

Contents

Statics and dynamics – what does it do for us?	i
Newton’s three laws	ii
Methods for forming equations of motion	iii
1 Math tools	1
1.1 Unit systems - SI and USA	1
1.2 Geometry: Ancient Euclid and modern vectors (see Chapters 2, 3, . . .).	2
1.3 Circles and their properties	2
1.4 Triangles and ratios of their sides (sine, cosine, tangent)	2
1.4.1 Unit circle concept of sine and cosine	2
1.4.2 Formulas involving sine and cosine	3
1.4.3 Sine and cosine as functions (Euler, circa 1730)	3
1.4.4 The amplitude-phase formulas for sine and cosine	4
1.4.5 The function $\text{atan2}(y, x)$	4
1.5 Types of scalars: Variable, specified, constant	4
1.6 Differentiation	5
1.6.1 Definition of an ordinary derivative of a scalar function	5
1.6.2 Definition of a partial derivative of a scalar function	5
1.6.3 Definition of the differential of an independent variable and scalar function	5
1.6.4 Short table of derivatives encountered in engineering	6
1.6.5 Good product rule for differentiation (for scalars, $\vec{\text{vectors}}$, matrices, . . .)	6
1.6.6 Quotient rule for derivatives (or use $\frac{u}{v} = uv^{-1}$ with exponent and product rules)	6
1.6.7 Chain rule for ordinary and partial derivatives	6
1.6.8 Example: Partial and ordinary differentiation	6
1.6.9 Implicit differentiation: A useful tool for calculating derivatives	7
1.7 Integration and a short table of integrals	7
1.8 Minimization and maximization (optimization)	8
1.9 Minimization and maximization with constraints (constrained optimization)	8
1.10 Solutions of <i>polynomial</i> equations (roots) quadratic equation	9
1.11 Computer solutions of algebraic and differential equations	10
1.12 Continuous solutions of <i>nonlinear</i> algebraic equations	10
2 Vectors ($\hat{=}$ $\vec{\mathbf{a}} + \vec{\mathbf{b}}$ $-\vec{\mathbf{b}}$ $\angle(\vec{\mathbf{a}}, \vec{\mathbf{b}})$ $\vec{\mathbf{a}} \cdot \vec{\mathbf{b}}$ $\vec{\mathbf{a}} \times \vec{\mathbf{b}}$)	11
2.1 Examples of scalars vectors and dyadics	11
2.2 Definition of a vector	12
2.3 Zero vector $\vec{\mathbf{0}}$, a vector whose magnitude is zero	12
2.4 Unit $\hat{\text{vectors}}$: Vectors with magnitude 1 and no units (typeset with a $\hat{\text{h}}$ at)	12
2.5 Equal vectors ($=$) vectors with the same magnitude and direction	13
2.6 Vector addition ($+$)	13
2.7 Vector multiplied or divided by a scalar ($*$ or $/$)	13
2.8 Vector negation and subtraction ($-$)	14

2.9	Vector dot product (\cdot)	14
2.9.1	Properties of the dot-product (\cdot)	14
2.9.2	Uses for the dot-product (\cdot)	15
2.9.3	Dot-products to change vector equations to scalar equations (see Hw 1.33)	15
2.9.4	Special case: Dot-products with orthogonal unit vectors (and matrix multiplication)	15
2.9.5	Examples: Vector dot-products (\cdot)	15
2.10	Vector cross product (\times)	16
2.10.1	Uses for the cross-product (\times) in geometry, statics, motion analysis,	16
2.10.2	Determinants and cross-products (with right-handed unit vectors)	16
2.10.3	Optional: Skew-symmetric matrices, cross products, and dot products	17
2.11	Scalar triple product ($\cdot \times$ or $\times \cdot$)	18
2.11.1	Scalar triple product and the volume of a tetrahedron	18
2.11.2	($\times \cdot$) to change vector equations to scalar equations (see Hw 1.33)	18
2.12	Words: Vectors vs. 1D matrices in the context of $\vec{F} = m \vec{a}$	18
3	Position vectors and vector geometry	19
3.1	What is a point, particle, position vector (see examples in Hw 3)	19
3.2	Area of a triangle	19
3.3	Geometry example: Length/distances, angle, surface area, volume	20
3.4	Measuring position: Cartesian, cylindrical, spherical coordinates	21
3.5	Distances: Point to line . . . (distance-traveled and speed is in Section 10.8)	21
3.6	Optional: Vector equation of a line or plane	21
3.7	Optional: Proof of law of cosines and law of sines with vectors	22
3.8	Optional: Proof of sine and cosine addition formulas with vectors	22
4	Vector basis	23
4.1	What is a vector basis?	23
4.2	Rigid and non-rigid bases	24
4.3	Non-orthogonal 3D vector basis	24
4.4	Creating various 3D bases from two non-parallel vectors	24
4.5	Concept: What is the vector vs. how is it expressed	25
4.6	Expressing a vector in terms of the basis $\vec{a}_1, \vec{a}_2, \vec{a}_3$	26
4.7	Right-handed (standard) and left-handed (unconventional) bases	26
4.8	Optional: Rigid frames and transformation matrices	27
4.9	Optional: Coordinate system vs. vector bases	28
5	Rotation matrices I	29
5.1	Uses for the rotation matrix ${}^aR^b$ (for geometry, statics, motion, stress . . .)	29
