; \vec{\mathbf{0}}$ into **one scalar** equation and subsequently solve the scalar equation for x.

Result: $$(2x-4)\,\widehat{\mathbf{p}} \;=\; \vec{\mathbf{0}} \quad \overset{??}{\Rightarrow} \quad (2x-4) \;=\; 0 \quad \Rightarrow \quad x \;=\; 2$$

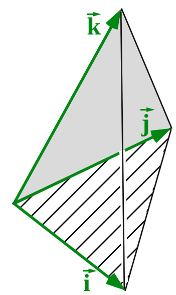
Show ***every*** vector operation (e.g., $+$, $*$, $\cdot$, $\times$) that changes the following **vector** equation into **three scalar** equations and subsequently solve the scalar equations for x, y, z.

$$(2\,x-4)\,\widehat{\mathbf{p}} \;+\; (3\,y-9)\,\widehat{\mathbf{q}} \;+\; (4\,z-16)\,\widehat{\mathbf{r}} \;=\; \vec{\mathbf{0}}$$

Result: $(2\,x-4) \;=\; 0$ $\quad (3\,y-9) \;=\; 0$ $\quad ($ ______ $) \;=\; 0$

$x \;=\; 2$ $\quad y \;=\; 3$ $\quad z \;=\; 4$

†Optional: The figure to the right shows three **non-orthogonal**, non-coplanar vectors $\vec{\mathbf{i}}, \vec{\mathbf{j}}, \vec{\mathbf{k}}$. Show **every** vector operation that changes the following **vector** equation into **three** uncoupled **scalar** equations and subsequently solve those scalar equations for x, y, z.



$$(2\,x-4)\,\vec{\mathbf{i}} \;+\; (3\,y-9)\,\vec{\mathbf{j}} \;+\; (4\,z-16)\,\vec{\mathbf{k}} \;=\; \vec{\mathbf{0}}$$

Result: $(2\,x-4) \;=\; 0$ $\quad (3\,y-9) \;=\; 0$ $\quad ($ ______ $) \;=\; 0$ $\quad$ Hint: think $\times$ $\cdot$, not matrix algebra.

$x \;=\; 2$ $\quad y \;=\; 3$ $\quad z \;=\; 4$

1.29 ♣ Number of independent scalar equations from 1 vector equation. **(Section 2.9.3)**

The **vector** equation shown right is useful for static analyses of a system S.

$$\vec{\mathbf{F}}^{S} \;=\; \vec{\mathbf{0}}$$

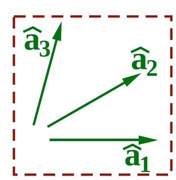
In the table to the right, box **all** integers that could be equal to the number of ***independent scalar*** equations produced by the previous vector equation. Hint: Hw 1.28. Related Hw 7.10.

System type	Integer(s)
1D (line)	0 1 2 3 4^+
2D (planar)	0 1 2 3 4^+
3D (spatial)	0 1 2 3 4^+

Note: 1D/linear means $\vec{\mathbf{F}}^{S}$ can be expressed in terms of one vector $\widehat{\mathbf{i}}$.
2D/planar means $\vec{\mathbf{F}}^{S}$ can be expressed in terms of two non-parallel unit vectors $\widehat{\mathbf{i}}$ and $\widehat{\mathbf{j}}$.
3D/spatial means $\vec{\mathbf{F}}^{S}$ can be expressed in terms of three non-coplanar unit vectors $\widehat{\mathbf{i}}$, $\widehat{\mathbf{j}}$, $\widehat{\mathbf{k}}$.

1.30 ♣ Vector concepts: Solving a vector equation **(just circle true or false and fill-in the blank).**

Consider the following vector equation written in terms of the scalars x, y, z and three unique non-orthogonal **coplanar** unit vectors $\widehat{\mathbf{a}}_1$, $\widehat{\mathbf{a}}_2$, $\widehat{\mathbf{a}}_3$.



$$(2\,x-4)\,\widehat{\mathbf{a}}_1 \;+\; (3\,y-9)\,\widehat{\mathbf{a}}_2 \;+\; (4\,z-16)\,\widehat{\mathbf{a}}_3 \;=\; \vec{\mathbf{0}}$$

The **unique** solution to this vector equation is $x=2$, $y=3$, $z=4$. **True/False**.

Explain: $\widehat{\mathbf{a}}_2$ can be expressed in terms of $\widehat{\mathbf{a}}_1$ and $\widehat{\mathbf{a}}_3$ (i.e., $\widehat{\mathbf{a}}_2$ is a linear combination of $\widehat{\mathbf{a}}_1$ and $\widehat{\mathbf{a}}_3$). Hence the vector equation produces ______ linearly independent scalar equations.

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1.31 ♣ Gibbs (≈1900 AD) **vectors revolutionizes Euclidean geometry** (300 BC). **(Sections 2.9.2, 2.10.1, 2.11.1)**

For each geometrical quantity shown right, circle the vector operation(s) (dot-product, cross-product, or both) that is ***most*** useful for their calculation.

Length: $\cdot$ $\times$	Angle: $\cdot$ $\times$
Area: $\cdot$ $\times$	Volume: $\boxed{\cdot}$ $\boxed{\times}$

1.32 ♣ Order of operations with vector dot products (•) and cross products (×). (Chapter 2)

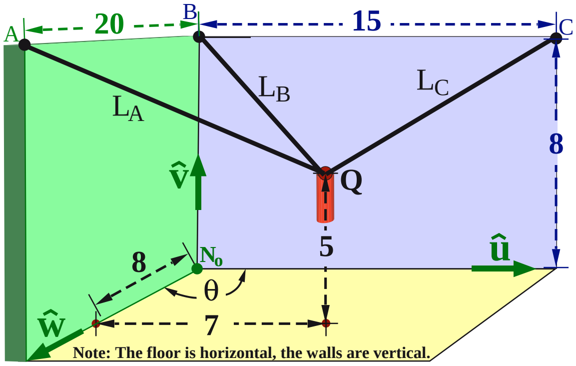
Create a valid expression by adding parentheses to each expression or ~~cross-out~~ the expression if it is inherently invalid.
Example: $3 * \vec{\mathbf{a}} + \vec{\mathbf{b}} \;\Rightarrow\; (3 * \vec{\mathbf{a}}) + \vec{\mathbf{b}}$.

