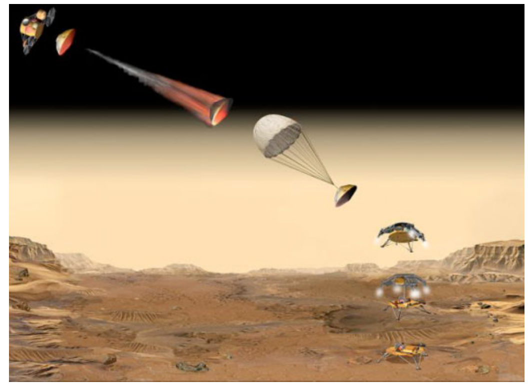


Chapter 2

Vectors ($\triangleq$ $=$ $-\vec{\mathbf{v}}$ $+$ $-$ $*$ $\cdot$ $\vec{\mathbf{v}}^2$ $\times$)

Examples in Hw 1, 2, 3

In 1881-1903, Gibbs developed ***vectors*** as a useful combination of magnitude and direction. Vectors are an important **geometrical tool** (for surveying, motion, optics, graphics, CAD, Finite Element Analysis, ...).

Symbol	Description	Details
$\vec{\mathbf{0}}$, $\hat{\mathbf{u}}$	Zero vector and unit vector.	Sections 2.3, 2.4
$+$ $-$ $*$	Vector addition, negation, subtraction, and scalar multiplication/division.	Sections 2.6 - 2.8
$\cdot$ $\times$	Vector dot product and cross product.	Sections 2.9, 2.10
$\frac{^F d}{dt}$	Vector differentiation.	Chapters 13, 14

2.1 Examples of scalars vectors and dyadics

- A ***scalar*** is a number, possibly with units (e.g., 7 $\frac{\text{m}}{\text{s}}$ or 9 kg), such as

time	density	volume	mass	potential energy	work
distance	speed	angle	weight	kinetic energy	temperature

- A ***vector*** is a quantity with magnitude and ***one*** associated direction. For example, a ***velocity vector*** has speed (how fast something moves) and direction (which way it is going). A ***force vector*** has magnitude (how hard something is pushed) and direction (which way it is shoved). Examples include:

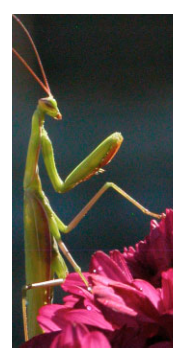
force	velocity	acceleration	translational momentum
torque	angular velocity	angular acceleration	angular momentum

In 1884, Gibbs re-defined ***vector*** and taught vectors using **90** lectures.

- A ***dyad*** is a quantity with magnitude and ***two*** associated directions. For example, ***stress*** associates with area and force (both regarded as vectors). A ***dyadic*** is the **sum of dyads**. For example, an ***inertia dyadic*** (Chapter 20) is the sum of dyads associated with moments and products of inertia.

2.2 Definition of a vector

A ***vector*** is defined as a quantity having ***magnitude*** and ***direction***.[a]
Vectors are represented pictorially with straight or curved arrows (examples below).
Vectors are typeset with an arrow and bold-faced font, e.g., $\vec{\mathbf{v}}$ denotes a vector.



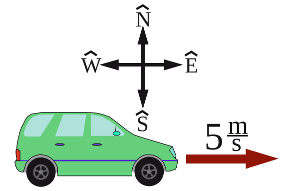
Courtesy Bro. Claude Rheaume. LaSalette.

Certain vectors have additional properties, e.g., a ***position vector*** $\vec{\mathbf{r}}$ has two associated points and units of length (e.g., meters) and a ***unit vector*** has magnitude 1 (no units).

[a] A vector's ***magnitude*** is a real non-negative scalar (e.g., 7 m/s). A vector's ***direction*** is its ***orientation*** and ***sense***. A vector is similar to a **ray** in direction, but a vector has finite magnitude. A vector is similar to a **line segment** in magnitude

and orientation, but a vector also has a ***sense*** (a fully defined direction).

Example of a vector: Consider the statement "the car is moving East at 5 $\frac{\text{m}}{\text{s}}$". It is convenient to represent the car's speed and direction with the velocity vector $\vec{\mathbf{v}} = 5\,\widehat{\mathbf{East}}$ (a hat designates the direction $\widehat{\mathbf{East}}$ as a ***unit vector***). The car's speed is always a real non-negative scalar denoted $|\vec{\mathbf{v}}|$ (the ***magnitude*** of $\vec{\mathbf{v}}$). The combination of ***magnitude*** and ***direction*** is a ***vector***.



The velocity of a car with speed 5 $\frac{\text{m}}{\text{s}}$ moving West can also be written as $\vec{\mathbf{v}} = \text{-}5\,\widehat{\mathbf{East}}$. The negative sign in $\text{-}5\,\widehat{\mathbf{East}}$ reverses vector $\vec{\mathbf{v}}$'s direction whereas $\vec{\mathbf{v}}$'s magnitude is $|\vec{\mathbf{v}}| = |\text{-}5\,\widehat{\mathbf{East}}| = 5\,\frac{\text{m}}{\text{s}}$.
Note: When $\vec{\mathbf{v}}$ is written as $\vec{\mathbf{v}} = \dot{x}\,\widehat{\mathbf{East}}$ where $\dot{x}$ is a scalar that can be **positive** or **zero** or **negative**, $\dot{x}$ is called the $\widehat{\mathbf{East}}$ ***measure*** of the vector $\vec{\mathbf{v}}$. The magnitude of $\vec{\mathbf{v}}$ is $|\vec{\mathbf{v}}| = \text{abs}(\dot{x})$ is inherently non-negative.