5.2	Rotation matrices and “squash rules”	30
5.3	Rotation matrices - why are they so important?	30
5.4	How to use a rotation matrix (angles, inverse, express, dot, cross)	31
5.5	Simple rotation matrices and the “hug rule”	32
5.6	Forming rotation matrices with matrix multiplication	33
5.7	What is an angle?	34
5.8	Optional: Proofs	35
5.8.1	Proof that a rotation matrix is orthonormal	35
5.8.2	Proof of relationship between $\hat{b}_x, \hat{b}_y, \hat{b}_z$ and $\hat{a}_x, \hat{a}_y, \hat{a}_z$ measures of a vector \vec{v}	35
5.8.3	Proof of skew-symmetric matrix relationship $\text{skew}[\vec{v}]_b = {}^bR^a \text{skew}[\vec{v}]_a {}^aR^b$	35

6	† Rotation matrices II	37
6.1	Other representations of orientation (see examples in Hw 7)	37
6.2	Euler rotation angles (24 combinations in the textbook's appendix of rotations)	38
6.2.1	Body xyz angles $\theta_1, \theta_2, \theta_3$ to rotation matrix R and angular velocity $\vec{\omega}$	38
6.2.2	Rotation matrix R to Body xyz angles $\theta_1, \theta_2, \theta_3$ (general algorithm in textbook appendix)	39
6.2.3	Optional: Space-fixed rotation sequences (see appendix)	39
6.3	Clinical case study of pelvis orientation (Euler angles vs. clinical angles)	40
6.3.1	Body yxz Euler angles [called TOR (tilt-obliquity-rotation) in biomechanics]	41
6.3.2	Body zxy Euler angles [called ROT (rotation-obliquity-tilt) in biomechanics]	41
6.3.3	Clinical angles (physically measurable angles)	42
6.4	Simple $\theta \hat{\lambda}$ rotation	44
6.4.1	Converting a simple $\theta \hat{\lambda}$ rotation to a rotation matrix	44
6.4.2	How a simple $\theta \hat{\lambda}$ rotation affects an arbitrary vector \vec{b} fixed in B	45
6.4.3	Rotation dyadic and rotation matrix for a simple $\theta \hat{\lambda}$ rotation	45
6.4.4	Special case: Rotation matrix from two vectors without θ and $\hat{\lambda}$ (example in Hw 7.5).	46
6.4.5	Optional: Converting a rotation matrix ${}^aR^b$ to a simple $\theta \hat{\lambda}$ rotation	46
6.5	Quaternion (Euler parameters) and $\theta \hat{\lambda}$ rotations	47
6.5.1	Calculating $\theta \hat{\lambda}$ from $\epsilon_0 \vec{e}$ (i.e., calculating $\theta \hat{\lambda}$ from a quaternion)	47
6.5.2	Quaternion (Euler parameters) and how a $\theta \hat{\lambda}$ rotation affects an arbitrary vector	48
6.5.3	Converting a quaternion (Euler parameters) to a rotation matrix	48
6.5.4	Converting a rotation matrix to a quaternion (or to $\theta \hat{\lambda}$ or to Rodrigues parameters)	48
6.5.5	Quaternion (Euler parameters) and inverse rotations	49
6.5.6	Quaternion (Euler parameters) and successive rotations	49
6.5.7	Converting Body xyz angles to a quaternion (Euler parameters)	49
6.5.8	Quaternion (Euler parameters) and small rotations	49
6.6	Rodrigues parameters (invented by Euler 1770, rediscovered by Rodrigues 1840)	50
6.7	Poisson parameters (9 elements of a rotation matrix)	50
6.7.1	Optional: Interdependencies in the elements of any rotation matrix R	50
6.7.2	Optional: Eigenvalues, eigenvectors and rotation matrices	50
6.8	Optional: Indirect determination of orientation	51
6.9	Optional: SVD orthogonalization of approximate rotation matrix	52
6.10	Optional: Various opinions about rotation matrix language	53
6.11	Optional: Proofs	54
6.11.1	Proof that equation (20) corresponds to an eigenvector of a rotation matrix	54
6.11.2	Geometrical proof of how a $\theta \hat{\lambda}$ rotation affects a vector fixed in B	54
6.11.3	Mathematical proof of how a $\theta \hat{\lambda}$ rotation affects a vector fixed in B	55
6.11.4	Proof: Converting a simple $\theta \hat{\lambda}$ rotation to a rotation matrix	55
6.11.5	Proof: Quaternion (Euler parameters) to rotation matrix	56
6.11.6	Proof: Rotation matrix to quaternion (Euler parameters) (algorithm in Section 6.5.4)	56
6.11.7	Proof: Converting Body xyz angles to a quaternion (Euler parameters)	57
7	Vector differentiation and integration	59
7.1	Definition: Derivative of a vector in a rigid basis (or reference frame)	59
7.2	What is a constant vector (i.e., a vector fixed in a reference frame)?	60
7.3	Derivative of a constant magnitude vector ($ \vec{v} $ is constant)	60
7.4	Properties of ordinary <u>or</u> partial derivatives of vectors	60
7.5	Optional: Expressing a vector in terms of a rigid vector basis	60
7.6	Example: Derivative of a vector	61
7.7	Optional: Differential of a vector in a rigid basis (or reference frame)	61
7.8	Optional: Integral of a vector in a rigid basis (or reference frame)	62
7.9	Differentiation concepts: Changes in magnitude and direction	62