$\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} + \vec{\mathbf{c}}$	$\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} \times \vec{\mathbf{c}}$	$\vec{\mathbf{a}} + 5 \times \vec{\mathbf{c}}$
$\vec{\mathbf{a}} \times \vec{\mathbf{b}} + \vec{\mathbf{c}}$	$\vec{\mathbf{a}} \times \vec{\mathbf{b}} \cdot \vec{\mathbf{c}}$	$\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} \cdot \vec{\mathbf{c}}$

1.33 † Microphone cable lengths (non-orthogonal walls) **"It's just geometry".** **Show work.**

A microphone Q is attached to three pegs A, B, C by three cables. Knowing the peg locations, microphone location, and the angle θ between the vertical walls, express L_A, L_B, L_C solely in terms of numbers and θ. Next, complete the table by calculating L_B when $\theta = 120°$.

Hint: To do this **efficiently**, use only unit vectors $\hat{\mathbf{u}}$, $\hat{\mathbf{v}}$, $\hat{\mathbf{w}}$.
Hint: Use the distributive property of the vector dot-product as shown in Section 2.9.1 and Homework 2.4.
Note: Synthesis problems are difficult. Think, talk, draw, sleep, walk, get help, ... (if necessary, read Section 3.4).



Distance between A and B	20 m
Distance between B and C	15 m
Distance between N_o and B	8 m
Distance along back wall (see picture)	7 m
Q's height above N_o	5 m
Distance along side wall (see picture)	8 m
L_A: Length of cable joining A and Q	16.9 m
L_B: Length of cable joining B and Q	8.1 m
L_C: Length of cable joining C and Q	14.2 m

$${}^{N_o}\vec{\mathbf{r}}^{Q} = 7\,\hat{\mathbf{u}} + 5\,\hat{\mathbf{v}} + 8\,\hat{\mathbf{w}}$$

Result: $L_A = \sqrt{202 - \boxed{}\cos(\theta)}$ $\quad L_B = \sqrt{122 + 112\cos(\theta)}$ $\quad L_C = \sqrt{\boxed{} - 128\,\boxed{}}$

Vocabulary: In this ***inverse kinematics*** analysis, the position of "end-effector" Q is known and you determine the cable lengths.

- Knowing β is defined as the angle between lines $\overline{B\,N_o}$ and $\overline{B\,Q}$, show $\beta \approx \boxed{68.33°}$.

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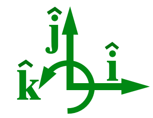
Show work – except for ♣ fill-in-blanks (print .pdf from www.MotionGenesis.com ⇒ Textbooks ⇒ Resources).

2.1 ♣ Right-handed, orthogonal, unit vectors. (Section 4.1)

Draw a set of right-handed orthogonal (mutually perpendicular) unit vectors consisting of $\hat{\mathbf{n}}_x$, $\hat{\mathbf{n}}_y$, $\hat{\mathbf{n}}_z$. In other words, draw $\hat{\mathbf{n}}_x$, $\hat{\mathbf{n}}_y$, $\hat{\mathbf{n}}_z$ so that $\hat{\mathbf{n}}_y$ is perpendicular (orthogonal) to $\hat{\mathbf{n}}_x$ and $\hat{\mathbf{n}}_z = \hat{\mathbf{n}}_x \times \hat{\mathbf{n}}_y$.

2.2 ♣ Adding and subtracting vectors. (Sections 2.6, 2.8)



Given: Vectors $\vec{\mathbf{p}}$ and $\vec{\mathbf{q}}$ expressed in terms of unit vectors $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, $\hat{\mathbf{k}}$. Form the vector sums and differences below.

$$\vec{\mathbf{p}} = a\,\hat{\mathbf{i}} + b\,\hat{\mathbf{j}} + c\,\hat{\mathbf{k}}$$
$$\vec{\mathbf{q}} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$$

Result: $\vec{\mathbf{p}} + \vec{\mathbf{q}} = (a+x)\,\hat{\mathbf{i}} + (\square)\,\hat{\mathbf{j}} + (\square)\,\hat{\mathbf{k}} \qquad \vec{\mathbf{p}} - \vec{\mathbf{q}} = (a-x)\,\hat{\mathbf{i}} + (\square)\,\hat{\mathbf{j}} + (\square)\,\hat{\mathbf{k}}$

2.3 ♣ Words: Physical vectors and column matrices. (Section 2.1, Hw 1.2)

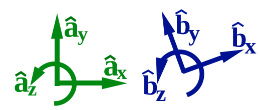
True/False As defined by Gibbs and for $\vec{\mathbf{F}} = m\,\vec{\mathbf{a}}$, physical vectors have magnitude and direction.

True/False In math (linear algebra), a column matrix is called a "vector".

True/False The physical vector $\hat{\mathbf{a}}_x + 2\,\hat{\mathbf{a}}_y + 3\,\hat{\mathbf{a}}_z$ can be written $[\hat{\mathbf{a}}_x \;\; \hat{\mathbf{a}}_y \;\; \hat{\mathbf{a}}_z] * \begin{bmatrix}1\\2\\3\end{bmatrix}$.
Note: $\hat{\mathbf{a}}_x$, $\hat{\mathbf{a}}_y$, $\hat{\mathbf{a}}_z$ are the orthogonal unit vectors shown below.

True/False The physical vector $\hat{\mathbf{a}}_x + 2\,\hat{\mathbf{a}}_y + 3\,\hat{\mathbf{a}}_z$ is equal to the column matrix $\begin{bmatrix}1\\2\\3\end{bmatrix}$.