2.3 Zero vector $\vec{0}$, a vector whose magnitude is zero

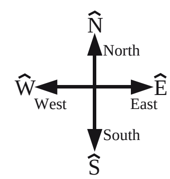
Addition with a zero vector:	$\textbf{any}\vec{\mathbf{V}}\textbf{ector} + \vec{\mathbf{0}} = \textbf{any}\vec{\mathbf{V}}\textbf{ector}$	
Dot product with a zero vector:	$\textbf{any}\vec{\mathbf{V}}\textbf{ector} \cdot \vec{\mathbf{0}} \underset{(2)}{=} 0$	$\vec{\mathbf{0}}$ is ***perpendicular*** to all vectors
Cross product with a zero vector:	$\textbf{any}\vec{\mathbf{V}}\textbf{ector} \times \vec{\mathbf{0}} \underset{(5)}{=} \vec{\mathbf{0}}$	$\vec{\mathbf{0}}$ is ***parallel*** to all vectors
Derivative of the ***zero vector***:	$\frac{{}^F d\,\vec{\mathbf{0}}}{dt} = \vec{\mathbf{0}}$	F is any reference frame

Vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ are said to be "***perpendicular***" if $\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = 0$ whereas $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ are "***parallel***" if $\vec{\mathbf{a}} \times \vec{\mathbf{b}} = \vec{\mathbf{0}}$.
Note: Some say $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ are "***parallel***" only if $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ have the same direction and "anti-parallel" if $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ have opposite directions.[1]

2.4 Unit $\hat{\text{v}}$ectors: Vectors with magnitude 1 and no units (typeset with a $\widehat{\text{hat}}$)

Unit vectors are "***sign posts***" (e.g., unit vectors $\widehat{\mathbf{N}}$, $\widehat{\mathbf{S}}$, $\widehat{\mathbf{W}}$, $\widehat{\mathbf{E}}$ for local Earth directions) chosen to simplify communication and calculations. Other useful "sign posts" are:

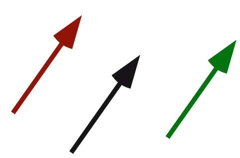
- Unit vector directed from one point to another point
- Unit vector directed locally vertical
- Unit vector tangent to a curve or perpendicular to a surface



A unit vector can be defined so it has the same direction as an arbitrary non-zero vector $\vec{\mathbf{v}}$ by dividing $\vec{\mathbf{v}}$ by $|\vec{\mathbf{v}}|$ (the magnitude of $\vec{\mathbf{v}}$).
To avoid divide-by-zero problems during numerical computation, approximate the unit vector with a "small" positive real number ϵ in the denominator.

$$\textbf{unit}\widehat{\mathbf{V}}\textbf{ector} = \frac{\vec{\mathbf{v}}}{|\vec{\mathbf{v}}|} \approx \frac{\vec{\mathbf{v}}}{|\vec{\mathbf{v}}| + \epsilon} \qquad (1)$$

2.5 Equal vectors (=) vectors with the same magnitude and direction



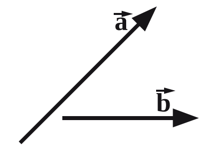
Shown right are three ***equal vectors***. Although each has a different location, the vectors are equal because they have the same magnitude and direction.

Some vectors have additional properties. For example, a position vector is associated with two points. Two position vectors are ***equal position vectors*** when, they have the same magnitude, same direction, and are associated with the same points. Two force vectors are ***equal force vectors*** when they have the same magnitude, direction, and point of application.

---

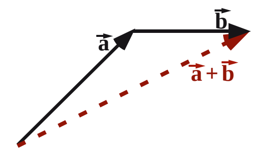
[1]The direction of a ***zero vector*** $\vec{\mathbf{0}}$ is arbitrary and may be regarded as having **any** direction so that $\vec{\mathbf{0}}$ is ***parallel*** to all vectors, $\vec{\mathbf{0}}$ is ***perpendicular*** to all vectors, all zero vectors are equal, and one may use the definite pronoun "the" instead of the indefinite "a" e.g., "the zero vector". It is improper to say the ***zero vector*** has no direction as a vector is **defined** to have both magnitude and direction. It is also improper to say a ***zero vector*** has all directions as a vector is defined to have a magnitude and **a** direction.

2.6 Vector addition (+)



As shown right, adding vectors $\vec{a} + \vec{b}$ produces a vector. First $\vec{b}$ is translated so its tail is at the tip of $\vec{a}$. Next, $\vec{a} + \vec{b}$ is drawn from the tail of $\vec{a}$ to the tip of the translated $\vec{b}$.

Translating $\vec{b}$ does *not* change the magnitude or direction of $\vec{b}$, and so produces an equal $\vec{b}$.



Properties of vector addition

Commutative property: $\vec{a} + \vec{b} \;=\; \vec{b} + \vec{a}$

Associative property: $(\vec{a} + \vec{b}) + \vec{c} \;=\; \vec{a} + (\vec{b} + \vec{c}) \;=\; \vec{a} + \vec{b} + \vec{c}$

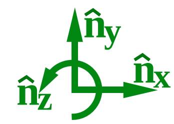
Addition of zero vector: $\vec{a} + \vec{0} \;=\; \vec{a}$

Vectors with different units do **not** add. Do **not** add a position vector (units of meters) with a force vector (units of Newtons).

Example: Vector addition (+) algebra

Shown right is how to add vectors $\vec{w}$ and $\vec{v}$ that are expressed in terms of orthogonal unit vectors $\hat{n}_x$, $\hat{n}_y$, $\hat{n}_z$.

$$\begin{array}{rcl} \vec{v} &=& 7\,\hat{n}_x + 5\,\hat{n}_y + 4\,\hat{n}_z \\ -\;\vec{w} &=& 2\,\hat{n}_x + 3\,\hat{n}_y + 2\,\hat{n}_z \\ \hline \vec{v}+\vec{w} &=& 9\,\hat{n}_x + 8\,\hat{n}_y + 6\,\hat{n}_z \end{array}$$

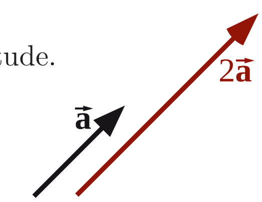


$$\vec{v} = \underbrace{x\,\hat{n}_x}_{\text{vector component}} + \underbrace{y\,\hat{n}_y}_{\text{vector component}}$$

Special names for parts of the generic vector $\vec{v}$.

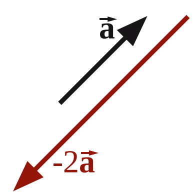
x is called the $\hat{n}_x$ ***scalar component*** (***measure***) of $\vec{v}$.
y is called the $\hat{n}_y$ ***scalar component*** (***measure***) of $\vec{v}$.

2.7 Vector multiplied or divided by a scalar ($*$ or /)



- Multiplying a vector by a **positive** number (other than 1) changes the vector's magnitude.
- Multiplying a vector by a **negative** number changes the vector's magnitude **and** reverses the ***sense*** of the vector.
- Dividing a vector $\vec{a}$ by a scalar s is defined as $\frac{\vec{a}}{s} \triangleq \frac{1}{s} * \vec{a}$.