7.10	Optional: Limit of a vector in a reference frame	62
7.11	Optional: Derivative of a scalar with respect to a vector (gradients)	63
7.11.1	Differential geometry: Normal and tangent to a circle	63
7.11.2	Differential geometry: Normal and tangent to an ellipse	63
7.11.3	Differential geometry: Normal (gradient) to an ellipsoid	63
7.12	Optional: Proof of vector dot-product derivative property	64
7.13	Optional: Proof magnitude-direction properties for vector derivative	64
8	Angular velocity & angular acceleration	65
8.1	Angular velocity concepts: Moon and Earth celestial systems	65
8.2	What is a reference frame? (also see Section 8.2.3).	66
8.2.1	What is a Newtonian (inertial/fixed) reference frame? $\vec{F} = m\vec{a}$	66
8.2.2	Orientation of a rigid body (or reference frame)	66
8.2.3	Differences between a vector basis, rigid frame, and reference frame	66
8.3	Defining property of angular velocity (golden rule for vector differentiation)	67
8.3.1	Example: Angular velocity and vector differentiation	67
8.3.2	Example: Vector differentiation and angular momentum – spinning body	68
8.3.3	Simple angular velocity	68
8.3.4	Angular velocity negative property	69
8.3.5	Angular velocity addition theorem (<i>squash rule for adding angular velocities</i>)	69
8.3.6	Angular velocity example: Welding robot	69
8.3.7	Angular velocity example: Chaotic plate pendulum	69
8.3.8	Angular velocity – a property of a rigid object	70
8.3.9	Optional: Mitiguy’s formula for angular velocity	70
8.3.10	Optional: Angular velocity and orthogonal basis vectors	70
8.3.11	Optional: Angular velocity and rotation matrices	71
8.4	Angular acceleration	72
8.4.1	Angular acceleration addition theorem (<i>\vec{a} squash rule only works in 2D!</i>)	72
8.4.2	Angular acceleration negative property	72
8.4.3	Angular acceleration example: Chaotic plate pendulum	72
8.4.4	Optional: 2^{nd} time-derivative of a vector in a reference frame (or rigid vector basis)	72
8.5	Optional: Angular velocity proofs	73
8.5.1	Proof of existence and uniqueness of the defining property of angular velocity	73
8.5.2	Proof of the new formula for angular velocity	73
8.5.3	Proof of angular velocity and orthogonal basis vectors	74
8.5.4	Proof of simple angular velocity	74
8.5.5	Proof of angular velocity addition theorem	75
8.5.6	Proof of angular velocity negative property	75
8.5.7	Proof of 2^{nd} time-derivative of any vector	75
9	†(Advanced) Angular velocity and kinematical ODEs	77
9.1	Time-derivative of yaw-pitch-roll (Euler) angles and angular velocity $\vec{\omega}$	77
9.2	Angular velocity and $\theta \hat{\lambda}$ rotations ($\vec{\omega}$ in terms of $\theta \hat{\lambda}$ and $\frac{d\hat{\lambda}}{dt}$)	78
9.2.1	Simple angular velocity – the special case $\vec{\omega} = \theta \hat{\lambda}$	78
9.2.2	Optional: $\dot{\theta}$ and $\frac{d\hat{\lambda}}{dt}$ in terms of angular velocity $\vec{\omega}$ (kinematical ODEs)	78
9.3	Quaternions (Euler parameters) and angular velocity/acceleration	79
9.3.1	Quaternion derivatives in terms of angular velocity $\vec{\omega}$ and angular acceleration $\vec{\alpha}$	79
9.3.2	Angular velocity $\vec{\omega}$ and angular acceleration $\vec{\alpha}$ in terms of quaternion derivatives	80
9.3.3	Constraints associated with quaternions (Euler parameters)	80
9.3.4	Optional: Quaternions (Euler vector/parameters) and small, slow rotations	80

9.4	Optional: Rodrigues vector/parameters and angular velocity $\vec{\omega}$	80
9.5	Rotation matrix time-derivative \dot{R} and angular velocity $\vec{\omega}$	81
9.5.1	Rotation matrix time-derivatives as $\dot{R} = \text{skew}[\vec{\omega}]_b R$ or $\dot{R} = R \text{skew}[\vec{\omega}]_a$	81
9.6	Optional: Proofs	82
9.6.1	Proof: Angular velocity $\vec{\omega}$ in terms of $\dot{\theta}$, $\frac{B d\hat{\lambda}}{dt}$	82
9.6.2	Proof: ODEs relating $\dot{\theta}$ and $\frac{B d\hat{\lambda}}{dt}$ to angular velocity $\vec{\omega}$	83
9.6.3	Proof: ODEs for quaternions to angular velocity $\vec{\omega}$ and angular acceleration $\vec{\alpha}$	83
9.6.4	Proof: Angular velocity $\vec{\omega}$ and angular acceleration $\vec{\alpha}$ in terms of quaternions	84
9.6.5	Proof: ODEs relating Rodrigues parameters $\frac{B d\vec{p}}{dt}$ to angular velocity $\vec{\omega}$	84
9.6.6	Proof: Angular velocity $\vec{\omega}$ in terms of Rodrigues parameters $\frac{B d\vec{p}}{dt}$	85
9.6.7	Proof: Rotation matrix time-derivative \dot{R} in terms of angular velocity $\vec{\omega}$	85
9.6.8	Optional: Angular acceleration and 2^{nd} -derivative of rotation matrix \ddot{R}	86