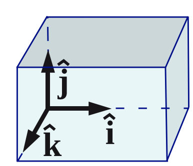
True/False $\hat{\mathbf{a}}_x + 2\,\hat{\mathbf{a}}_y + 3\,\hat{\mathbf{a}}_z \;+\; 4\,\hat{\mathbf{b}}_x + 5\,\hat{\mathbf{b}}_y + 6\,\hat{\mathbf{b}}_z \;=\; \begin{bmatrix}1\\2\\3\end{bmatrix} + \begin{bmatrix}4\\5\\6\end{bmatrix} = \begin{bmatrix}5\\7\\9\end{bmatrix}$
($\hat{\mathbf{a}}_x$, $\hat{\mathbf{a}}_y$, $\hat{\mathbf{a}}_z$ and $\hat{\mathbf{b}}_x$, $\hat{\mathbf{b}}_y$, $\hat{\mathbf{b}}_z$ are shown right).



- Complete the following statement with one equal sign $\boxed{=}$ and one not-equal sign $\boxed{\neq}$.

$$\hat{\mathbf{a}}_x + 2\,\hat{\mathbf{a}}_y + 3\,\hat{\mathbf{a}}_z \;\square\; [\hat{\mathbf{a}}_x \;\; \hat{\mathbf{a}}_y \;\; \hat{\mathbf{a}}_z] * \begin{bmatrix}1\\2\\3\end{bmatrix} \;\square\; \begin{bmatrix}1\\2\\3\end{bmatrix}$$

2.4 ♣ Dot products with orthogonal unit vectors. (Sections 2.9, 2.9.4)



Given: Vectors $\vec{\mathbf{v}}$ and $\vec{\mathbf{w}}$ expressed in terms of right-handed orthogonal unit vectors $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, $\hat{\mathbf{k}}$, with: $\vec{\mathbf{v}} \cdot \vec{\mathbf{w}} = \underbrace{(a\,\hat{\mathbf{i}} + b\,\hat{\mathbf{j}} + c\,\hat{\mathbf{k}})}_{\vec{\mathbf{v}}} \cdot \underbrace{(x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}})}_{\vec{\mathbf{w}}}$

- Use the ***distributive property*** for dot products to write $\vec{\mathbf{v}} \cdot \vec{\mathbf{w}}$ in terms of $\hat{\mathbf{i}} \cdot \hat{\mathbf{i}}$, $\hat{\mathbf{i}} \cdot \hat{\mathbf{j}}$, etc. Next, use the ***definition*** of the dot product to calculate $\hat{\mathbf{i}} \cdot \hat{\mathbf{i}}$, $\hat{\mathbf{i}} \cdot \hat{\mathbf{j}}$, etc. (below-right).

Result:

$$\begin{aligned}\vec{\mathbf{v}} \cdot \vec{\mathbf{w}} = \; & a\,x\;\hat{\mathbf{i}} \cdot \hat{\mathbf{i}} + a\,y\;\hat{\mathbf{i}} \cdot \hat{\mathbf{j}} + \square\;\hat{\mathbf{i}} \cdot \hat{\mathbf{k}} \\ & + b\,x\;\hat{\mathbf{j}} \cdot \hat{\mathbf{i}} + b\,y\;\square \cdot \square + \square\;\square \cdot \square \\ & + c\,x\;\hat{\mathbf{k}} \cdot \hat{\mathbf{i}} + c\,y\;\square \cdot \square + \square\;\square \cdot \square\end{aligned}$$

$\hat{\mathbf{i}} \cdot \hat{\mathbf{i}} = 1$	$\hat{\mathbf{i}} \cdot \hat{\mathbf{j}} = \square$	$\hat{\mathbf{i}} \cdot \hat{\mathbf{k}} = \square$
$\hat{\mathbf{j}} \cdot \hat{\mathbf{i}} = 0$	$\hat{\mathbf{j}} \cdot \hat{\mathbf{j}} = \square$	$\hat{\mathbf{j}} \cdot \hat{\mathbf{k}} = \square$
$\hat{\mathbf{k}} \cdot \hat{\mathbf{i}} = 0$	$\hat{\mathbf{k}} \cdot \hat{\mathbf{j}} = \square$	$\hat{\mathbf{k}} \cdot \hat{\mathbf{k}} = \square$

- Simplify $\vec{\mathbf{v}} \cdot \vec{\mathbf{w}}$ and use its special dot-product formula for the calculations that follow.

Result: $\boxed{\vec{\mathbf{v}} \cdot \vec{\mathbf{w}} = a\,x + b\,y + \square}$ Use this special dot-product formula to calculate $\vec{\mathbf{v}} \cdot \vec{\mathbf{w}}$ when $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, $\hat{\mathbf{k}}$ are **orthogonal unit** vectors.

Given	Calculate		
$\vec{\mathbf{p}} = 2\,\hat{\mathbf{i}} + 3\,\hat{\mathbf{j}} + 4\,\hat{\mathbf{k}}$	$\vec{\mathbf{p}} \cdot \vec{\mathbf{q}} = 2\,x + 3\,y + \square\,z$	$\vec{\mathbf{p}} \cdot \vec{\mathbf{p}} = 29$	$\lvert\vec{\mathbf{p}}\rvert = \sqrt{29}$
$\vec{\mathbf{q}} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$	$\vec{\mathbf{p}} \cdot \vec{\mathbf{r}} = \square$	$\vec{\mathbf{q}} \cdot \vec{\mathbf{q}} = x^2 + \square + \square$	$\lvert\vec{\mathbf{q}}\rvert = \sqrt{\square}$
$\vec{\mathbf{r}} = 5\,\hat{\mathbf{i}} - 6\,\hat{\mathbf{j}} + 7\,\hat{\mathbf{k}}$	$\vec{\mathbf{q}} \cdot \vec{\mathbf{r}} = \square$	$\vec{\mathbf{r}} \cdot \vec{\mathbf{r}} = \square$	$\lvert\vec{\mathbf{r}}\rvert = \sqrt{110}$