Properties of multiplication of a vector by a scalar s_1 or s_2



Commutative property: $s_1\,\vec{a} \;=\; \vec{a}\,s_1$

Associative property: $s_1\,(s_2\,\vec{a}) \;=\; (s_1\,s_2)\,\vec{a} \;=\; s_2\,(s_1\,\vec{a}) \;=\; s_1\,s_2\,\vec{a}$

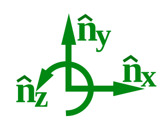
Distributive property: $(s_1 + s_2)\,\vec{a} \;=\; s_1\,\vec{a} + s_2\,\vec{a}$ $\qquad s_1\,(\vec{a} + \vec{b}) \;=\; s_1\,\vec{a} + s_1\,\vec{b}$

Multiplication by zero: $0 * \vec{a} \;=\; \vec{0}$

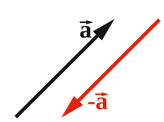
Example: Vector scalar multiplication and division ($*$ and /)

Given: $\vec{v} \;=\; 7\,\hat{n}_x + 5\,\hat{n}_y + 4\,\hat{n}_z$
then: $5\,\vec{v} \;=\; 35\,\hat{n}_x + 25\,\hat{n}_y + 20\,\hat{n}_z$

and $\frac{\vec{v}}{-2} \;=\; -3.5\,\hat{n}_x \;-\; 2.5\,\hat{n}_y \;-\; 2\,\hat{n}_z$



2.8 Vector negation and subtraction (−)

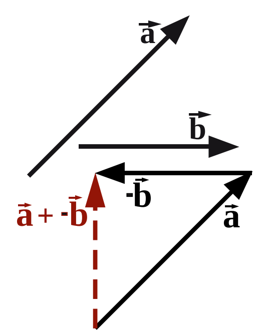


Negation: As shown right, negating a vector (multiplying by -1) reverses the vector's ***sense*** (it points in the opposite direction). Negation does not change the vector's magnitude or orientation.

Subtraction: As the drawing to the right shows, subtracting a vector $\vec{b}$ from a vector $\vec{a}$ is simply addition and negation.[a]

$$\vec{a} - \vec{b} \;\triangleq\; \vec{a} + -\vec{b}$$

[a] In most/all mathematics, subtraction is defined as negation and addition.



After negating vector $\vec{b}$, it is translated so the tail of $-\vec{b}$ is at the tip of $\vec{a}$.
Next, vector $\vec{a} + -\vec{b}$ is drawn from the tail of $\vec{a}$ to the tip of the translated $-\vec{b}$.

Example: Vector subtraction ($\vec{v} - \vec{w}$)

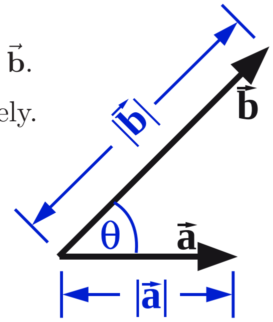
It is easy to subtract vectors that are expressed in terms of orthogonal unit vectors $\hat{n}_x$, $\hat{n}_y$, $\hat{n}_z$.

$$\begin{array}{rcl} \vec{v} &=& 7\,\hat{n}_x + 5\,\hat{n}_y + 4\,\hat{n}_z \\ -\;\vec{w} &=& 2\,\hat{n}_x + 3\,\hat{n}_y + 2\,\hat{n}_z \\ \hline \vec{v}-\vec{w} &=& 5\,\hat{n}_x + 2\,\hat{n}_y + 2\,\hat{n}_z \end{array}$$

2.9 Vector dot product (·)

Equation (2) defines the ***dot product*** of vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$.

- $|\vec{\mathbf{a}}|$ and $|\vec{\mathbf{b}}|$ are the magnitudes of $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$, respectively.
- θ is the smallest angle between $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ $(0 \le \theta \le \pi)$.



$$\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} \triangleq |\vec{\mathbf{a}}|\,|\vec{\mathbf{b}}|\,\cos(\theta) \qquad (2)$$

Equation (3) is a rearrangement of equation (2) that is useful for calculating the angle θ between two vectors.

$$\cos(\theta) \underset{(2)}{=} \frac{\vec{\mathbf{a}} \cdot \vec{\mathbf{b}}}{|\vec{\mathbf{a}}|\;|\vec{\mathbf{b}}|} \qquad (3)$$

Use **acos** to calculate θ.

Note: $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ are "***perpendicular***" when $\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = 0$.

Note: Dot-products encapsulate the ***law of cosines***.

Equation (2) shows $\vec{\mathbf{v}} \cdot \vec{\mathbf{v}} = |\vec{\mathbf{v}}|^2$. Hence, the dot product can calculate a vector's ***magnitude*** as shown for $|\vec{\mathbf{v}}|$ in equation (4).

Equation (4) also defines ***vector exponentiation*** $\vec{\mathbf{v}}^n$ (vector $\vec{\mathbf{v}}$ raised to scalar power n) as a non-negative scalar.

$$\begin{aligned} \vec{\mathbf{v}}^2 &\triangleq |\vec{\mathbf{v}}|^2 = \vec{\mathbf{v}} \cdot \vec{\mathbf{v}} \\ |\vec{\mathbf{v}}| &= +\sqrt{\vec{\mathbf{v}} \cdot \vec{\mathbf{v}}} \\ \vec{\mathbf{v}}^n &\triangleq |\vec{\mathbf{v}}|^n = +(\vec{\mathbf{v}} \cdot \vec{\mathbf{v}})^{\frac{n}{2}} \end{aligned} \qquad (4)$$

Example: Kinetic energy $\mathrm{K} = \frac{1}{2}\,\mathrm{m}\,\vec{\mathbf{v}}^2 \underset{(4)}{=} \frac{1}{2}\,\mathrm{m}\,\vec{\mathbf{v}} \cdot \vec{\mathbf{v}}$

2.9.1 Properties of the dot-product (·)

Dot product with a zero vector	$\vec{\mathbf{a}} \cdot \vec{\mathbf{0}} = 0$	
Dot product of ***perpendicular*** vectors	$\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = 0$	if $\vec{\mathbf{a}} \perp \vec{\mathbf{b}}$
Dot product of parallel vectors	$\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = \pm\,|\vec{\mathbf{a}}|\,|\vec{\mathbf{b}}|$	if $\vec{\mathbf{a}} \parallel \vec{\mathbf{b}}$
Dot product with vectors scaled by s_1 and s_2	$s_1\,\vec{\mathbf{a}} \cdot s_2\,\vec{\mathbf{b}} = s_1\,s_2\,(\vec{\mathbf{a}} \cdot \vec{\mathbf{b}})$	
Commutative property	$\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = \vec{\mathbf{b}} \cdot \vec{\mathbf{a}}$	
Distributive property	$\vec{\mathbf{a}} \cdot (\vec{\mathbf{b}} + \vec{\mathbf{c}}) = \vec{\mathbf{a}} \cdot \vec{\mathbf{b}} + \vec{\mathbf{a}} \cdot \vec{\mathbf{c}}$	
Distributive property	$(\vec{\mathbf{a}} + \vec{\mathbf{b}}) \cdot (\vec{\mathbf{c}} + \vec{\mathbf{d}}) = \vec{\mathbf{a}} \cdot \vec{\mathbf{c}} + \vec{\mathbf{a}} \cdot \vec{\mathbf{d}} + \vec{\mathbf{b}} \cdot \vec{\mathbf{c}} + \vec{\mathbf{b}} \cdot \vec{\mathbf{d}}$	

Note: The distributive property for dot-products and cross-products is proved in [33, pgs. 23-24, 32-34].