10	Points: Velocity and acceleration	87
10.1	Definition of a point's velocity and acceleration	88
10.2	Velocity and acceleration of two points <u>fixed</u> on a rigid body	90
10.3	Velocity and acceleration of a point <u>moving</u> on a rigid body	91
10.4	Velocity and acceleration of a point via another point	92
10.5	Relative velocity and relative acceleration of a point	93
10.6	Elongation (separation-speed) and elongation-rate	93
10.7	Relationship between position, velocity, and acceleration	94
10.7.1	Constant acceleration along a curve or line (also see Section 10.8)	94
10.7.2	Circular motion and centripetal acceleration – a special case	94
10.7.3	Analogy between translational motion and planar (2D) rotational motion	94
10.7.4	Example: Free-fall of a sky-diver with constant downward acceleration	95
10.8	Speed and distance-traveled in a reference frame (see Hw 9.4, 9.7, 15.5)	95
10.9	Optional: Acceleration vocabulary	96
10.10	Optional: Definition of jerk (snap, crackle, pop)	96
10.11	Optional: Jacobian definitions and notation	97
10.11.1	Velocity and angular velocity in terms of their Jacobians.	97
10.11.2	Generalized forces in terms of velocity and angular velocity Jacobians.	97
10.11.3	Motion constraints, generalized forces, and Jacobians.	98
10.11.4	Equations of motion in terms of constraint Jacobians	98
10.11.5	Example: Position and velocity Jacobians (cylindrical coordinates)	98
10.11.6	Example: Jacobians and their time-derivatives (robotic satellite tracker)	99
10.12	Velocity and acceleration proofs	100
10.12.1	Proof of equivalence of the two relative velocity/acceleration definitions	100
10.12.2	Proof of relative velocity/acceleration formulas for coincident points	101
10.12.3	Proof of minimum velocity magnitude/location and instant center (for Section 10.2)	101
11	Constraints: Rods, rolling, gears, ...	103
11.1	Number of degrees of freedom (an integer characterizing motion)	104
11.2	Rods, ropes, and separators (example in Section 11.14)	104
11.3	Linear actuator	104
11.4	Ball and socket joint	104
11.5	Revolute joint (also called a hinge or pin joint)	105
11.6	Revolute motor (revolute joint with additional angular constraint)	105
11.7	Cylindrical joint (revolute joint on straight slot)	105
11.8	Ball and socket in slot	105

11.9 Prismatic joint (also called a slider joint or square-slot joint)	106
11.10 Universal joint (Hooke's joint)	106
11.11 Contact (an intermittent inequality constraint)	106
11.12 Sliding, rolling, and pure rolling (contact)	107
11.12.1 Example: Velocity, sliding, rolling, and two points fixed on a rigid object	107
11.12.2 Example: Acceleration, rolling, and two points fixed on a rigid object	107
11.12.3 Example: Rolling motions of a cylinder inside a cylinder	108
11.13 Gears and constraint equations	109
11.14 Constraints associated with cables supporting a platform	111
11.15 Optional: Constraint stabilization	112
12 Particles (points with mass)	113
12.1 $\vec{F} = m\vec{a}$ recipe for translational motion (examples: Sections 12.11, 12.12, Hw 12)	114
12.2 Systems with particles	114
12.3 Translational (linear) momentum for particles	114
12.4 Angular momentum and shift theorem for particles	115
12.5 Kinetic energy for particles	116
12.6 Effective force for particles	117
12.7 Moment of effective force and shift theorem for particles	117
12.8 Optional: Analogies between momentum and effective force	118
12.9 Shift theorems for moments of momentum, force, inertia, ...	118
12.10 Child on a swing: Momentum, effective force, and energy	119
12.11 Example: $\vec{F} = m\vec{a}$ recipe for translational motion (rocket-sled)	120
12.12 Example: $\vec{F} = m\vec{a}$ recipe for translational motion (particle in spinning slot)	121
12.13 Optional: Proofs for momentum and effective force	122
12.13.1 Proof of translational (linear) momentum of a system of particles	122
12.13.2 Proof of effective force and its relationship to translational momentum	122
12.13.3 Proof of shift theorem for angular momentum	123
12.14 Optional: A different definition of angular momentum	123
13 Mass, center of mass, centroid	125
13.1 Mass	125
13.2 Center of mass and centroid	125
13.3 Figures and their length, area, and volume	127
13.4 Integrals for mass, center of mass, and centroid	127
13.5 Optional: Experiments, tables, CAD/CAE, and medical scanning	128
13.6 Optional: Measuring mass	128
13.7 Optional: The language and etymology of "mass"	128
13.8 Optional: What is mass?	129
13.9 Optional: Mass, energy, and physics	129