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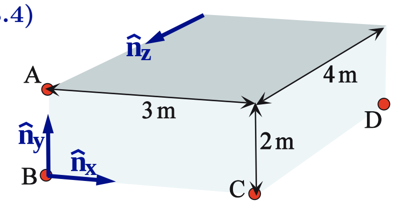
2.5 ♣ Perpendicular vectors. **(Note: $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ are orthogonal unit vectors). (Section 2.9)**

Draw two vectors $\vec{\mathbf{v}}$ and $\vec{\mathbf{w}}$ that are perpendicular. Hence, $\vec{\mathbf{v}} \cdot \vec{\mathbf{w}} = \square$.
When $\vec{\mathbf{v}} = x\,\hat{\mathbf{i}} + 2\,\hat{\mathbf{j}} + 3\,\hat{\mathbf{k}}$ is perpendicular to $\vec{\mathbf{w}} = 4\,\hat{\mathbf{i}} + 5\,\hat{\mathbf{j}} + 6\,\hat{\mathbf{k}}$, $x = \square$.

2.6 Dot products to calculate distance and angles. **(Sections 2.9, 3.4)**

The figure to the right shows a block with sides of length 2 m, 3 m, 4 m and points A, B, C located at corners. Right-handed orthogonal unit vectors $\hat{\mathbf{n}}_x$, $\hat{\mathbf{n}}_y$, $\hat{\mathbf{n}}_z$ are directed with $\hat{\mathbf{n}}_x$ from B to C and $\hat{\mathbf{n}}_y$ from B to A.



(a) Express $\vec{\mathbf{r}}$ (position from A to C) in terms of $\hat{\mathbf{n}}_x$, $\hat{\mathbf{n}}_y$, $\hat{\mathbf{n}}_z$ and find a numerical value for $|\vec{\mathbf{r}}|^2$. Next calculate the distance d between A to C (magnitude of $\vec{\mathbf{r}}$).
Result: $\vec{\mathbf{r}} = \square\,\hat{\mathbf{n}}_x - \square\,\hat{\mathbf{n}}_y$ $\quad |\vec{\mathbf{r}}|^2 \underset{(2.4)}{=} \vec{\mathbf{r}} \cdot \vec{\mathbf{r}} = \square\ \text{m}^2$ $\quad \text{d} = \sqrt{\square}\ \text{m}$

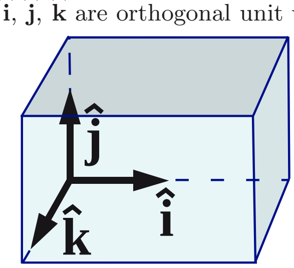
(b) Calculate the unit vector $\hat{\mathbf{u}}$ directed from A to C and the unit vector $\hat{\mathbf{v}}$ directed from A to D.
Result:
$$\hat{\mathbf{u}} = \frac{3\,\hat{\mathbf{n}}_x - \square\,\hat{\mathbf{n}}_y}{\sqrt{\square}} \qquad \hat{\mathbf{v}} = \frac{\square\,\hat{\mathbf{n}}_x - \square\,\hat{\mathbf{n}}_y - \square\,\hat{\mathbf{n}}_z}{\sqrt{\square}}$$

(c) Calculate $\angle BAC$ (angle between line $\overline{AB}$ and line $\overline{AC}$) and $\angle CAD$ (angle between line $\overline{AC}$ and line $\overline{AD}$).
Result: $\angle BAC = \square^\circ$ $\qquad \angle CAD = \boxed{47.97^\circ}$

2.7 ♣ Construct a unit vector $\hat{\mathbf{u}}$ in the direction of each vector given below. **(Section 2.9.2)**

Vector	Unit vector $\hat{\mathbf{u}}$
$3\,\hat{\mathbf{i}}$	$\hat{\mathbf{i}}$
$-3\,\hat{\mathbf{i}}$	___
$3\,\hat{\mathbf{i}} - 4\,\hat{\mathbf{j}}$	$\frac{\square - \square}{\square}$
$3\,\hat{\mathbf{i}} - 4\,\hat{\mathbf{j}} + 12\,\hat{\mathbf{k}}$	$\frac{\square}{\square}$
$c\,\hat{\mathbf{i}}$ c is a real non-zero number	$\frac{c}{\square}\,\hat{\mathbf{i}}$ or $\text{sign}(c)\,\hat{\mathbf{i}}$

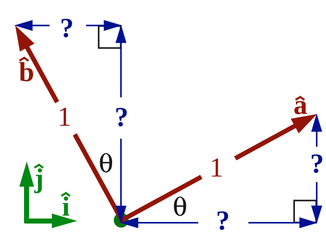
Note: $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ are orthogonal unit vectors.



Ensure your last answer agrees with your first two answers, e.g., if $c = 3$ or $c = -3$.

2.8 ♣ Vector components: Sine and cosine. **(Section 1.4)**

- **Replace** each **?** in the figure to the right with $\sin(\theta)$ or $\cos(\theta)$.
- Use vector addition to express $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ in terms of $\sin(\theta)$, $\cos(\theta)$, $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$.



Result:
$$\hat{\mathbf{a}} = \square\,\hat{\mathbf{i}} + \square\,\hat{\mathbf{j}}$$
$$\hat{\mathbf{b}} = \square\,\hat{\mathbf{i}} + \cos(\theta)\,\hat{\mathbf{j}}$$

Reminder:
SohCahToa

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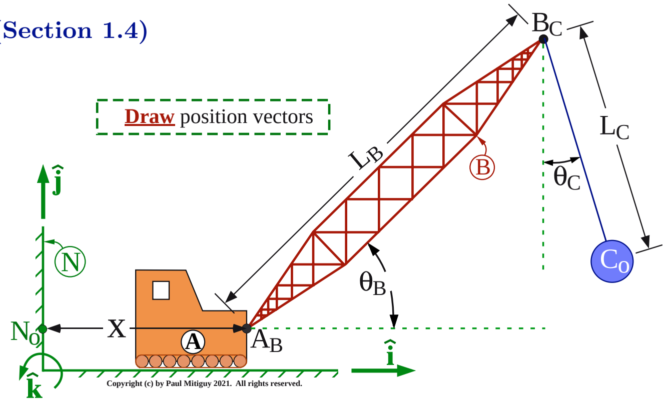
2.9 ♣ **Vector components for a crane-boom.** **(Section 1.4)**

Shown right is a crane whose cab A supports a boom B that swings a wrecking ball C_o.