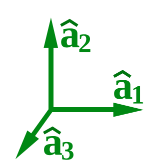
2.9.2 Uses for the dot-product (·)

- Calculating an ***angle*** between two vectors [see equation (3) and example in Section 3.4]
- Determining when two vectors are ***perpendicular***, e.g., $\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = 0$.
- Calculating a vector's ***magnitude*** [see equation (4) and ***distance*** examples in Sections 3.1 and 3.4].
- Changing a ***vector equation*** into a ***scalar equation*** (see Hw 2.28).
- Calculating a ***unit vector*** in the direction of a vector $\vec{\mathbf{v}}$ [from equation (1)] $\quad \textbf{unit}\vec{\textbf{V}}\textbf{ector} \underset{(1)}{=} \dfrac{\vec{\mathbf{v}}}{|\vec{\mathbf{v}}|}$
- ***Projection*** of a vector $\vec{\mathbf{v}}$ in the direction of $\vec{\mathbf{b}}$, defined as: $\quad \vec{\mathbf{v}} \cdot \dfrac{\vec{\mathbf{b}}}{|\vec{\mathbf{b}}|}$
  See Section 4.2 for ***projections***, ***measures***, ***coefficients***, ***components***.
  See Section 3.4 for a distance measure from a point to a plane.

2.9.3 Dot-products to change vector equations to scalar equations (see Hw 1.28)

One way to form up to three linearly independent scalar equations from the vector equation $\vec{\mathbf{v}} = \vec{\mathbf{0}}$ is by dot-multiplying $\vec{\mathbf{v}} = \vec{\mathbf{0}}$ with three orthogonal unit vectors $\hat{\mathbf{a}}_1$, $\hat{\mathbf{a}}_2$, $\hat{\mathbf{a}}_3$, i.e.,

$$\text{if} \quad \vec{\mathbf{v}} = \vec{\mathbf{0}} \quad \Rightarrow \quad \vec{\mathbf{v}} \cdot \hat{\mathbf{a}}_1 = 0 \qquad \vec{\mathbf{v}} \cdot \hat{\mathbf{a}}_2 = 0 \qquad \vec{\mathbf{v}} \cdot \hat{\mathbf{a}}_3 = 0$$

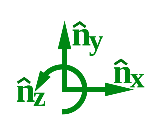


Section 2.11.2 describes another way to form three ***different*** scalar equations from $\vec{\mathbf{v}} = \vec{\mathbf{0}}$.

2.9.4 Special case: Dot-products with orthogonal unit vectors

When $\hat{\mathbf{n}}_{\mathrm{x}}$, $\hat{\mathbf{n}}_{\mathrm{y}}$, $\hat{\mathbf{n}}_{\mathrm{z}}$ are **orthogonal unit** vectors, it can be shown (see Hw 2.4)

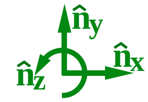
$$(a_x\,\hat{\mathbf{n}}_{\mathrm{x}} + a_y\,\hat{\mathbf{n}}_{\mathrm{y}} + a_z\,\hat{\mathbf{n}}_{\mathrm{z}}) \cdot (b_x\,\hat{\mathbf{n}}_{\mathrm{x}} + b_y\,\hat{\mathbf{n}}_{\mathrm{y}} + b_z\,\hat{\mathbf{n}}_{\mathrm{z}}) = a_x\,b_x + a_y\,b_y + a_z\,b_z$$



2.9.5 Examples: Vector dot-products $(\cdot)$

Shown below is how to use dot-products when vectors $\vec{v}$ and $\vec{w}$ are expressed in terms of orthogonal unit vectors $\hat{n}_x$, $\hat{n}_y$, $\hat{n}_z$.

$$\vec{v} = 7\,\hat{n}_x + 5\,\hat{n}_y + 4\,\hat{n}_z \qquad \vec{w} = 2\,\hat{n}_x + 3\,\hat{n}_y + 2\,\hat{n}_z$$

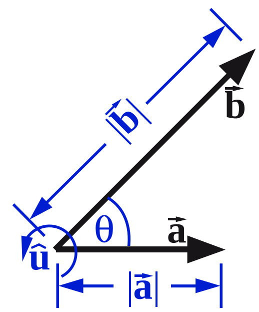


$\hat{n}_x$ measure of $\vec{v}$	$\vec{v} \cdot \hat{n}_x = 7$ (**measures** how much of $\vec{v}$ is in the $\hat{n}_x$ direction).
$\vec{v} \cdot \vec{v} = 7^2 + 5^2 + 4^2 = 90$	$\lvert\vec{v}\rvert = \sqrt{90} \approx 9.4868$
$\vec{w} \cdot \vec{w} = 2^2 + 3^2 + 2^2 = 17$	$\lvert\vec{w}\rvert = \sqrt{17} \approx 4.1231$
Unit vector in the direction of $\vec{v}$:	$\frac{\vec{v}}{\lvert\vec{v}\rvert} = \frac{7\,\hat{n}_x + 5\,\hat{n}_y + 4\,\hat{n}_z}{\sqrt{90}} \approx 0.738\,\hat{n}_x + 0.527\,\hat{n}_y + 0.422\,\hat{n}_z$
Unit vector in the direction of $\vec{w}$:	$\frac{\vec{w}}{\lvert\vec{w}\rvert} = \frac{2\,\hat{n}_x + 3\,\hat{n}_y + 2\,\hat{n}_z}{\sqrt{17}} \approx 0.485\,\hat{n}_x + 0.728\,\hat{n}_y + 0.485\,\hat{n}_z$
$\vec{v} \cdot \vec{w} = 7*2 + 5*3 + 4*2 = 37$	$\angle(\vec{v}, \vec{w}) = \text{acos}\left(\frac{37}{\sqrt{90}\,\sqrt{17}}\right) \approx 0.33 \text{ rad} \approx 18.93°$

2.10 Vector cross product $(\times)$

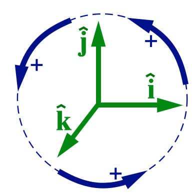
The ***cross product*** of a vector $\vec{a}$ with a vector $\vec{b}$ is defined in equation (5).