14 Concepts: Moments/products of inertia	131
14.1 Moment of inertia (also see Section 16.3)	131
14.1.1 Shift theorem for moment of inertia (parallel axis theorem)	132
14.1.2 Example: Moments of inertia of a particle (repeated in Hw 13.2)	132
14.1.3 Example: Moment of inertia concepts (repeated in Hw 13.3)	132
14.1.4 Demo: How moment of inertia affects rolling objects	133
14.1.5 Demo: How moment of inertia affects a spinning book (repeated in Hw 13.4)	133
14.2 Products of inertia (also see Section 16.4)	133
14.2.1 Shift theorem for product of inertia	134
14.2.2 Example: Products of inertia of a particle (repeated in Hw 13.2)	134
14.2.3 Conceptual example of products of inertia (repeated in Hw 13.9)	134

14.2.4	Demo: How product of inertia affects a rotating rattleback (repeated in Hw 13.18)	134
14.2.5	Two sign conventions (\pm) for products of inertia	135
14.3	Optional: Radius of gyration and moment of inertia	136
14.3.1	Concept: Radius of gyration as matter concentrated in a thin-ring	136
14.3.2	Concept: Radius of gyration as a single particle	136
14.3.3	Context: Uses for radius of gyration	136
14.3.4	Calculation: Shift theorem for radius of gyration (parallel axis theorem)	136
15	Dyadics	137
15.1	Expressing a dyadic in terms of basis $\vec{a}_1, \vec{a}_2, \vec{a}_3$	138
15.2	Derivative of a dyadic in a rigid basis (or a reference frame)	139
15.3	Dyadics and 3×3 matrices	139
15.4	Dyadic symmetry	140
15.5	Dyadic principal directions, eigenvalues, and eigenvectors	140
15.6	Optional: Dyadic inverse and positive-definite dyadics	140
15.7	Optional: Vector \times dyadic and skew-symmetric matrices	141
15.8	Optional: Proof of unit dyadic expressed in orthogonal basis	141
16	Inertia dyadics	143
16.1	Inertia dyadic: A “suitcase” for moments and products of inertia	143
16.1.1	Inertia dyadic for a particle or system	143
16.1.2	Inertia dyadics for continuous systems	143
16.1.3	Shift theorem for an inertia dyadic (for an arbitrary system S)	144
16.1.4	Example: Inertia dyadic of a particle (also see example in Section 16.6)	144
16.2	Inertia scalars in terms of an inertia dyadic	144
16.3	Moments of inertia and inertia dyadics (also see Section 14.1)	145
16.4	Products of inertia and inertia dyadics (also see Section 14.2)	145
16.5	Inertia matrix (a symmetric matrix)	146
16.6	Example: Inertia properties for a child on a swing	147
16.7	Example: Mass properties of 3 particles on a parallelepiped	147
16.8	Optional: Axis-symmetric and triaxially-symmetric inertia	148
16.9	Motivating concepts for inertia dyadics	149
16.10	Optional: Inertia vector	150
16.11	Optional: Principal inertia directions – an eigen-problem	150
16.12	Optional: Proof of inertia dyadic shift theorem	151
16.13	Optional: Summations and the inertia dyadic (useful for proofs)	151
16.14	Nonnegative sums and moment of inertia inequalities	152
16.15	Proof of product of inertia inequalities	152
17	Rigid bodies	153
17.1	What is a rigid body? (see examples in Hw 15, laws of motion in Chapter 22).	153
17.2	Angular momentum of a rigid body (also see Sections 22.5, 22.8)	154
17.2.1	Conservation of angular momentum for axis-symmetric rigid body (more Section 22.8).	154
17.3	Effective force of a rigid body	155
17.4	Moment of effective force of a rigid body	155
17.5	Kinetic energy of a rigid body	156
17.6	$\vec{F} = m\vec{a}$ recipe for translational motion of a rigid body B in a Newtonian frame N	156
17.7	$\vec{M} = I\vec{\alpha}$ recipe for rotational motion of a rigid body B via a point B_p of B	156
17.8	Example: Momentum of an inverted pendulum on cart	157
17.9	Example: Kinetic energy of inverted pendulum on cart	157
17.10	Example: Momentum, effective force, & kinetic energy of a simplified aircraft	158
17.11	Optional: Proof of angular momentum for a rigid body	159

17.12	Optional: Proof of moment of effective force for a rigid body	160
17.13	Optional: Proof of kinetic energy for a rigid body	161
17.14	Optional: Proof of conservation of angular momentum for an axis-symmetric rigid body	162
18	Force and resultant	163
18.1	Force and the law of action/reaction	163
18.2	Statics, dynamics, and resultant forces	164
18.3	Resultant of a set of vectors (e.g., forces on a point, body, or system)	164
18.4	Resultant force on a system (internal force cancellation)	164
18.5	Force and effective force – two types of forces in $\vec{F} = m\vec{a}$	165
18.6	Contact and distance forces	165
18.7	Applied and constraint forces	165
18.8	Optional: Conditions for force to exist (section under ongoing discussion)	165