Right-handed orthogonal unit vectors $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ are directed with $\hat{\mathbf{i}}$ horizontally-right, $\hat{\mathbf{j}}$ vertically-upward, and $\hat{\mathbf{k}}$ outward-normal to the plane containing points N_o, A_B, B_C, C_o.

Draw each position vector listed below and then use your knowledge of sine/cosine to resolve these vectors into $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ components.

Position from N_o to A_B $\quad {}^{N_o}\vec{\mathbf{r}}^{A_B} = \square\,\hat{\mathbf{i}} + \square\,\hat{\mathbf{j}}$

Position from A_B to B_C $\quad {}^{A_B}\vec{\mathbf{r}}^{B_C} = \square\,\hat{\mathbf{i}} + \square\,\hat{\mathbf{j}}$

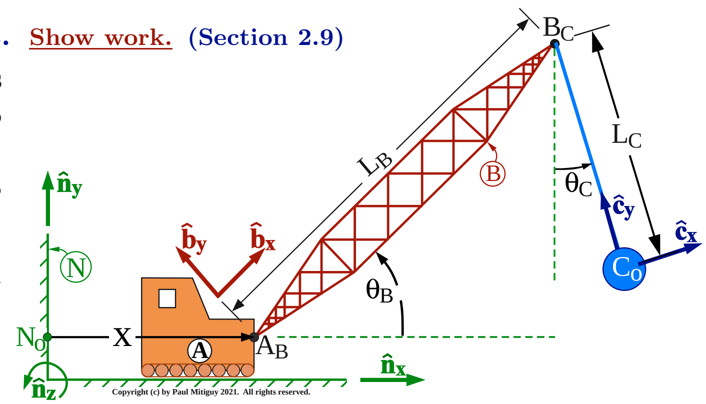
Position from B_C to C_o $\quad {}^{B_C}\vec{\mathbf{r}}^{C_o} = \square\,\hat{\mathbf{i}} + \square\,\hat{\mathbf{j}}$

Position from N_o to B_C $\quad {}^{N_o}\vec{\mathbf{r}}^{B_C} = [\square + \square]\,\hat{\mathbf{i}} + \square\,\hat{\mathbf{j}}$

Position from N_o to C_o $\quad {}^{N_o}\vec{\mathbf{r}}^{C_o} = [\square]\,\hat{\mathbf{i}} + [L_B \sin(\theta_B) - L_C \cos(\theta_C)]\,\hat{\mathbf{j}}$

2.10 Dot products and distance calculations. **Show work.** **(Section 2.9)**

Shown right is a crane whose cab A supports a boom B that swings a wrecking ball C_o. To prevent the wrecking ball from hitting a car, the distance between N_o and point B_C (the tip of the boom) must be controlled.



To start this problem, express $\vec{\mathbf{r}}$ (the position vector from N_o to B_C) in terms of x, L_B, $\hat{\mathbf{n}}_x$, $\hat{\mathbf{b}}_x$.

Result: $\vec{\mathbf{r}} = \square\,\hat{\mathbf{n}}_x + \square\,\hat{\mathbf{b}}_x$

- ***Without*** resolving $\vec{\mathbf{r}}$ into $\hat{\mathbf{n}}_x$ and $\hat{\mathbf{n}}_y$ components, use $|\vec{\mathbf{r}}| = \sqrt{\vec{\mathbf{r}} \cdot \vec{\mathbf{r}}}$ [from equation (3.1)] and the distributive property to calculate the distance between N_o and B_C in terms of x, L_B, θ_B.

Result: (if stumped, hint below).[1] **Optional:** Calculate $|\vec{\mathbf{r}}|$ when $x = 20$ m, $L_B = 10$ m, $\theta_B = 30°$.

$$\text{Distance between } N_o \text{ and } B_C\text{:} \quad |\vec{\mathbf{r}}| = \sqrt{\square^2 + \square^2 + 2\,x\,L_B\,\cos(\theta_B)} \approx 29.1 \text{ m}$$

- Homework 2.9 showed $\vec{\mathbf{r}}$ can be expressed as $\vec{\mathbf{r}} = [x + L_B \cos(\theta_B)]\,\hat{\mathbf{n}}_x + L_B \sin(\theta_B)\,\hat{\mathbf{n}}_y$. Use this expression to verify your previous result for $|\vec{\mathbf{r}}| = \sqrt{\vec{\mathbf{r}} \cdot \vec{\mathbf{r}}}$.

Result: $|\vec{\mathbf{r}}|$ simplifies to the previous result but uses an inefficient process and $\sin^2(\theta_B) + \cos^2(\theta_B) = 1$.

[1] Hint: The distributive property for vector dot-multiplication is $(\vec{\mathbf{a}} + \vec{\mathbf{b}}) \cdot (\vec{\mathbf{c}} + \vec{\mathbf{d}}) = \vec{\mathbf{a}} \cdot \vec{\mathbf{c}} + \vec{\mathbf{a}} \cdot \vec{\mathbf{d}} + \vec{\mathbf{b}} \cdot \vec{\mathbf{c}} + \vec{\mathbf{b}} \cdot \vec{\mathbf{d}}$. Use the distributive property to express $\vec{\mathbf{r}} \cdot \vec{\mathbf{r}}$ in terms of x, L_B, and $\hat{\mathbf{n}}_x \cdot \hat{\mathbf{b}}_x$. Thereafter, use the ***dot-product definition*** of $(\hat{\mathbf{n}}_x \cdot \hat{\mathbf{b}}_x)$ to form $\vec{\mathbf{r}} \cdot \vec{\mathbf{r}} = \square^2 + \square^2 + 2\,x\,L_B\,(\hat{\mathbf{n}}_x \cdot \hat{\mathbf{b}}_x) \underset{(2.2)}{=} \square^2 + \square^2 + 2\,x\,L_B\,\cos(\square)$.

