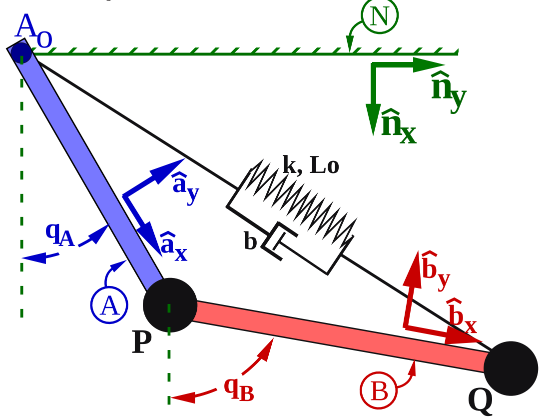
## 20.9 Optional: Statics and dynamics for spring-damper double pendulum.

The following figure shows two light rigid rods $A$ and $B$ and a spring-damper that support two particles $P$ and $Q$ in a Newtonian reference frame $N$. Rod $A$ connects with frictionless revolute joints to $N$ and $B$ at points $A_o$ and $P$, respectively. Right-handed sets of orthogonal unit vectors $\widehat{\mathbf{n}}_i$, $\widehat{\mathbf{a}}_i$, $\widehat{\mathbf{b}}_i$ $(i = x, y, z)$ are fixed in $N$, $A$, $B$, with $\widehat{\mathbf{n}}_x$ vertically-downward, $\widehat{\mathbf{a}}_x$ directed from $A_o$ to $P$, $\widehat{\mathbf{b}}_x$ directed from $P$ to $Q$, and $\widehat{\mathbf{n}}_z = \widehat{\mathbf{a}}_z = \widehat{\mathbf{b}}_z$ parallel to the revolute joints' axes.

| Quantity | Symbol | Value |
|---|---|---|
| Mass of $P$ | $m^P$ | 10 kg |
| Mass of $Q$ | $m^Q$ | 20 kg |
| Earth's gravitational constant | $g$ | $9.8\ \frac{\text{m}}{\text{s}^2}$ |
| Distance from $A_o$ to $P$ | $L_A$ | 1 m |
| Distance from $P$ to $Q$ | $L_B$ | 2 m |
| Spring's natural length | $L_o$ | 1 m |
| Linear spring constant | $k$ | $200\ \frac{\text{N}}{\text{m}}$ |
| Linear damping constant (force) | $b$ | $100\ \frac{\text{N*s}}{\text{m}}$ |
| Linear damping constant (torques) | $c$ | $100\ \frac{\text{N m s}}{\text{rad}}$ |
| Angle from $\widehat{\mathbf{n}}_x$ to $\widehat{\mathbf{a}}_x$ with $+\widehat{\mathbf{n}}_z$ sense | $q_A$ | Variable |
| Angle from $\widehat{\mathbf{n}}_x$ to $\widehat{\mathbf{b}}_x$ with $+\widehat{\mathbf{n}}_z$ sense | $q_B$ | Variable |




- Form statics equations governing $q_A$ and $q_B$ (when damping has stopped the system's motion).[1]

Determine four static solutions for $q_A$ and $q_B$ between -180° and 180°.

**Result:** (Using intuition/guessing, circle the **stable** solutions).

$$L_{\text{Spring}} = \sqrt{L_A^2 + L_B^2 + 2\,L_A\,L_B\,\cos(q_A - q_B)} \qquad s = L_{\text{Spring}} - L_o$$

$$\text{Static}_1 = L_A\left[L_B\,k\,s\,\frac{\sin(q_A - q_B)}{L_{\text{Spring}}} - g\,(m^P + m^Q)\,\sin(q_A)\right] = 0$$

$$\text{Static}_2 = L_B\left[-L_A\,k\,s\,\frac{\sin(q_A - q_B)}{L_{\text{Spring}}} - g\,(m^Q\,\sin(q_B)\right] = 0$$

**Static solutions**

| # | $q_A$ | $q_B$ |
|---|---|---|
| 1 | 0° | 0° |
| 2 | -50.2° | 35.2° |
| 3 | 50.2° | -35.2° |
| 4 | 180° | 180° |

- Form dynamics equations governing $\ddot{q}_A$ and $\ddot{q}_B$ (use air damping torque $\vec{\mathbf{T}}^A = -c\,\dot{q}_A\,\widehat{\mathbf{n}}_z$ and $\vec{\mathbf{T}}^B = -c\,\dot{q}_B$).

**Result:**

$$\dot{s} = -L_A L_B \sin(q_A - q_B)\,(\dot{q}_A - \dot{q}_B)\,/\,L_{\text{Spring}}$$

$$\text{Static}_1 + b\,L_A L_B \frac{\sin(q_A - q_B)}{L_{\text{Spring}}}\dot{s} - c\,\dot{q}_A = (m^P + m^Q) L_A^2\,\ddot{q}_A + m^Q L_A L_B\,\cos(q_A - q_B)\,\ddot{q}_B + m^Q L_A L_B\,\sin(q_A - q_B)\,\dot{q}_B^{\,2}$$

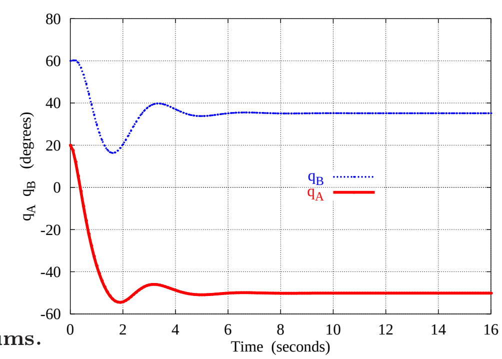
$$\text{Static}_2 - b\,L_A L_B \frac{\sin(q_A - q_B)}{L_{\text{Spring}}}\dot{s} - c\,\dot{q}_B = m^Q\,L_A\,L_B\,\cos(q_A - q_B)\,\ddot{q}_A + m^Q\,L_B^2\,\ddot{q}_B - m^Q\,L_A\,L_B\,\sin(q_A - q_B)\,\dot{q}_A^{\,2}$$

- Plot $q_A$ and $q_B$ for $0 \le t \le 16$ sec when the system is released from ***rest*** with $q_A = 20°$ and $q_B = 60°$.

Verify the following static solution for $q_A$, $q_B$, and the $\widehat{\mathbf{n}}_x$ and $\widehat{\mathbf{n}}_y$ measures of the reaction force on $A_o$.

**Result:** $q_A(t{=}16) \approx -50.2°$ $\qquad q_B(t{=}16) \approx 35.2°$

$F_x(t{=}16) \approx -294$ N $\qquad F_y(t{=}16) \approx 0$ N




- **Optional:** Form a numerical integration energy checking function and verify it remains approximately constant.

Solution at **www.MotionGenesis.com** ⇒ **Get Started** ⇒ **Pendulums**.

[1] Consider using ***MG road-maps*** or ***potential energy*** minimization for $q_A$ and $q_B$, or ***Kane's equations*** for ***generalized speeds*** $\dot{q}_A$, $\dot{q}_B$, or ***Lagrange's equations*** for ***generalized coordinates*** $q_A$, $q_B$.