18.9	Optional: The definition and philosophy of force	166
18.10	Optional: Force, mass, gravity, space, time, reference frames	166
19	Moment and torque	167
19.1	Moment of a vector	167
19.1.1	Moment of a set of vectors	167
19.1.2	Shift theorem for the moment of a set of vectors	167
19.1.3	$\hat{\mathbf{u}} \cdot \vec{M}^{S/P} = \hat{\mathbf{u}} \cdot \vec{M}^{S/O}$ if $\hat{\mathbf{u}}$ is parallel to line \overline{OP} (useful for MG road-maps in Section 23.1)	168
19.1.4	Moment arm of a vector about a point (example at end of Section 19.6)	168
19.2	Moment of internal forces (a separate law of mechanics?)	168
19.3	Statics, dynamics, and moments of forces	168
19.4	Definition of static equilibrium	169
19.5	Torque of a set of vectors (moment of a couple)	169
19.6	Neuromuscular biomechanics example: Muscle tension for curling	170
19.7	Optional: Proof of shift theorem for moment of a set of vectors	172
19.8	Optional: Minimum moment magnitude and central axis (wrench)	172
20	Replacement of forces and bound vectors	173
20.1	Replacement of distance forces (not a unique process)	173
20.2	Replacement of contact forces	174
20.3	Applicability of equivalence/replacement?	174
20.4	Replacement of forces associated with constraints	175
20.5	Optional: Proof of action/reaction for contact forces	176
21	Encyclopedia of applied force and torque	177
21.1	Weight, mass, and gravity	177
21.2	Uniform gravity on a particle or body (see appendix for g for planets)	177
21.3	Universal gravitational attraction between two particles	178
21.4	Electrostatic forces and Coulomb’s law for charged particles	179
21.5	DC (direct current) permanent magnet motors	179
21.6	Kinetic friction and the Continuous Friction Law	180
21.7	Translational spring between two points	181
21.8	Rotational (torsional) spring	181
21.9	Translational damper between two points	182
21.10	Rotational (torsional) damper	182
21.11	Light linear springs and dampers in <i>series</i>	183
21.12	Light linear springs and dampers in <i>parallel</i>	183
21.13	Viscous damper between two surfaces	184
21.14	Fluid forces (lift and drag)	184

21.14.1 Demo: Experimental determination of drag forces on a coffee filter	185
21.14.2 Closed-form solutions with aerodynamic drag	185
21.15 Baseball aerodynamic lift/drag/torque - a case study in philosophy	186
21.16 Optional: Gravitational attraction between a particle and a body	189
21.17 Optional: Gravitational attraction between two bodies	190
21.18 Optional: Center of gravity	191
21.19 Optional: Proofs for spring and gravity forces	192
21.19.1 Optional: Proof of light linear springs and dampers in <i>series</i>	192
21.19.2 Optional: Proof of light linear springs and dampers in <i>parallel</i>	192
21.19.3 Optional: Proof for uniform gravity on a body or set of particles	192
22 Equations of motion	193
22.1 Dynamics methods: Newton, Euler, D'Alembert, Lagrange, Kane	194
22.2 Newton's law of motion for a particle $\vec{\mathbf{F}} = m \vec{\mathbf{a}}$ (also see Chapter 12)	195
22.3 Newton's equation for a system	195
22.4 Translational (linear) momentum principle	195
22.5 Angular momentum principle	195
22.6 Euler's equations for a rigid body (3D)	196
22.7 Impulse/momentum	196
22.8 Conservation of translational/angular momentum	197
22.9 Optional: Proof: Translational momentum principle	197
22.10 Proof: Angular momentum principle, Euler's rigid body equation	198
23 D'Alembert's method with MG road-maps	201
23.1 MG road-maps for efficient statics and dynamics	202
23.1.1 MG road-map: Projectile motion (2D)	203
23.1.2 MG road-map: Rigid body pendulum (2D)	203
23.1.3 MG road-map: Inverted pendulum on cart (x and θ) (2D)	203
23.1.4 MG road-map: Rotating rigid body (3D)	204
23.1.5 MG road-map: Bridge crane equations of motion (2D)	204
23.1.6 MG road-map: Particle on spinning slot (2D)	204
23.1.7 MG road-map: Motion of a chaotic double pendulum (3D)	205
23.1.8 MG road-map: Particle pendulum (2D) – angle and tension	205
24 Power and work	207
24.1 Power/kinetic energy-rate principles	207
24.2 Why the power/kinetic energy-rate principle is such a useful tool.	208
24.3 Power of a force, set of forces, or torque on a rigid body	209
24.4 Forces that do not contribute to power (workless forces)	209
24.5 Example: Power/kinetic energy-rate to size a torque-motor	210
24.6 Power/kinetic energy-rate for a commercial spring scale	211
24.7 Definition of work with an integral or differential equation	211
24.8 Work/kinetic energy principle	212
24.9 Optional: Proofs with power	212
24.9.1 Optional: Proof of the power/kinetic energy-rate principle	212