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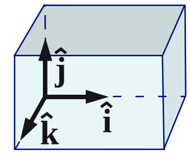
2.11 ♣ **Cross products with right-handed orthogonal unit vectors.** **(Section 2.10)**



Given: Vectors $\vec{v}$ and $\vec{w}$ expressed in terms of right-handed orthogonal unit vectors $\hat{i}$, $\hat{j}$, $\hat{k}$, with: $\vec{v} \times \vec{w} = \underbrace{(a\,\hat{i} + b\,\hat{j} + c\,\hat{k})}_{\vec{v}} \times \underbrace{(x\,\hat{i} + y\,\hat{j} + z\,\hat{k})}_{\vec{w}}$

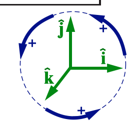
- Use the ***distributive property*** for cross products to write $\vec{v} \times \vec{w}$ in terms of $\hat{i} \times \hat{i}$, $\hat{i} \times \hat{j}$, etc. Next, use the ***definition*** of the cross product to calculate $\hat{i} \times \hat{i}$, $\hat{i} \times \hat{j}$, etc. (below-right).

Result:

$$\begin{aligned} \vec{v} \times \vec{w} = {} & a\,x\;\hat{i} \times \hat{i} \;+\; a\,y\;\hat{i} \times \hat{j} \;+\; \square\;\hat{i} \times \hat{k} \\ & + b\,x\;\hat{j} \times \hat{i} \;+\; b\,y\;\square \times \square \;+\; \square\;\square \times \square \\ & + c\,x\;\hat{k} \times \hat{i} \;+\; c\,y\;\square \times \square \;+\; \square\;\square \times \square \end{aligned}$$

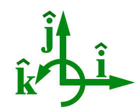
$\hat{i} \times \hat{i} = \vec{0}$	$\hat{i} \times \hat{j} = \hat{k}$	$\hat{i} \times \hat{k} = -\hat{j}$
$\hat{j} \times \hat{i} = \square$	$\hat{j} \times \hat{j} = \square$	$\hat{j} \times \hat{k} = \square$
$\hat{k} \times \hat{i} = \square$	$\hat{k} \times \hat{j} = \square$	$\hat{k} \times \hat{k} = \square$

- Combine your previous results to calculate $\vec{v} \times \vec{w}$ in terms of a, b, c, x, y, z.



Result: $\vec{v} \times \vec{w} = (b\,z - \square)\,\hat{i} + (\square - a\,z)\,\hat{j} + (\square)\,\hat{k}$

2.12 ♣ **Cross products and determinants** **(orthogonal unit vectors).** **(Section 2.10.2)**



Shown right are arbitrary vectors $\vec{v}$ and $\vec{w}$ expressed in terms of right-handed orthogonal unit vectors $\hat{i}$, $\hat{j}$, $\hat{k}$. Show that calculating $\vec{v} \times \vec{w}$ with the ***distributive property*** of the cross product (seen in Hw 2.11) happens to be equal to the ***determinant*** of the matrix shown to the right.

$$\vec{v} = a\,\hat{i} + b\,\hat{j} + c\,\hat{k}$$
$$\vec{w} = x\,\hat{i} + y\,\hat{j} + z\,\hat{k}$$
$$\vec{v} \times \vec{w} = \det \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ a & b & c \\ x & y & z \end{bmatrix}$$

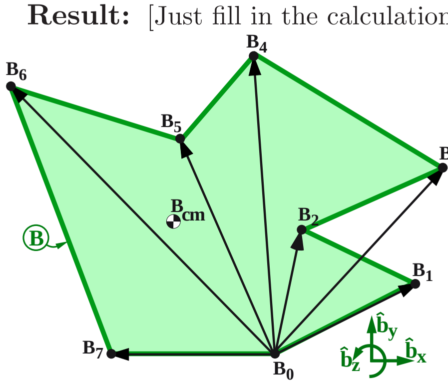
Result: $\vec{v} \times \vec{w} = (b\,z - \square)\,\hat{i} + (\square - a\,z)\,\hat{j} + (\square)\,\hat{k}$

2.13 ♣ **Cross products: Commercial area calculation algorithm** **(surveying).** **(Sections 2.10.1, 3.3)**

Complex **planar objects** such as the polygon B below can be decomposed into triangles for important planar measurements (e.g., farming acreage, building costs, and mass and area properties of 2D objects).

- Calculate $\vec{\mathbf{A}}_2$ and $\vec{\mathbf{A}}_4$, the vector-areas of triangles $B_0\,B_2\,B_3$ and $B_0\,B_4\,B_5$.
- Account for overlapped areas with **positive** and **negative** vector areas.

Result: [Just fill in the calculations for $\vec{\mathbf{A}}_2$, $\vec{\mathbf{A}}_4$, and $\mathbf{A}$ using eqn (3.4)].



$$\vec{r}_1 = {}^{B_0}\vec{r}^{\,B_1} = 2.0\,\hat{b}_x + 2.0\,\hat{b}_y$$
$$\vec{r}_2 = {}^{B_0}\vec{r}^{\,B_2} = 0.5\,\hat{b}_x + 2.5\,\hat{b}_y$$
$$\vec{r}_3 = {}^{B_0}\vec{r}^{\,B_3} = 3.0\,\hat{b}_x + 4.0\,\hat{b}_y$$
$$\vec{r}_4 = {}^{B_0}\vec{r}^{\,B_4} = -0.5\,\hat{b}_x + 7.0\,\hat{b}_y$$
$$\vec{r}_5 = {}^{B_0}\vec{r}^{\,B_5} = -1.0\,\hat{b}_x + 5.0\,\hat{b}_y$$
$$\vec{r}_6 = {}^{B_0}\vec{r}^{\,B_6} = -3.0\,\hat{b}_x + 6.0\,\hat{b}_y$$
$$\vec{r}_8 = {}^{B_0}\vec{r}^{\,B_8} = -2.0\,\hat{b}_x + 0.0\,\hat{b}_y$$