- $|\vec{a}|$ and $|\vec{b}|$ are the magnitudes of $\vec{a}$ and $\vec{b}$, respectively
- θ is the smallest angle between $\vec{a}$ and $\vec{b}$ $(0 \le \theta \le \pi)$.
- $\hat{u}$ is the unit vector **perpendicular** to both $\vec{a}$ and $\vec{b}$.
  The direction of $\hat{u}$ is determined by the ***right-hand rule***.
  The right-hand rule is a convention like driving on the right-hand side of the road.
  Note: $|\vec{a}|\,|\vec{b}|\,\sin(\theta)$ [the coefficient of $\hat{u}$ in equation (5)] is inherently non-negative because $\sin(\theta) \ge 0$ since $0 \le \theta \le \pi$. Hence, $|\vec{a} \times \vec{b}| = |\vec{a}|\,|\vec{b}|\,\sin(\theta)$.



$$\vec{a} \times \vec{b} \triangleq |\vec{a}|\,|\vec{b}|\,\sin(\theta)\,\hat{u} \qquad (5)$$

Properties of the cross-product $(\times)$



Cross product with a zero vector	$\vec{a} \times \vec{0} = \vec{0}$
Cross product of a vector with itself	$\vec{a} \times \vec{a} = \vec{0}$
Cross product of ***parallel*** vectors	$\vec{a} \times \vec{b} = \vec{0}$ if $\vec{a} \parallel \vec{b}$
Cross product of scaled vectors	$s_1\,\vec{a} \times s_2\,\vec{b} = s_1\,s_2\,(\vec{a} \times \vec{b})$
Distributive property	$\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$
Cross products are ***not*** associative	$\vec{a} \times (\vec{b} \times \vec{c}) \ne (\vec{a} \times \vec{b}) \times \vec{c}$

Cross products are ***not*** commutative.

$$\vec{a} \times \vec{b} = -\vec{b} \times \vec{a} \qquad (6)$$

Vector triple cross product (bac-cab).

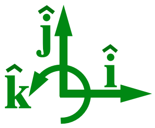
$$\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}\,(\vec{a} \cdot \vec{c}) - \vec{c}\,(\vec{a} \cdot \vec{b}) \qquad (7)$$

A mnemonic for eqn (7) $\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}\,(\vec{a} \cdot \vec{c}) - \vec{c}\,(\vec{a} \cdot \vec{b})$ is "back cab" - as in were you born in the back of a cab? Many proofs of this formula resolve $\vec{a}$, $\vec{b}$, and $\vec{c}$ into orthogonal unit vectors (e.g., $\hat{n}_x$, $\hat{n}_y$, $\hat{n}_z$) and equate components.

2.10.1 Uses for the cross-product $(\times)$ in geometry, statics, motion analysis, ...

- ***Moment*** of a force such as $\vec{r} \times \vec{F}$ (details in Section 10.1).
- ***Velocity/acceleration*** formulas [see eqns (15.3, 15.4)] $\vec{v} = \vec{\omega} \times \vec{r}$ and $\vec{a} = \vec{\alpha} \times \vec{r} + \vec{\omega} \times (\vec{\omega} \times \vec{r})$.
- ***Perpendicular*** vectors, e.g., $\vec{a} \times \vec{b}$ is perpendicular to both $\vec{a}$ and $\vec{b}$.
- ***Distance*** between a point and a line (see Section 3.2 and example in Section 3.4).
- ***Area of a triangle*** with sides $\vec{a}$ and $\vec{b}$ (see Sections 3.3, 3.4 and Hw 2.13). $\vec{\Delta}(\vec{a}, \vec{b}) = \frac{1}{2}\,\vec{a} \times \vec{b}$.

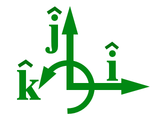
2.10.2 Determinants and cross-products (with right-handed unit vectors)



When vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ are expressed in terms of **orthogonal unit** vectors $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, $\hat{\mathbf{k}}$, it can be shown (Hw 2.12) that $\vec{\mathbf{a}} \times \vec{\mathbf{b}}$ happens to equal the ***determinant*** of an associated matrix.

$$\left.\begin{aligned}\vec{\mathbf{a}} &= a_x\,\hat{\mathbf{i}} + a_y\,\hat{\mathbf{j}} + a_z\,\hat{\mathbf{k}}\\ \vec{\mathbf{b}} &= b_x\,\hat{\mathbf{i}} + b_y\,\hat{\mathbf{j}} + b_z\,\hat{\mathbf{k}}\end{aligned}\right\} \qquad \vec{\mathbf{a}} \times \vec{\mathbf{b}} \;=\; \det\begin{bmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ a_x & a_y & a_z\\ b_x & b_y & b_z\end{bmatrix} \;=\; \begin{aligned}&(a_y\,b_z - a_z\,b_y)\,\hat{\mathbf{i}}\\ -\;&(a_x\,b_z - a_z\,b_x)\,\hat{\mathbf{j}}\\ +\;&(a_x\,b_y - a_y\,b_x)\,\hat{\mathbf{k}}\end{aligned} \tag{8}$$

Examples: Vector cross-products (×) with determinants.



The following shows how to use cross-products with the vectors $\vec{\mathbf{v}}$ and $\vec{\mathbf{w}}$, each which is expressed in terms of the orthogonal unit vectors $\hat{\mathbf{n}}_{\mathrm{x}}$, $\hat{\mathbf{n}}_{\mathrm{y}}$, $\hat{\mathbf{n}}_{\mathrm{z}}$ shown to the right.

$$\left.\begin{aligned}\vec{\mathbf{v}} &= 7\,\hat{\mathbf{i}} + 5\,\hat{\mathbf{j}} + 4\,\hat{\mathbf{k}}\\ \vec{\mathbf{w}} &= 2\,\hat{\mathbf{i}} + 3\,\hat{\mathbf{j}} + 2\,\hat{\mathbf{k}}\end{aligned}\right\} \qquad \vec{\mathbf{v}} \times \vec{\mathbf{w}} \;=\; \det\begin{bmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ 7 & 5 & 4\\ 2 & 3 & 2\end{bmatrix} \;=\; \text{-}2\,\hat{\mathbf{i}} \;-\; 6\,\hat{\mathbf{j}} \;+\; 11\,\hat{\mathbf{k}}$$

$$\text{Scalar triple product:}\quad (2\,\hat{\mathbf{i}} + 3\,\hat{\mathbf{j}} + 4\,\hat{\mathbf{k}}) \cdot (\vec{\mathbf{v}} \times \vec{\mathbf{w}}) \;=\; \det\begin{bmatrix}2 & 3 & 4\\ 7 & 5 & 4\\ 2 & 3 & 2\end{bmatrix} \;=\; 22$$