24.9.2 Optional: Proof of the rotational power/kinetic energy-rate principle	213
24.9.3 Optional: Proof of the translational power/kinetic energy-rate principle	213
24.9.4 Optional: Proof for power of a set of forces (or <u>torque</u>) on a rigid body B	213

25 Potential energy and energy conservation	215
25.1 Work/kinetic energy principle (summarized from Section 24.8)	215
25.2 Conservation of mechanical energy (special case of work/kinetic energy)	215
25.3 Potential energy - a special type of work for “conservative” forces	216
25.3.1 Potential energy for a constant force or a uniform gravitational field	216
25.3.2 Potential energy for central forces (e.g., springs and inverse-square-law gravity)	217
25.3.3 Potential energy for a simple rotational spring	217
25.4 Force and gradient of potential energy	218
25.5 Optional: Proofs of potential energy	218
25.5.1 Optional: Proof of potential energy for a constant force in reference frame N	218
25.5.2 Optional: Proof of potential energy for a central force	218
25.5.3 Optional: Proof of applied force as negative gradient of potential energy	219
25.5.4 The proof of generalized forces \mathcal{F}_r from potential energy U is in Section 26.9	219
26 Kane’s method	221
26.1 Kane’s equation of motion (dynamics)	221
26.1.1 Examples of Kane’s method (also see examples in Hw 21, 22)	221
26.1.2 Comparison of the methods of Kane and Lagrange (also see Section 28.8)	221
26.2 Generalized speeds and partial velocities (for Kane’s method)	222
26.3 Kane’s method (with proof)	223
26.4 Generalized force - useful for Lagrange and Kane’s equations	224
26.4.1 Generalized force for a force, set of forces, or torque on a rigid body	224
26.4.2 Generalized forces, potential energy, static equilibrium	225
26.4.3 Non-contributing forces (forces that do not contribute to generalized force \mathcal{F}_r)	225
26.4.4 Examples of generalized forces (see guided problems in Hw 21, 22)	225
26.5 Generalized effective force of a particle or rigid body (handling $m\mathbf{\ddot{a}}$)	226
26.5.1 Generalized effective force of a system of particles	226
26.5.2 Generalized effective force of a rigid body	226
26.5.3 Optional: Generalized effective force’s relationship to kinetic energy	227
26.5.4 Optional: Efficiency of generalized effective force vs. kinetic energy	227
26.6 Reminder: The three sets of equations in Kane’s method	227
26.7 Optional: Generalized momentum of a particle and rigid body	228
26.7.1 Generalized momentum of a system of particles and kinetic energy	228
26.7.2 Generalized momentum of a rigid body	228
26.8 Optional: Proof of generalized effective force for a rigid body	229
26.9 Optional: Proof of generalized forces from potential energy	229
27 Lagrange’s method	231
27.1 Lagrange’s equations of motion	231
27.2 Optional: Lagrange multipliers λ for constraints (useful math abstraction)	231
27.3 Optional: Novel proof of Lagrange’s equations of motion (Mitiguy)	232
27.4 Optional: Interchange properties of partial/ordinary derivatives	233
27.5 Optional: Inefficiencies in Lagrange’s equations	234
27.6 Optional: Virtual work	234
27.6.1 Virtual work for a force or set of forces	234
27.6.2 Virtual work and generalized force	235
27.6.3 Virtual work and static equilibrium	235
27.7 Optional: Virtual displacements/angular displacements	235

28 Force and motion scalars: Generalized coordinates/speeds, ...	237
28.1 Force and impulse scalars - pushes and pulls	237
28.2 State and state variables	237
28.3 Position scalars and generalized coordinates	237
28.4 Velocity scalars and generalized speeds (also see Section 28.8)	238
28.5 Kinematical differential equations (ODEs)	238
28.6 Velocity written in terms of velocity scalars (unconstrained)	239
28.6.1 Velocity scalars, motion constraints, and degrees of freedom	239
28.6.2 Optional: Motion constraints and independent/dependent velocity scalars	239
28.6.3 Optional: Velocity written in terms of velocity scalars (with constraints)	240
28.7 Research topic: Acceleration scalars	240
28.8 Optional: Guidelines for choosing efficient velocity scalars	241
29 MIPS: Classic particle pendulum	245
29.1 Modeling the classic particle pendulum	245
29.2 Identifiers for the classic particle pendulum	246
29.3 Physics: Equations of motion of the classic particle pendulum	246
29.3.1 $\vec{F} = m\vec{a}$ for the particle pendulum	247
29.3.2 Angular momentum principle for the particle pendulum	247
29.3.3 Euler's rigid body equation for the particle pendulum	248