$\vec{\mathbf{A}}_1$	=	$\frac{1}{2}\,(\vec{r}_1 \times \vec{r}_2)$	=	$2\,\hat{b}_z$
$\vec{\mathbf{A}}_2$	=	$\frac{1}{2}\,(\vec{r}_2 \times \vec{r}_3)$	=	$\square.75\,\hat{b}_z$
$\vec{\mathbf{A}}_3$	=	...	=	$11.5\,\hat{b}_z$
$\vec{\mathbf{A}}_4$	=	...	=	$\square.25\,\hat{b}_z$
$\vec{\mathbf{A}}_5$	=	...	=	$4.5\,\hat{b}_z$
$\vec{\mathbf{A}}_6$	=	$\frac{1}{2}\,(\vec{r}_6 \times \vec{r}_7)$	=	$6\,\hat{b}_z$
$\vec{\mathbf{A}}$	=	$\sum_{i=1}^{6} \vec{\mathbf{A}}_i$	=	$\square$
Area	=	$\lvert\vec{\mathbf{A}}\rvert$	=	23.5

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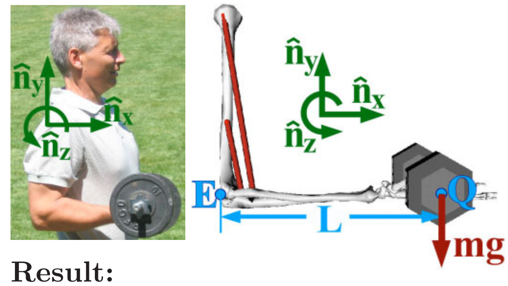
2.14 Biomechanics: Gravity moment for curling $\vec{\mathbf{M}} = \vec{\mathbf{r}} \times \vec{\mathbf{F}}$ (Section 2.10)

The figures to the right show an athlete curling a dumbbell (modeled as a particle Q of mass m). The forearm connects to the upper arm at the elbow (point E). Orthogonal unit vectors $\widehat{\mathbf{n}}_x$, $\widehat{\mathbf{n}}_y$, $\widehat{\mathbf{n}}_z$ are directed with $\widehat{\mathbf{n}}_y$ vertically upward and $\widehat{\mathbf{n}}_x$ from E to Q.



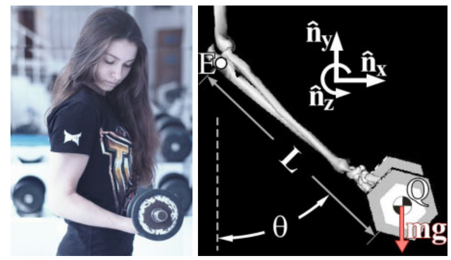
Description	Symbol	Type
Earth's gravitational constant	g	$g \approx 9.8 \frac{\text{m}}{\text{s}^2}$
Mass of dumbbell Q	m	Positive constant
Distance between elbow E and Q	L	Positive constant

The moment of gravity forces on Q about E is $\vec{\mathbf{M}} = \vec{\mathbf{r}} \times \vec{\mathbf{F}}$ where $\vec{\mathbf{F}} = \text{-m}\, g\, \widehat{\mathbf{n}}_y$. Express $\vec{\mathbf{M}}$ in terms of m, g, L, $\widehat{\mathbf{n}}_z$.

Result: $\vec{\mathbf{M}} = \vec{\mathbf{r}} \times \vec{\mathbf{F}} =$ ________

Now consider the forearm making an angle θ with downward vertical. Form $\vec{\mathbf{M}}$ and its magnitude $|\vec{\mathbf{M}}|$. Determine the values of θ $(0 \le \theta \le 135°)$ that produce maximum and minimum $|\vec{\mathbf{M}}|$.

To simplify $|\vec{\mathbf{M}}|$, note m, g, L are positive and for $0 \le \theta \le 135°$, $\sin(\theta) \ge 0$.



Result: (in terms of m, g, L, θ, $\widehat{\mathbf{n}}_z$).

$\vec{\mathbf{M}} = \vec{\mathbf{r}} \times \vec{\mathbf{F}} =$ ________ $\qquad |\vec{\mathbf{M}}| =$ ________

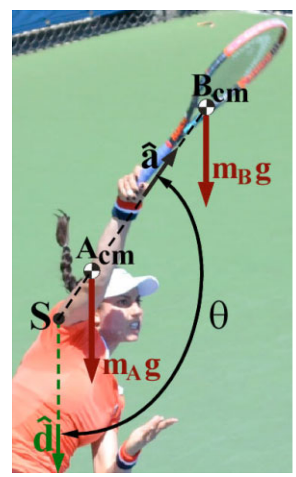
Max $|\vec{\mathbf{M}}| =$ ____ at $\theta =$ ____°

Min $|\vec{\mathbf{M}}| =$ ____ at $\theta =$ ____°

Optional: Modeling the elbow as a revolute joint, draw a ***free-body diagram (FBD)*** of the system consisting of the forearm and dumbbell.

2.15 Biomechanics: Gravity force and moment for tennis $\vec{\mathbf{M}} = \vec{\mathbf{r}} \times \vec{\mathbf{F}}$ (Section 2.10)

Shown right is an athlete whose arm A swings a tennis racquet B. Point S (shoulder), A_{cm} (A's center of mass), and B_{cm} (B's center of mass) lie along a line parallel to a unit vector $\widehat{\mathbf{a}}$. The unit vector $\widehat{\mathbf{d}}$ is vertically-downward ↓.