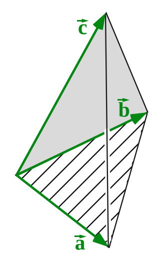
2.11 Optional: Scalar triple product (· × or × ·)

The ***scalar triple product*** of vectors $\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$, $\vec{\mathbf{c}}$ is the scalar defined in the various ways shown below.

$$\text{ScalarTripleProduct} \;\triangleq\; \boxed{\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} \times \vec{\mathbf{c}} \;=\; \vec{\mathbf{a}} \times \vec{\mathbf{b}} \cdot \vec{\mathbf{c}}} \;=\; \vec{\mathbf{b}} \cdot \vec{\mathbf{c}} \times \vec{\mathbf{a}} \;=\; \vec{\mathbf{b}} \times \vec{\mathbf{c}} \cdot \vec{\mathbf{a}} \tag{9}$$

Although parentheses help clarify equation (9) e.g., $\vec{\mathbf{a}} \cdot (\vec{\mathbf{b}} \times \vec{\mathbf{c}})$ instead of $\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} \times \vec{\mathbf{c}}$, the parentheses are unnecessary because the cross product $\vec{\mathbf{b}} \times \vec{\mathbf{c}}$ **must** be performed before the dot product (for a sensible result to be produced).

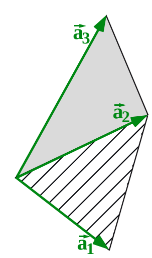
2.11.1 Scalar triple product and the volume of a tetrahedron



For a tetrahedron whose sides are described by the vectors $\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$, $\vec{\mathbf{c}}$ (sides of length $|\vec{\mathbf{a}}|$, $|\vec{\mathbf{b}}|$, $|\vec{\mathbf{c}}|$), a geometrical interpretation of $\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} \times \vec{\mathbf{c}}$ is the ***volume of the parallelepiped***. This formula helps calculate mass and volume of generic 3D shapes (e.g., for highway cut/fill calculations and CAD/CAE solid modeling). A tetrahedron's volume is calculated in Section 3.4.

$$\boxed{\text{Tetrahedron Volume} \;=\; \tfrac{1}{6}\,\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} \times \vec{\mathbf{c}} \;=\; \tfrac{1}{6}\,\vec{\mathbf{a}} \times \vec{\mathbf{b}} \cdot \vec{\mathbf{c}} \underset{(3.4)}{=} \tfrac{1}{3}\,\vec{\mathbf{\Delta}}(\vec{\mathbf{a}}, \vec{\mathbf{b}}) \cdot \vec{\mathbf{c}}} \tag{10}$$

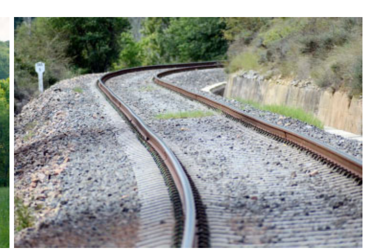
2.11.2 (× ·) to change vector equations to scalar equations (see Hw 1.28)



Section 2.9.3 showed one method to form scalar equations from the vector equation $\vec{\mathbf{v}} = \vec{\mathbf{0}}$. A 2^{nd} method expresses $\vec{\mathbf{v}}$ in terms of three non-coplanar (but not necessarily orthogonal or unit) vectors $\vec{\mathbf{a}}_1$, $\vec{\mathbf{a}}_2$, $\vec{\mathbf{a}}_3$, and writes the equally valid (but generally different) set of linearly independent scalar equations shown below [proved by directly by substituting $\vec{\mathbf{v}} = \vec{\mathbf{0}}$ into eqn (4.2)].

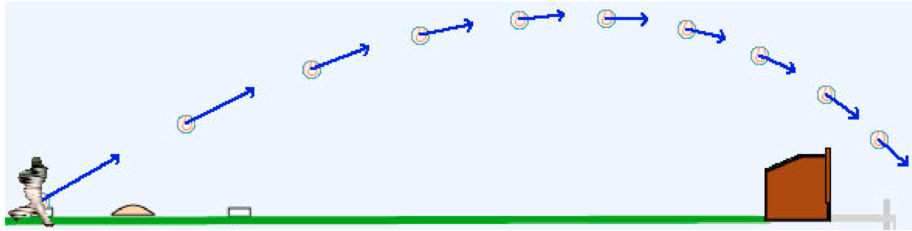
$$\text{Method 2:}\quad \text{if}\quad \vec{\mathbf{v}} \;=\; v_1\,\vec{\mathbf{a}}_1 \;+\; v_2\,\vec{\mathbf{a}}_2 \;+\; v_3\,\vec{\mathbf{a}}_3 \;=\; \vec{\mathbf{0}} \quad\Rightarrow\quad \boxed{v_1 = 0 \qquad v_2 = 0 \qquad v_3 = 0}$$





Courtesy Accuray Inc. Vectors are widely useful, e.g., in medical robotics, cut/fill calculations for highway & railway construction, ...

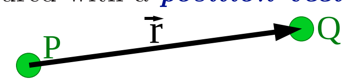
Chapter 3



Position vectors and vector geometry

3.1 Position of a point (or particle) (see examples in Hw 3)

A ***point*** is a location in space with no spatial dimension (no height, width, or depth). A ***particle*** is a point with mass (all particles are points, but not all points are particles). ***center of mass*** is a ***point*** that plays a central role for gravity and $\vec{\mathbf{F}} = \mathrm{m}\,\vec{\mathbf{a}}$. A point's location can be measured with a ***position vector*** that characterizes its position from another point.