29.3.4 Kinetic energy for the particle pendulum	248
29.3.5 Power/kinetic energy-rate principle for the particle pendulum	248
29.3.6 Conservation of mechanical energy for the particle pendulum	249
29.3.7 Kane's method for the particle pendulum	249
29.3.8 Lagrange's method for the particle pendulum	249
29.4 Solution of the classic particle pendulum ODE	249
29.4.1 Numerical solution of pendulum ODE via MotionGenesis and/or MATLAB [®]	250
29.4.2 Optional: Exact (closed-form) solution of the classic particle pendulum ODE	250
29.4.3 Simplification and analytical solution of the classic particle pendulum ODE	250
29.5 Interpretation of results for the classic particle pendulum	250
30 Example: Inverted pendulum on cart	251
30.1 Kinematics (space and time)	251
30.2 Rotation matrix, angular velocity, angular acceleration	251
30.3 Position vectors, velocity, acceleration	252
30.4 Forces, moments, and 2D free-body diagrams (FBD)	253
30.5 Mass, center of mass, inertia (required by dynamics)	253
30.6 Newton/Euler laws of motion for A and B separately (inefficient)	253
30.7 Dynamics of a rigid body with simple angular velocity (special 2D case)	254
30.8 Optional: Angular momentum principle (2D alternative to Section 30.7)	254
30.9 Optional: Angular momentum principle (3D alternative to Section 30.7)	254
30.10 Equations of motion via MG road-maps/D'Alembert (efficient)	254
30.11 Generalized forces for Lagrange/Kane's equations of motion	255
30.12 Lagrange's equations of motion (described in Chapter 27)	255
30.13 Kane's equations of motion (described in Chapter 26)	256
30.14 Matrix form of equations of motion (for solution, controls, ...)	256
31 Solving for forces and motion. Forward, inverse, and mixed dynamics.	257
31.1 Matrix form of equations of motion	257
31.2 Inverse dynamics: Calculating knee joint torque	258
31.3 Inverse dynamics: Calculating hip and knee joint torque	258
31.4 Solving for forces and motion with operational tasks	259

31.4.1	Inverse dynamics: Force/torques from specified end-effector motion	259
31.4.2	Example: Inverse dynamics with operational task to hip and knee joint torques . . .	260
32	Feed-forward control	261
32.1	Feed-forward control	261
32.1.1	Example: Feed-forward control of a rocket-sled's <u>velocity</u>	262
32.1.2	Example: Feed-forward control of a rocket-sled's <u>position</u>	263
32.1.3	Example: Leg press Feed-forward control with operational task	263
32.1.4	Example: Feed-forward control with additional actuator (over-actuated)	264
32.2	Optional: k_p , k_d , and analytical solutions of 2^{nd} -order ODEs	264
32.3	Derivation/insights into the mathematics of feed-forward control	265
33	Flexible Bodies (Overview)	267
Appendices and index		269
Appendix:	Mass and geometry properties of common objects	269
Appendix of	constants	274
Rotation tables	and the angle θ	275
Rotation tables	and angles θ_1 and θ_2	276
Rotation tables	and angles $\theta_1, \theta_2, \theta_3$	278
Summary of	equations	282
Bibliography	287
HOMEWORK		291
Homework 1:	Vectors (basis independent)	293
Homework 2:	Vector addition and dot/cross-products (with basis)	301
Homework 3:	Optional/Advanced: Position vectors and geometry	307
Homework 4:	Vector bases and rotation matrices I	315
Homework 5:	Vector differentiation	327
Homework 6:	Angular velocity and angular acceleration	335
Homework 7:	Optional/Advanced: Vector bases and rotation matrices II	347
Homework 8:	Velocity and acceleration I	355
Homework 9:	Optional: Velocity and acceleration II	371
Homework 10:	Constraints I	381
Homework 11:	Constraints II	389
Homework 12:	Particle mass, translational/angular momentum, kinetic energy, etc.	405
Homework 13:	Moments and products of inertia, dynamic Celt (rattleback)	427
Homework 14:	Optional: Inertia dyadics	437
Homework 15:	Rigid body: Momentum, energy, and equations of motion.	445
Homework 16:	Optional: Forces, force models, and statics	463
Homework 17:	Optional: Moments, torques, and static equilibrium	471
Homework 18:	$\vec{F} = m\vec{a}$ for translational motion (MG road-maps)	487
Homework 19:	$\vec{M} = \frac{d\vec{H}}{dt} + \dots$ for rotational motion (MG road-maps)	493
Homework 20:	Optional: Power, work, potential energy, conservation of energy	507
Homework 21:	Kane/Lagrange dynamics (unconstrained)	517
Homework 22:	Kane/Lagrange dynamics (with constraints)	527
Homework 23:	Optional: Dynamics with control	547
Homework 24:	Optional: MIPS simulation project	549
Energy integrals	and Hamiltonian at: www.MotionGenesis.com \Rightarrow Textbooks \Rightarrow Resources .	i