Description	Symbol	Type
Earth's gravitational constant	g	$g \approx 9.8 \frac{\text{m}}{\text{s}^2}$
Mass of A, mass of B	m_A, m_B	Positive constants
Distances between S and A_{cm} and S and B_{cm}	L_A, L_B	Positive constants
Angle between $\widehat{\mathbf{a}}$ and $\widehat{\mathbf{d}}$	θ	$0 \le \theta \le 180°$

- Form $\vec{\mathbf{F}}_{\text{gravity}}$ (the net force on A and B due to Earth's gravity).
- Form $|\vec{\mathbf{M}}|$ (the magnitude of the moment of those gravity forces about S).

  Note: $\vec{\mathbf{M}} = {}^{S}\vec{\mathbf{r}}^{A_{cm}} \times \text{m}_A\, g\, \widehat{\mathbf{d}} + {}^{S}\vec{\mathbf{r}}^{B_{cm}} \times \text{m}_B\, g\, \widehat{\mathbf{d}}$.

Result:

$\vec{\mathbf{F}}_{\text{gravity}} = ($ ________ $)\, \widehat{\mathbf{d}}$

$|\vec{\mathbf{M}}| =$ ________

Optional: Modeling the athlete grip of the racquet as a weld, draw a ***free-body diagram (FBD)*** of the racquet. Next, choose a model for the shoulder joint and draw a ***FBD*** of the system consisting of the arm and racquet.

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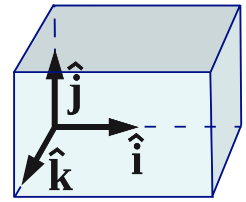
2.16 **Scalar triple product with bases** (Section 2.11).

The figure shows right-handed orthogonal unit vectors $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, $\hat{\mathbf{k}}$.



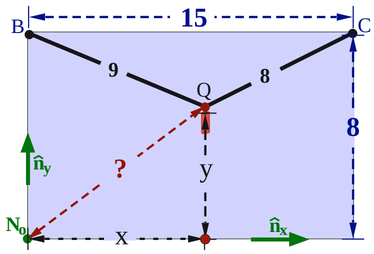
Given	Calculate
$\vec{\mathbf{u}} = 2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 4\hat{\mathbf{k}}$	$\vec{\mathbf{u}} \times \vec{\mathbf{v}} \cdot \vec{\mathbf{u}} = \square$
$\vec{\mathbf{v}} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$	$\vec{\mathbf{u}} \times \vec{\mathbf{v}} \cdot \vec{\mathbf{w}} = \square\, z - \square\, x - 6\,y$
$\vec{\mathbf{w}} = 5\hat{\mathbf{i}} - 6\hat{\mathbf{j}} + 7\hat{\mathbf{k}}$	$\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} \times \vec{\mathbf{w}} = \square\, z - 45\,x - \square\, y$

Note: Although the order of operations in $\vec{\mathbf{u}} \times \vec{\mathbf{v}} \cdot \vec{\mathbf{u}}$ is unambiguous, parentheses may clarify your work.

$\vec{\mathbf{u}} \times \vec{\mathbf{v}} \cdot \vec{\mathbf{w}} = \vec{\mathbf{u}} \cdot \vec{\mathbf{v}} \times \vec{\mathbf{w}}$ and it is OK to switch $\cdot$ and $\times$ in scalar triple products. **True/False**.

2.17 **Locating a microphone** (2D). **Show work**. (Section 1.4)

A microphone Q is attached to two pegs B and C by two cables. Knowing the peg locations, cable lengths, and points B, C, Q, N_o all lie in the same plane, determine the distance between Q and N_o. Do the problem with Euclidean geometry (e.g., law of cosines), then try vectors (see Hw 1.33).

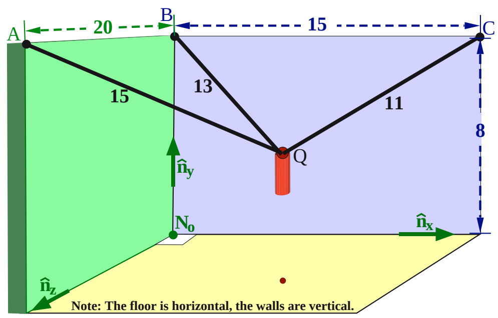


Distance between B to C		15 m
Distance between N_o to B	h	8 m
Length of cable joining B and Q	L_B	9 m
Length of cable joining C and Q	L_C	8 m
Distance between N_o and Q		9.01 m

Note: Although there are two mathematical answers to this problem, one is above the ceiling by ≈ 12 m and requires the cables to be in compression.

2.18 **†Locating a microphone** (3D).

A microphone Q is attached to three pegs A, B, and C by three cables. Knowing the peg locations and cable lengths, determine the distance between Q and point N_o. **Show work.** (If needed, hint below).[2]



Distance between A to B		20 m
Distance between B to C		15 m
Distance between N_o to B	h	8 m
Length of cable joining A and Q	L_A	15 m
Length of cable joining B and Q	L_B	13 m
Length of cable joining C and Q	L_C	11 m
Distance between N_o and Q If Q is above ceiling, distance ≈ 17 m		13.3 m

Note: This is part of the process of a camera targeting a football/baseball in a stadium or laser targeting cancer or ...

Vocabulary: In this ***forward kinematics*** analysis, the cable lengths are known and you determine the position of "end-effector" Q.

[2] Hint: See Homework 1.33 or Section 3.4. Introduce unknowns x, y, z so Q's position from N_o is $x\,\hat{\mathbf{n}}_x + y\,\hat{\mathbf{n}}_y + z\,\hat{\mathbf{n}}_z$. Although nonlinear algebraic equations are usually solved with a computer, these can also be solved "by-hand". Solution at www.MotionGenesis.com ⇒ Get Started ⇒ **2D/3D geometry**. Alternatively, go to www.WolframAlpha.com and type

Solve `x^2 + (-20+z)^2 + (-8+y)^2 = 225,    x^2 + z^2 + (-8+y)^2 = 169,    z^2 + (-15+x)^2 + (-8+y)^2 = 121`

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