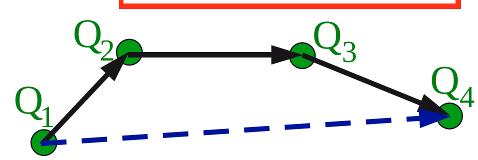


Distance between two points

$$\left|{}^{P}\vec{\mathbf{r}}^{Q}\right| = +\sqrt{{}^{P}\vec{\mathbf{r}}^{Q} \cdot {}^{P}\vec{\mathbf{r}}^{Q}} \qquad (1)$$

A position vector is defined by its properties: two points associated with a vector having units of length. For points P and Q, the symbol ${}^{P}\vec{\mathbf{r}}^{Q}$ denotes the position from P to Q.
The magnitude $\left|{}^{P}\vec{\mathbf{r}}^{Q}\right|$ is the ***distance*** between P and Q.

Usually, a position vector is formed by **inspection** or **vector addition**. For example, shown right are points Q_1, Q_2, Q_3, Q_4. The position vector ${}^{Q_1}\vec{\mathbf{r}}^{Q_4}$ (from Q_1 to Q_4) is formed by vector addition as shown in equation (2).

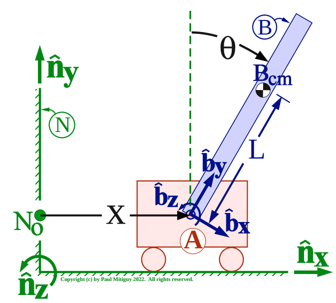


$${}^{Q_1}\vec{\mathbf{r}}^{Q_4} = {}^{Q_1}\vec{\mathbf{r}}^{Q_2} + {}^{Q_2}\vec{\mathbf{r}}^{Q_3} + {}^{Q_3}\vec{\mathbf{r}}^{Q_4} \qquad (2)$$

Squash rule for adding position vectors

Position vectors are **very useful for geometry**, e.g., for ***angles*** [Hw 2.4 4.21], ***distance*** [Hw 2.10], ***area***, and ***location*** [Hw 2.18, 4.22].

Example: Position vector and distance (inverted pendulum on cart)



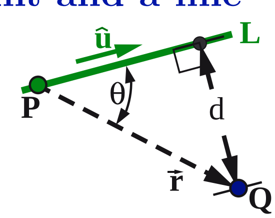
The ***position vector*** position from N_o to B_{cm} from point N_o to point B_{cm} is determined by visual inspection (from the figure to the right) and vector addition.

- Visual inspection: Position from N_o to A: ${}^{N_o}\vec{\mathbf{r}}^{A} = x\,\hat{\mathbf{n}}_x$ Position from A to B_{cm}: ${}^{A}\vec{\mathbf{r}}^{B_{cm}} = L\,\hat{\mathbf{b}}_y$
- Vector addition: ${}^{N_o}\vec{\mathbf{r}}^{B_{cm}} = {}^{N_o}\vec{\mathbf{r}}^{A} + {}^{A}\vec{\mathbf{r}}^{B_{cm}} = x\,\hat{\mathbf{n}}_x + L\,\hat{\mathbf{b}}_y$
- The ***distance*** d between N_o and B_{cm} is $\left|{}^{N_o}\vec{\mathbf{r}}^{B_{cm}}\right|$ (the magnitude of ${}^{N_o}\vec{\mathbf{r}}^{B_{cm}}$).

$$\mathrm{d} \underset{(1)}{=} +\sqrt{{}^{N_o}\vec{\mathbf{r}}^{B_{cm}} \cdot {}^{N_o}\vec{\mathbf{r}}^{B_{cm}}} = +\sqrt{(x\,\hat{\mathbf{n}}_x + L\,\hat{\mathbf{b}}_y) \cdot (x\,\hat{\mathbf{n}}_x + L\,\hat{\mathbf{b}}_y)} = +\sqrt{x^2 + 2\,x\,L\,\sin(\theta) + L^2}$$

3.2 Distance between a point and a line

A line L passes through a point P and is parallel to the unit vector $\hat{\mathbf{u}}$. The distance d between line L and a point Q can be calculated as shown right. Other distance calculations are in Section 3.4.



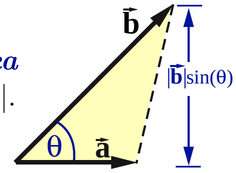
Distance between point Q and line L

$$\mathrm{d} = \left|\vec{\mathbf{r}} \times \hat{\mathbf{u}}\right| \underset{(2.5)}{=} \left|\vec{\mathbf{r}}\right| \sin(\theta) \qquad (3)$$

where $\vec{\mathbf{r}} = {}^{P}\vec{\mathbf{r}}^{Q}$

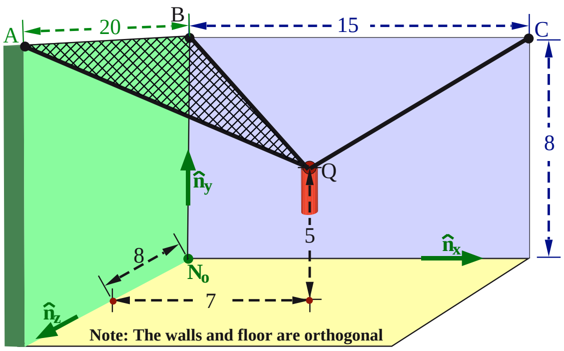
3.3 Area of a triangle

Eqn (4) calculates the vector or scalar ***area of a triangle*** with sides of length $|\vec{\mathbf{a}}|$, $|\vec{\mathbf{b}}|$.
Surveying: Hw 2.13 and Section 3.4 show how to use cross-products to calculate area.



$$\begin{aligned} \text{Vector area} \quad \vec{\boldsymbol{\Delta}}(\vec{\mathbf{a}}, \vec{\mathbf{b}}) &= \tfrac{1}{2}\ \vec{\mathbf{a}} \times \vec{\mathbf{b}} \\ \text{Scalar area} \quad \Delta(\vec{\mathbf{a}}, \vec{\mathbf{b}}) &= \tfrac{1}{2}\,|\vec{\mathbf{a}} \times \vec{\mathbf{b}}| \\ \tfrac{1}{2}\ \text{base} * \text{height} &= \tfrac{1}{2}\,|\vec{\mathbf{a}}|\,|\vec{\mathbf{b}}|\ \sin(\theta) \end{aligned} \qquad (4)$$

3.4 Geometry example: Length/distances, angle, surface area, volume



Three cables attach a microphone Q to pegs A, B, C.
Given: Peg and microphone locations from a point N_o.

Distance between A and B	20 m
Distance between B and C	15 m
Distance between N_o and B	8 m
Distance along back wall (see picture)	7 m
Q's height above N_o	5 m
Distance along side wall (see picture)	8 m

$$^{N_o}\vec{\mathbf{r}}^{\,Q} = 7\,\hat{\mathbf{n}}_x + 5\,\hat{\mathbf{n}}_y + 8\,\hat{\mathbf{n}}_z$$

Length L_A of the cable joining A and Q.

- Form the position from N_o to A (inspection). $\quad {}^{N_o}\vec{\mathbf{r}}^{\,A} = 8\,\hat{\mathbf{n}}_y + 20\,\hat{\mathbf{n}}_z$
- Form the position from A to Q (vector addition). $\quad {}^{A}\vec{\mathbf{r}}^{\,Q} = {}^{A}\vec{\mathbf{r}}^{\,N_o} + {}^{N_o}\vec{\mathbf{r}}^{\,Q} = 7\,\hat{\mathbf{n}}_x - 3\,\hat{\mathbf{n}}_y - 12\,\hat{\mathbf{n}}_z$
- Calculate ${}^{A}\vec{\mathbf{r}}^{\,Q} \cdot {}^{A}\vec{\mathbf{r}}^{\,Q}$. $\quad (7\,\hat{\mathbf{n}}_x - 3\,\hat{\mathbf{n}}_y - 12\,\hat{\mathbf{n}}_z) \cdot (7\,\hat{\mathbf{n}}_x - 3\,\hat{\mathbf{n}}_y - 12\,\hat{\mathbf{n}}_z) = 202$
- Form $|{}^{A}\vec{\mathbf{r}}^{\,Q}|$, the magnitude of ${}^{A}\vec{\mathbf{r}}^{\,Q}$. $\quad L_A = |{}^{A}\vec{\mathbf{r}}^{\,Q}| = \sqrt{{}^{A}\vec{\mathbf{r}}^{\,Q} \cdot {}^{A}\vec{\mathbf{r}}^{\,Q}} = \sqrt{202} \approx 14.2$

Angle ϕ between lines $\overline{AQ}$ and $\overline{AB}$.

- To solve for ϕ, rearrange the dot-product: $\quad {}^{A}\vec{\mathbf{r}}^{\,Q} \cdot {}^{A}\vec{\mathbf{r}}^{\,B} \triangleq |{}^{A}\vec{\mathbf{r}}^{\,Q}|\,|{}^{A}\vec{\mathbf{r}}^{\,B}|\ \cos(\phi)$

$$\cos(\phi) = \frac{{}^{A}\vec{\mathbf{r}}^{\,Q} \cdot {}^{A}\vec{\mathbf{r}}^{\,B}}{|{}^{A}\vec{\mathbf{r}}^{\,Q}|\,|{}^{A}\vec{\mathbf{r}}^{\,B}|} = \frac{(7\,\hat{\mathbf{n}}_x - 3\,\hat{\mathbf{n}}_y - 12\,\hat{\mathbf{n}}_z) \cdot (\text{-}20\,\hat{\mathbf{n}}_z)}{(14.2) * (20)} = \frac{240}{284} \Rightarrow \phi = \text{acos}\left(\frac{240}{284}\right) \approx \begin{cases} 0.564\ \text{rad} \\ 32.32^\circ \end{cases}$$

Surface area $\vec{\boldsymbol{\Delta}}$ and unit vector $\hat{\mathbf{u}}$ perpendicular to Δ_{ABQ}.

- Visual inspection: Form the position vectors from N_o to Q, N_o to B, and B to A.
  ${}^{N_o}\vec{\mathbf{r}}^{\,Q} = 7\,\hat{\mathbf{n}}_x + 5\,\hat{\mathbf{n}}_y + 8\,\hat{\mathbf{n}}_z \qquad {}^{N_o}\vec{\mathbf{r}}^{\,B} = 8\,\hat{\mathbf{n}}_y \qquad {}^{B}\vec{\mathbf{r}}^{\,A} = 20\,\hat{\mathbf{n}}_z$
- Vector addition: The position from B to Q is: ${}^{B}\vec{\mathbf{r}}^{\,Q} = {}^{B}\vec{\mathbf{r}}^{\,N_o} + {}^{N_o}\vec{\mathbf{r}}^{\,Q} = 7\,\hat{\mathbf{n}}_x - 3\,\hat{\mathbf{n}}_y + 8\,\hat{\mathbf{n}}_z$
- The ***vector area*** is: $\vec{\boldsymbol{\Delta}} \underset{(4)}{\triangleq} \tfrac{1}{2}\,{}^{B}\vec{\mathbf{r}}^{\,A} \times {}^{B}\vec{\mathbf{r}}^{\,Q} = \tfrac{1}{2}\,(20\,\hat{\mathbf{n}}_z) \times (7\,\hat{\mathbf{n}}_x - 3\,\hat{\mathbf{n}}_y + 8\,\hat{\mathbf{n}}_z) = 30\,\hat{\mathbf{n}}_x + 70\,\hat{\mathbf{n}}_y$
- The scalar area is: $|\vec{\boldsymbol{\Delta}}| = \sqrt{\vec{\boldsymbol{\Delta}} \cdot \vec{\boldsymbol{\Delta}}} = \sqrt{30^2 + 70^2} \approx 76.16$
- The unit normal $\hat{\mathbf{u}}$ in the direction of $\vec{\boldsymbol{\Delta}}$ is: $\hat{\mathbf{u}} = \dfrac{\vec{\boldsymbol{\Delta}}}{|\vec{\boldsymbol{\Delta}}|} \approx 0.394\,\hat{\mathbf{n}}_x + 0.919\,\hat{\mathbf{n}}_y$

Distance between point Q and line $\overline{BC}$. $\quad d \underset{(3)}{=} |{}^{B}\vec{\mathbf{r}}^{\,Q} \times \hat{\mathbf{n}}_x| = |8\,\hat{\mathbf{n}}_y - 5\,\hat{\mathbf{n}}_z| = \sqrt{89}$

$+\hat{\mathbf{u}}$ distance from point N_o to plane ABQ. $\quad \delta = \hat{\mathbf{u}} \cdot {}^{N_o}\vec{\mathbf{r}}^{\,B} = (0.394\,\hat{\mathbf{n}}_x + 0.919\,\hat{\mathbf{n}}_y) \cdot 8\,\hat{\mathbf{n}}_y \approx 7.35$

Volume of tetrahedron N_o, A, B, Q. $\quad \text{Volume} \underset{(2.10)}{=} \tfrac{1}{3}\,(\text{-}\vec{\boldsymbol{\Delta}} \cdot {}^{B}\vec{\mathbf{r}}^{\,N_o}) = (\text{-}30\,\hat{\mathbf{n}}_x - 70\,\hat{\mathbf{n}}_y) \cdot (\text{-}8\,\hat{\mathbf{n}}_y) \approx 186.7$